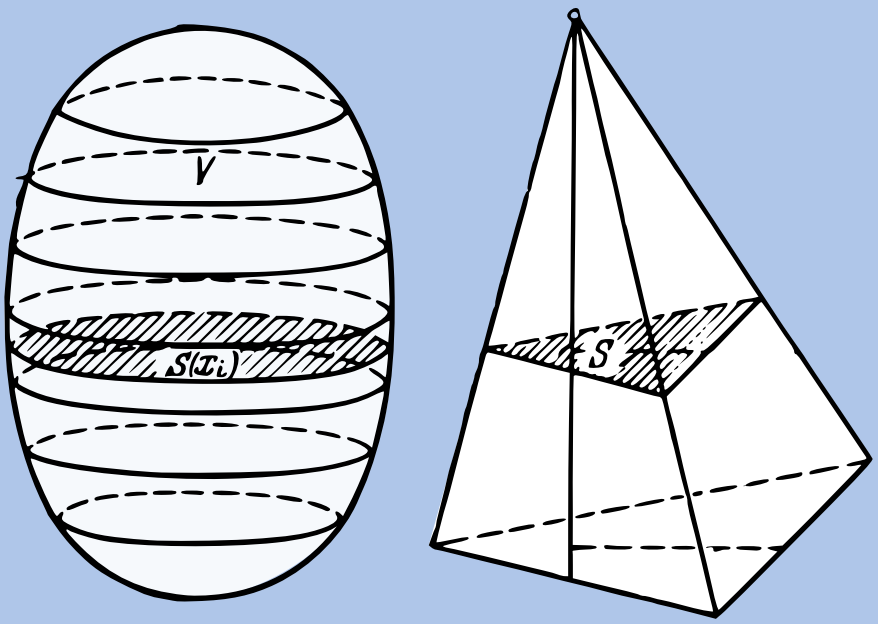
V.A. Kudryavtsev, B.R. Demidovich

A Brief Course of HIGHER MATHEMATICS



Mir Publishers Moscow

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Краткий курс высшей математики

Издательство «Наука» Москва

A Brief Course of Higher Mathematics

V. A. Kudryavtsev and B. P. Demidovich

Translated from the Russian by Leonid Levant

Mir Publishers Moscow

First published 1981

Revised from the 1978 Russian edition

На английском языке

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Introduction

Frederick Engels wrote in 'Anti-Düring': "Pure mathematics deals with the spatial forms and quantity relations of the real world—that is, with material which is very real indeed". Accordingly, mathematics, in general is divided into Geometry and Analysis.

Mathematics originated in hoary antiquity. Its first period has been named 'Elementary Mathematics' and is characterized by the following features; (a) immovability of objects under consideration, (b) non-utilization of the idea of infinity, (c) absence of general methods.

The rapid growth of production, engineering, and the natural sciences in the seventeenth and eighteenth centuries necessitated the creation of mathematical means suitable for studying functionally related variable quantities. There arose a new mathematics, the so-called 'Higher Mathematics', with its various branches: *analytical geometry*, *differential and integral calculus*, *theory of differential equations*, etc.

Analytical geometry provided a general method for solving geometrical problems (*the method of coordinates*). Powerful new notions, such as the *infinitesimal*, the *derivative*, the *integral*, an *infinite series*, and so on, which found their origin in differential and integral calculus not only enabled mathematicians to cope with ease with the famous ancient problems concerning the tangents, areas, volumes of solids, etc., but also supplied the means for solving even

more difficult problems previously considered unsolvable. Urgent problems (for those times) arising from mechanics, physics, and engineering were successfully solved with the aid of differential equations.

Here is a brief historical review of the development of higher mathematics.

The foundations of analytical geometry were provided by the famous French philosopher and mathematician, René Descartes (1596-1650). The superb English mathematician, physicist and astronomer, Isaac Newton (1642-1727), and the great German philosopher and mathematician, Gottfried Wilhelm von Leibniz (1646-1716), invented differential and integral calculus independently; though we should not forget that the ideas for the new calculus can be found in the publications of their numerous predecessors*. An outstanding role in creating classical mathematical analysis was played by Leonhard Euler (1707-1783), Joseph Louis Lagrange (1736-1813), Carl Friedrich Gauss (1777-1855), Augustin Louis Cauchy (1789-1857), Karl Theodor Wilhelm Weierstrass (1815-1897), and other well-known scientists.

It being impossible for us to completely elucidate the role Russian scientists played in the development of higher mathematics, we will confine ourselves to mentioning the most important contributions by our countrymen**.

Fundamental discoveries in almost all fields of mathematical knowledge and its applications are credited to the great mathematician L. Euler, member of the Academy in St. Petersburg. His works have retained their importance until the present day.

The Russian mathematician of genius, N. I. Lobachevsky (1792-1856), brought about a genuine revolution in geometry by creating a new science—the Lobachevskian geometry. His ideas were so new that his contemporaries did not understand them and only comparatively recently was his theory generally recognized and developed further. Lobachevsky also contributed to mathematical analysis.

---

* See G. M. Fichtenholz, *Foundations of Mathematical Analysis*, v. I, Chapter 14 (A Historical Essay on the Origin of the Principal Ideas of Mathematical Analysis), Nauka, 1968 (in Russian).

** For more detail see B. N. Delone, *Mathematics and Its Development in Russia*, Pravda Publishing House, 1948; and B. V. Gnedenko, *Essays on the History of Mathematics in Russia*, Gostekhizdat, Moscow, 1946 (in Russian).

The outstanding academician M. V. Ostrogradsky (1801-1861) derived an important relation in the theory of multiple integrals which is widely used in many applications.

The great Russian scientist, P. L. Chebyshev (1821-1894), by publishing his remarkable works on the theory of mechanisms, founded a new branch of mathematics—the theory of the best approximation of functions; it is still being strongly developed today. Chebyshev obtained classic results in many other branches of mathematics including probability theory, number theory, and integral calculus. He was the first to achieve success in solving the most difficult problem of the distribution of simple numbers, one which had remained unsolved for more than two thousand years.

Chebyshev was the founder of the Petersburg school of mathematics, one of the strongest schools in the world, including among its brilliant members A. A. Markov, A. M. Lyapunov, V. A. Steklov, A. N. Krylov, and others.

The well-known Russian mathematician, S. V. Kovalevskaya (1850-1891), obtained important results in the theory of differential equations in theoretical mechanics.

Soviet mathematicians take an active part in the development of science. Thanks to the presence in the USSR of a large number of outstanding scientists in all fields of modern mathematics, assisted by their active followers, the achievements of Soviet mathematicians are abreast of the most advanced mathematical thought.

The aim of this text is to set forth the essentials of higher mathematics and their applications in various fields. At present higher mathematics serves as the theoretical foundation for most branches of the natural, applied and engineering sciences. Therefore, every natural scientist must necessarily master its methods to be able to apply them for practical purposes.

Chapter I

The Rectangular Coordinate System in the Plane and Its Application to Simple Problems

Sec. 1. Rectangular Coordinates of a Point in the Plane

The *coordinates of a point* in the plane are defined as the numbers determining the position of this point in a plane.

In the plane the *rectangular Cartesian coordinates** are introduced in the following way: a point O (the *origin of coordinates* or, simply, the *origin*) and two mutually perpendicular directed straight lines Ox and Oy (the *coordinate axes*) are chosen in this plane (Fig. 1). For the sake of convenience we assume that the axis Ox (the *axis of abscissas*, or the x-axis) is horizontal and is directed from left to right, and the axis Oy (the *axis of ordinates*, or the y-axis) is vertical and directed upwards. Thus, the axis Oy is rotated anticlockwise with respect to the axis Ox through an angle of 90°**. Besides, a scale unit (a unit of length) is chosen to measure distances.

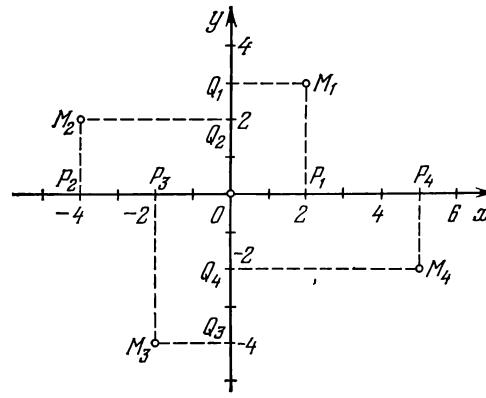


FIG. 1

For a given point M let us consider the following two numbers: the *abscissa* x and *ordinate* y of this point.

---

* Cartesian system of coordinates may be rectangular or oblique; here we consider only a rectangular system of coordinates.

René Descartes (1596-1650), celebrated French philosopher and mathematician. Cartesius is the latinized version of his name.

** Generally speaking, the arrangement of coordinate axes as well as their positive and negative directions are chosen arbitrarily.

The *abscissa* x is defined as a number expressing (in a certain scale) the distance of a point from the axis of ordinates taken with the plus sign if the point lies to the right of the axis of ordinates, and with the minus sign if the point lies to the left of this axis.

The *ordinate* y is defined as a number expressing in a certain scale (usually in the same as taken for the abscissa) the distance of a point from the axis of abscissas taken with the plus sign if the point lies above the axis of abscissas, and with the minus sign if the point lies below this axis.

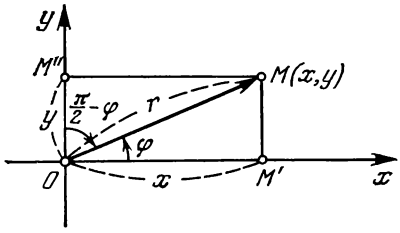


FIG. 2

The two numbers x and y are called the *rectangular coordinates* (or, simply, *coordinates*) of point M, since they completely determine the position of the point in a plane, namely, *to any pair of the numbers x and y there corresponds a unique point, these numbers being its coordinates; and conversely, any point of a plane has a pair of definite coordinates x and y*. If a point M has the coordinates x and y, then this is denoted in the following way: $M(x, y)$ (the abscissa x is followed by the ordinate y). In writing down the coordinates, the plus sign, as usual, may be omitted.

The x- and y-axes separate the plane into four compartments called the *quadrants*. Numbering the quadrants (I, II, III, and IV) in the anticlockwise direction, starting from the quadrant where both coordinates are positive, we get the following table indicating the signs of the coordinates of points located in the different quadrants:

	I	II	III	IV
x	+	−	−	+
y	+	+	−	−

The line segment OM joining the origin O to the point M (Fig. 2) is termed the *radius vector* of this point. Denoting by φ the angle formed by the segment OM with the positive direction of the x-axis, and by r its length, from the triangles OMM' and OMM'' we get

for the point M located in the first quadrant

$$\left.\begin{aligned} x &= r\cos\varphi, \\ y &= r\cos\left(\frac{\pi}{2}-\varphi\right) = r\sin\varphi. \end{aligned}\right\} \tag{1-1}$$

It is obvious that these formulas are valid for the coordinates of points situated in all the quadrants. Hence, the sign of the abscissa x of the point M coincides with the sign of the cosine, and the sign of its ordinate y with the sign of the sine in the appropriate quadrant.

It is easy to see that if a point lies on the axis of abscissas, then its ordinate y is equal to zero; if a point lies on the axis of ordinates, then its abscissa x is zero, and conversely. Consequently, if a point coincides with the origin, then both of its coordinates are equal to zero.

Example. The points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$, $M_3(x_3, y_3)$ and $M_4(x_4, y_4)$ in Fig. 1 have the following coordinates:

(1) $x_1 = Q_1M_1 = OP_1 = +2$; $\quad y_1 = P_1M_1 = OQ_1 = +3$;

(2) $x_2 = Q_2M_2 = OP_2 = -4$; $\quad y_2 = P_2M_2 = OQ_2 = +2$;

(3) $x_3 = Q_3M_3 = OP_3 = -2$; $\quad y_3 = P_3M_3 = OQ_3 = -4$;

(4) $x_4 = Q_4M_4 = OP_4 = +5$; $\quad y_4 = P_4M_4 = OQ_4 = -2$.

For the sake of brevity, henceforth we shall call the rectangular Cartesian coordinates simply *rectangular coordinates.*

Below we are going to consider a number of simplest problems solved by using the rectangular coordinates in the plane.

Sec. 2. Transformation of Rectangular Coordinates

When solving problems, it is sometimes advantageous, instead of a given rectangular coordinate system Oxy, to choose another rectangular coordinate system $O'x'y'$ oriented in a certain way with respect to the first one. For instance, for the purpose of interplanetary journeys we may use the coordinate system associated with the centre of the Earth (the so-called *geocentric* coordinate system); but it is more convenient to take advantage of the coordinate system associated with the centre of the Sun (the *heliocentric* coordinate system).

There arises a problem of transformation to new coordinates, i.e. the problem of substituting one ("old") coordinate system by another ("new") system.

We shall first consider the simplest case (Fig. 3) when the axes of the new coordinate system $O'x'y'$ are parallel to the corresponding

axes of the old coordinate system Oxy and are in the same directions (this case is called the *translation of a coordinate system*).

Suppose the origin of the new coordinate system (point O') has the coordinates (a, b) in the old coordinate system. The point M with the old coordinates (x, y) will have certain new coordinates designated for clarity $[x', y']$. From Fig. 3 we get directly

$$x' = x - a, \qquad y' = y - b, \tag{2-1}$$

i.e. *the new coordinates of a point are equal to its old coordinates less the old coordinates of the new origin.*

Conversely, from (2-1) we find

$$x = x' + a, \qquad y = y' + b. \tag{2-2}$$

Let now the new coordinate system $Ox'y'$ be rotated with respect to the old system Oxy through an angle α, with the origin O remaining unchanged (Fig. 4). That is, $\angle x'Ox = \alpha$, α being regarded as positive if the rotation is accomplished *anticlockwise*, and as negative in the opposite case (*rotation of the coordinate system*).

Let us denote by β the angle formed by the radius vector $r = OM$ of the point M with the axis Ox'. Then, taking into account the sign

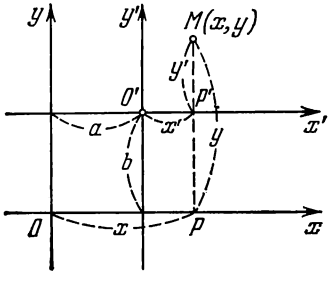


FIG. 3

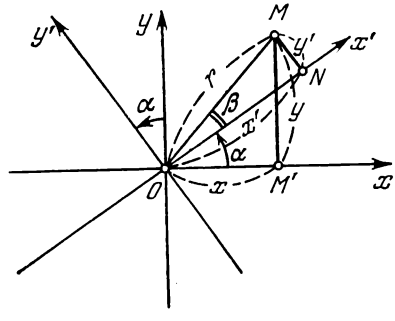


FIG. 4

of the angle β*, the line segment OM will make with the axis Ox an angle $\alpha + \beta$. Whence on the basis of formulas (1-1) for any location of the point M we have

$$x = r\cos(\alpha + \beta) = r\cos\alpha\cos\beta - r\sin\alpha\sin\beta \tag{2-3}$$

and

$$y = r\sin(\alpha + \beta) = r\sin\alpha\cos\beta + r\cos\alpha\sin\beta. \tag{2-4}$$

* Here the angle β is regarded as positive if the radius vector OM is rotated about the axis Ox' anticlockwise, and negative if it is rotated about this axis clockwise.

Since the new coordinates of the point M are, obviously,

$$x' = r\cos\beta, \qquad y' = r\sin\beta, \tag{2-5}$$

from (2-3) and (2-4) we get

$$\left.\begin{aligned} x &= x'\cos\alpha - y'\sin\alpha, \\ y &= x'\sin\alpha + y'\cos\alpha. \end{aligned}\right\} \tag{2-6}$$

When memorizing formulas (2-6) note the *lack of order* in the expression for x (cosine in front of sine, minus sign between terms on the right). On the contrary, there is *complete order* in the expression for y (first the sine, then the cosine, and a plus sign between them).

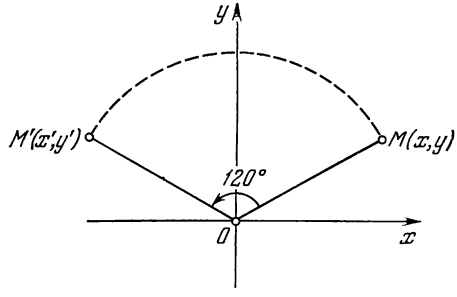


FIG. 5

Formulas (2-6) express the old coordinates x and y of the point M in terms of its new coordinates x' and y'. To express the new coordinates x' and y' in terms of the old ones x and y, it is sufficient to solve system (2-6) with respect to x' and y'. But we may proceed simpler, namely, to take the system $Ox'y'$ for the "old" one, and the system Oxy for the "new" one. Then, taking into consideration that the second system is rotated with respect to the first one through an angle $-\alpha$, replacing in formulas (2-6) x' and y' respectively by x and y, and conversely, and taking into consideration that $\cos(-\alpha) = \cos\alpha$, $\sin(-\alpha) = -\sin\alpha$, we shall have

$$\left.\begin{aligned} x' &= x\cos\alpha + y\sin\alpha, \\ y' &= -x\sin\alpha + y\cos\alpha. \end{aligned}\right\} \tag{2-7}$$

Finally, in the general case, when point $O'(a, b)$ is the new origin, and the axis $O'x'$ forms an angle α with the axis Ox, combining (2-2) and (2-6), we find

$$\left.\begin{aligned} x &= a + x'\cos\alpha - y'\sin\alpha, \\ y &= b + x'\sin\alpha + y'\cos\alpha. \end{aligned}\right\} \tag{2-8}$$

Analogously, from (2-1) and (2-7) we get

$$\left.\begin{aligned} x' &= (x-a)\cos\alpha + (y-b)\sin\alpha, \\ y' &= -(x-a)\sin\alpha + (y-b)\cos\alpha. \end{aligned}\right\} \tag{2-9}$$

From (2-8) and (2-9) it follows that the formulas for transition from one rectangular coordinate system to another rectangular coordinate

system are *linear functions* of both the new and old coordinates, i.e. they contain these coordinates in the first power.

Example. The line segment OM whose end-point M has the coordinates (x, y) is rotated anticlockwise through an angle $\alpha = 120°$ (Fig. 5). Find the coordinates x' and y' of the new position M' of the point M.

Supposing that the point M is associated with the moving coordinate system $Ox'y'$, by virtue of formulas (2-6), we shall have

$$x' = x\cos 120° - y\sin 120° = -\frac{1}{2}(x + y\sqrt{3}),$$

$$y' = x\sin 120° + y\cos 120° = \frac{1}{2}(x\sqrt{3} - y).$$

Sec. 3. The Distance Between Two Points in the Plane

(1) We first find the distance r of the point $M(x, y)$ from the origin O (0, 0) (Fig. 6).

Obviously, the distance $r = OM$ is the hypotenuse of the right-angled triangle OMM' with the legs $OM' = |x|$ and $M'M = |y|$.

Using the Pythagorean theorem, we get

$$r = \sqrt{x^2 + y^2}. \tag{3-1}$$

Thus, *the distance of a point from the origin is equal to the square root of the sum of the squares of the coordinates of this point.*

(2) In the general case, for instance, for the points $A(x_1, y_1)$ and

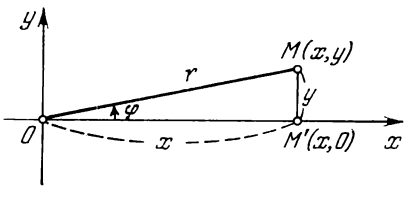


FIG. 6

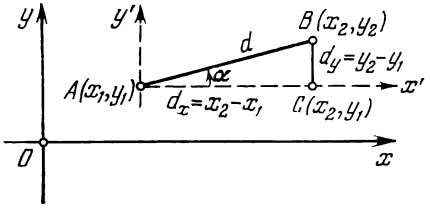


FIG. 7

$B(x_2, y_2)$ (Fig. 7), it is required to find the distance $d = AB$ between these points.

Let us choose a new coordinate system $Ax'y'$ whose origin coincides with the point A and whose axes are parallel to the old axes, being in the same direction with them, respectively. Then in the new coordinate system the points A and B will have the following coordinates: A [0, 0] and B $[x_2 - x_1, y_2 - y_1]$ (see Sec. 2). Hence,

on the basis of (3-1), we get

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \tag{3-2}$$

i.e. *the distance between two arbitrarily chosen points of a plane is equal to the square root of the sum of the squares of the differences of like coordinates of these points.*

Note. Formula (3-2) also yields the *length of the line segment AB.*

It is easy to determine the direction of this segment. From the right-angled triangle ABC we have

$$\left.\begin{aligned} d_x &= d\cos\alpha = x_2 - x_1, \\ d_y &= d\sin\alpha = y_2 - y_1 \end{aligned}\right\} \tag{3-3}$$

(d_x and d_y are called the *projections* of the line segment AB on the coordinate axes Oxy). Whence we get

$$\cos\alpha = \frac{x_2 - x_1}{d}, \quad \sin\alpha = \frac{y_2 - y_1}{d}, \quad \tan\alpha = \frac{y_2 - y_1}{x_2 - x_1},$$

where d is determined by formula (3-2).

Example. A tank moved across the country from point A (—30, 80) to point B (50, 20) (relative to a coordinate system Oxy); the coordinates are given in kilometres. Find the distance d covered by the tank if it moved without changing the direction.

Applying (3-2), we have

$$d = \sqrt{(50 + 30)^2 + (20 - 80)^2} = \sqrt{6400 + 3600} = 100 \text{ km}.$$

Sec. 4. Dividing a Line Segment in a Given Ratio

Suppose the line segment AB (Fig. 8) joining the points A (x_1, y_1) and B (x_2, y_2) is divided by the point C into two segments AC and CB, and the ratio of AC to CB is equal to l $(l \geqslant 0)$:

$$\frac{AC}{CB} = l. \tag{4-1}$$

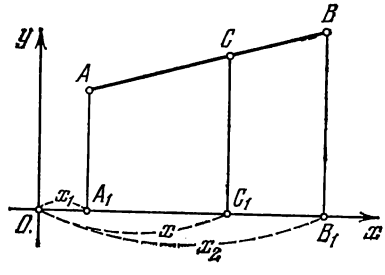


FIG. 8

It is required to express the coordinates x and y of the point C (x, y) in terms of the coordinates of the end-points of the line segment AB.

We drop perpendiculars AA_1, BB_1, and CC_1 onto the axis Ox from the points A, B, and C, respectively. Now we have three parallel lines A_1A, C_1C, B_1B intersecting the sides of the angle (not shown in the drawing) formed by the lines AB and Ox. As is known from elementary geometry, a pencil of parallel straight lines cuts

the sides of an angle into proportional parts; therefore

$$\frac{A_1C_1}{C_1B_1} = \frac{AC}{CB},$$

whence, by (4-1), we shall have

$$\frac{A_1C_1}{C_1B_1} = l. \tag{4-2}$$

As is obvious from Fig. 8,

$$A_1C_1 = OC_1 - OA_1 = x - x_1,$$

$$C_1B_1 = OB_1 - OC_1 = x_2 - x.$$

Substituting these expressions into (4-2), we get

$$\frac{x-x_1}{x_2-x} = l. \tag{4-3}$$

Solving (4-3) with respect to the unknown abscissa x, we shall have

$$x = \frac{x_1+lx_2}{1+l};$$

analogously,

$$y = \frac{y_1+ly_2}{1+l}.$$

Thus, the coordinates of the point $C(x, y)$ dividing the line segment AB in the ratio l (as measured from A to B) are determined by the formulas

$$\left.\begin{aligned} x &= \frac{x_1+lx_2}{1+l}, \\ y &= \frac{y_1+ly_2}{1+l}. \end{aligned}\right\} \tag{4-4}$$

If the point C bisects the segment AB, then $AC = CB$ and, consequently,

$$l = \frac{AC}{CB} = 1.$$

Denoting the coordinates of the midpoint of AB by $\bar{x}$, $\bar{y}$, we get by virtue of (4-4)

$$\left.\begin{aligned} \bar{x} &= \frac{x_1+x_2}{2}, \\ \bar{y} &= \frac{y_1+y_2}{2}, \end{aligned}\right\} \tag{4-5}$$

i.e. *the coordinates of the midpoint of a line segment are equal to the half-sums of the corresponding coordinates of its end-points.*

Note. When deriving formulas (4-4) and (4-5), we proceeded from the assumption that the end-points A and B of the line segment AB lie in the first quadrant, and, hence, the coordinates of the points A and B are positive. It is easy to prove that formulas (4-4) and (4-5) will also hold when one or both end-points of AB lie in other quadrants and, consequently, some or all coordinates of the points A and B turn out to be negative.

Example. Compute the coordinates of the point $C(x, y)$ dividing the line segment AB between the points $A(-5, -3)$ and $B(4, -6)$ in the ratio $\frac{AC}{CB} = \frac{3}{2}$.

In this case $l = \frac{3}{2}$ and, hence,

$$x = \frac{-5+\frac{3}{2}\cdot 4}{1+\frac{3}{2}} = \frac{2}{5}, \qquad y = \frac{-3+\frac{3}{2}\cdot(-6)}{1+\frac{3}{2}} = -4\frac{4}{5}.$$

Sec. 5. The Area of a Triangle

Let it be required to find the area S of the triangle ABC (Fig. 9) with the vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.

Let $AB = c$, $AC = b$, and the angles formed by these sides with the x-axis are equal to α and β, respectively.

On the basis of Sec. 3 (see the note) we have

$$\left.\begin{aligned} A'B' &= c_x = c\cos\alpha = x_2 - x_1, \\ A''B'' &= c_y = c\sin\alpha = y_2 - y_1 \end{aligned}\right\} \tag{5-1}$$

and

$$\left.\begin{aligned} A'C' &= b_x = b\cos\beta = x_3 - x_1, \\ A''C'' &= b_y = b\sin\beta = y_3 - y_1. \end{aligned}\right\} \tag{5-2}$$

Let $\varphi = \angle CAB$; obviously (Fig. 9),

$$\varphi = \beta - \alpha.$$

Using the known trigonometric formula, we get

$$S = \frac{1}{2}bc\sin\varphi = \frac{1}{2}bc\sin(\beta-\alpha) =$$

$$= \frac{1}{2}bc(\sin\beta\cos\alpha - \cos\beta\sin\alpha) = \frac{1}{2}(b_y c_x - b_x c_y). \tag{5-3}$$

Hence, by (5-1) and (5-2), we have

$$S = \frac{1}{2}[(y_3-y_1)(x_2-x_1) - (x_3-x_1)(y_2-y_1)]. \tag{5-4}$$

It should be noted here that with the vertices arranged in a different way formula (5-4) may yield the area S of the triangle with a minus sign. Therefore the formula for the area of a triangle is usually written in the form

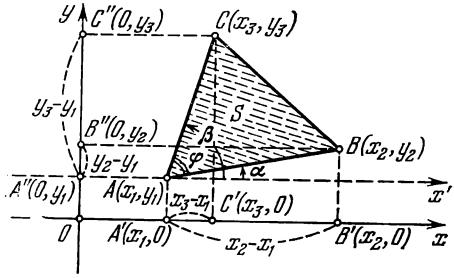


FIG. 9

$$S = \pm \frac{1}{2} [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)], \qquad (5\text{-}4')$$

where the sign is chosen so that a positive number is obtained for the required area.

Using the notion of a *second-order determinant*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

formula (5-4′) may be written in the form to be readily memorized

$$S = \pm \frac{1}{2} \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix} \qquad (5\text{-}5)$$

Formula (5-4′) becomes simplified if the point A (x_1, y_1) is situated at the origin. Namely, putting $x_1 = 0$, $y_1 = 0$, we get

$$S = \pm \frac{1}{2} (x_2 y_3 - x_3 y_2).$$

Let us note that if A, B, C are collinear points, then the area $S = 0$; and conversely, if $S = 0$, then the vertices A, B, and C are on a single straight line.

Example. A field has the form of a triangle with the vertices A $(-2, -1)$, B $(3, 5)$, and C $(-1, 4)$ (the dimensions are given in kilometres). Determine the area S of this field.

By (5-5), we have

$$S = \pm \frac{1}{2} \begin{vmatrix} 3+2 & -1+2 \\ 5+1 & 4+1 \end{vmatrix} = \pm \frac{1}{2} \begin{vmatrix} 5 & 1 \\ 6 & 5 \end{vmatrix} =$$

$$= \pm \frac{1}{2} (25 - 6) = 9.5 \text{ km}^2 = 950 \text{ ha}.$$

Note. The determination of the area of a polygon is reduced to determining the areas of triangles. Therefore, it is sufficient to divide a polygon into triangles whose areas are calculated by (5-4).

Exercises

1. Construct the following points: A (2, 3), B (—4, 1), C (2, —3), D (—2, —2), E (—5, 0).

2. Determine the coordinates of the vertices of an equilateral triangle located in the first quadrant with the side equal to 10 if one of its vertices coincides with the origin O, and the base of the triangle is situated on the x-axis.

3. Determine the coordinates of the point $M(x, y)$ symmetrical to the point A (1, 2) about: (a) the x-axis; (b) the y-axis; (c) the bisector of the first and third quadrants; (d) the bisector of the second and fourth quadrants.

4. A straight line MN is parallel to the axis of ordinates and lies to the right of it at a distance of 5 length units. Find the coordinates of the point A_1 symmetrical to the point A (2, 4) about the line MN, and coordinates of the point B_1 symmetrical to the point B (—1, 3) about the same line MN.

5. On the axis Ox find the point whose distance from the point A (3, 4) is equal to 5.

6. A line segment AB with the end-points A (2, 5) and B (4, 8) is divided by a point C in the ratio 2 : 3. Find the coordinates of the point C.

7. Point C (2, 3) divides a line segment in the ratio 1 : 2. Find the coordinates of the point B if it is known that the point A has the coordinates $x = 1$, $y = 2$.

8. The vertices of a triangle are A (—2, 0), B (6, 6), and C (1, —4). Find the length of the bisector drawn from the vertex A.

9. Placed at points A (—2, 1) and B (7, 4) are the masses $m_1 = 10$ g and $m_2 = 20$ g, respectively. Determine the coordinates of the centre of mass of this system.

10. Find the coordinates of the centroid N of the triangle ABC with the vertices: A (—2, 1), B (2, —1), C (4, 3). (The centroid of the triangle coincides with the point of intersection of its medians. This point is two-thirds of the distance from a vertex to the opposite side along a median.)

11. The line segment between the points $A(x_0, y_0)$ and $B(x, y)$ is divided into n equal parts. Determine the coordinates x_i, y_i ($i = 1, 2, \ldots, n - 1$) of the division points.

12. Compute the area of the triangle with the vertices A (—2, —2), B (—1, 3), and C (3, —1).

13. Show that the points A (—7, —3), B (—1, 1) and C (2, 3) are collinear.

14. The area of the triangle ABC with the vertices A (—2, 1), B (2, 2), C (4, y) equals 15. Determine the ordinate of vertex C.

15. A forest has the shape of a quadrilateral whose vertices are: A (0 m, 200 m), B (200 m, 100 m), C (500 m, 300 m), and D (100 m, 700 m). Find the area of the forest.

Chapter 2

The Equation of a Line

Sec. 6. Sets

A *set* $X = \{x, x', x'', \ldots\}$ is understood as a collection of some *elements* (or members) $x, x', x'', \ldots$. If x is a member of a set X, then we write

$$x \in X$$

(we read $\in$ as "is a member of, or belongs to", hence, the above expression is read as "x belongs to X"); if y is not a member of this set, then we write

$$y \notin X$$

(which is read as "y does not belong to X").

Example 1. X, the set of all students in a given lecture-room.

Example 2. $X = \{1, 2, 3, \ldots\}$, the set of natural numbers.

It is convenient to introduce the notion of the empty (or null) set, i.e. a set that contains no elements, which is denoted by ϕ. In particular, this frees us from the necessity to prove each time the existence of at least one element belonging to a given set.

Example 3. The set of three-headed people is empty.

The sets X and X' are considered *to be equal*, written

$$X = X'$$

if they consist of precisely the same elements.

Definition 1. *A set Y comprising a part of the elements belonging to the set X or coinciding with it is called a subset of X; in this case we write*

$$Y \subset X \tag{6-1}$$

and read as "Y is contained in X".

An empty set is conventionally regarded as a subset of any set. Graphically the idea of subset is illustrated in Fig. 10.

If the symbol $\forall$ is understood as "for any", then relationship (6-1) is equivalent to the following:

$$\forall\, y \in Y \Rightarrow y \in X, \tag{6-1'}$$

where the arrow $\Rightarrow$ substitutes the word "follows".

Example 4. Let X be the set of all students of the first course, and Y the set of all girl students of the first course. Obviously,

$$Y \subset X.$$

If $Y \subset X$ and $X \subset Y$, then, obviously,

$$X = Y.$$

Definition 2. *The union (or sum) of two sets X and Y (written $X \cup Y$) is defined as the set (or collection) of all the elements belonging to at*

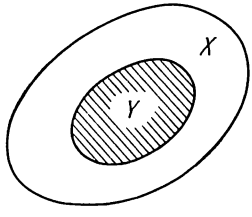


FIG. 10

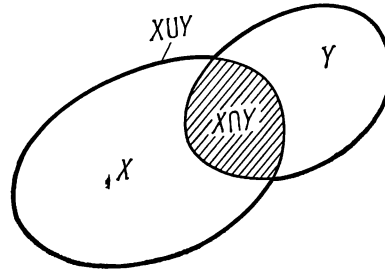


FIG. 11

least one of the given sets, i.e. to X or Y, or to both sets (see Fig. 11).

The *union* of more than two sets is defined analogously. For instance, the union $X \cup Y \cup Z$ of three sets is understood as the set of all elements belonging at least to one of the sets X, Y, Z. Logically, the symbol for the union of sets corresponds to the copulative "or".

Example 5. $\{1, 2, 3\} \cup \{2, 3, 4\} = \{1, 2, 3, 4\}$.

Definition 3. *The set of all elements which belong to both X and Y is called the intersection (or product) of the sets X and Y (the common part of the sets), written $X \cap Y$* (Fig. 11).

Thus, logically the symbol for the intersection corresponds to the conjunction "and". If the sets X and Y have no common elements, then their intersection is empty

$$X \cap Y = \phi.$$

The *intersection* of more than two sets is defined similarly. For instance, the intersection $X \cap Y \cap Z$ of three sets is understood as

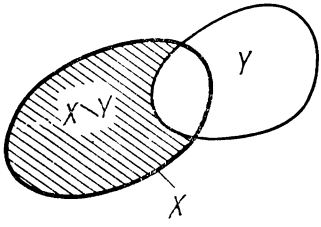


FIG. 12

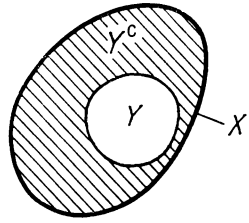


FIG. 13

the set of all elements which belong simultaneously to the sets X, Y, and Z.

Example 6.

$\{1, 2, 3\} \cap \{2, 3, 4\} = \{2, 3\}$.

Definition 4. *The set consisting of all elements of X that do not belong to Y is called the difference between the sets X and Y and is denoted $X \setminus Y$* (Fig. 12).

If

$$Y \subset X,$$

then the set

$$Y^{\mathbf{C}} = X \setminus Y$$

is called the *complement* of the set Y to the set X (Fig. 13).

Obviously,

$$Y \cup Y^{\mathbf{C}} = X, \qquad Y \cap Y^{\mathbf{C}} = \phi.$$

Example 7.

$\{1, 2, 3\} \setminus \{2, 3, 4\} = \{1\}$.

Sec. 7. The Method of Coordinates in the Plane

It was shown in Chapter 1 how, using rectangular coordinates, we can solve geometrical problems algebraically.

The branch of mathematics which studies the properties of geometrical figures with the aid of algebraic methods of reasoning is named *analytical geometry* (Cartesian, or coordinate geometry).

Analytical geometry grew out of the need for establishing uniform techniques for solving geometrical problems, the aim being to apply them to the study of curves, which are of particular importance in

practical problems. This aim was achieved in the coordinate method. In this method, calculations are fundamental, while constructions play a subordinate role.

The origins of the coordinate method go back to the works of the ancient Greek mathematicians, in particular Apollonius (3-2 century B.C.). The coordinate method was systematically elaborated in the first half of the 17th century in the works of Fermat* and Descartes**. However, they considered only plane curves. It was Euler*** who first applied the coordinate method in a systematic study of space curves and surfaces.

In the preceding chapter we used the coordinate method for solving a number of important but particular problems. Now we are going to start a systematic study of how analytical geometry solves the *general problem* consisting in the determination of the shape, position, and properties of a given line using the methods of mathematical analysis.

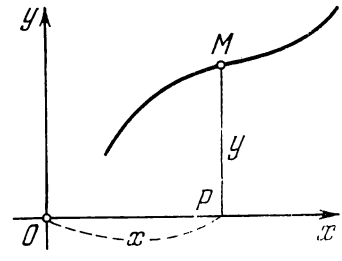


FIG. 14

Suppose we have a line in the plane (Fig. 14). The coordinates x and y of a point lying on this line are not completely arbitrary. They must meet certain limitations stipulated by the geometrical properties of a given line. The fact that the numbers x and y are the coordinates of a point lying on a given line is written analytically in the form of an equation. This equation is said to be the *equation of a line in the plane.*

By representing each point in the plane by its coordinates, and each line by an equation that relates the running (moving, or current) coordinates, we reduce geometrical problems to analytical (computational) problems. Hence, the name "analytical geometry".

Thus, the essence of the coordinate method in the plane consists in that any plane line is associated with its equation (or, more

* Pierre Fermat (1601-1655), celebrated French mathematician, one of the forerunners of Newton and Leibniz in developing the differential calculus; made a great contribution in the theory of numbers. Most of Fermat's works (including those on analytical geometry) were not published during the author's lifetime.

** The year 1637, which saw the publication of his *Geometrie*, an appendix to his philosophical treatise, is taken to be the date of birth of analytical geometry.

*** Leonhard Euler (1707-1783), gifted Swiss mathematician, wrote over 800 scientific papers and made important discoveries in all of the physico-mathematical sciences. He spent most of his life as a member of the Academies in Berlin and in St. Petersburg (now Leningrad).

precisely, the class of equivalent equations), and then the properties of this line are studied analytically investigating the corresponding equation.

Sec. 8. The Line as a Set of Points

A line in the plane is usually specified as a *set of points* possessing certain geometrical properties inherent exclusively in them.

Example 1. A circle of radius R (Fig. 15) is a set of all points of the plane situated at a distance R from one of its points O (*the centre of the circle*).

In other words, situated on the circumference of a circle are those and only those points whose distance from the centre of the circle is equal to its radius.

Example 2. The bisector of the angle ABC (Fig. 16) is a set of all points lying inside the angle and equidistant from its sides.

This statement asserts that: (1) for any point M lying on the bisector BD the lengths of the perpendiculars MP and MQ dropped

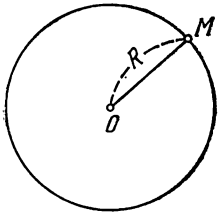


FIG. 15

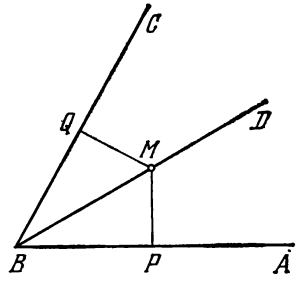


FIG. 16

onto the sides BA and BC, respectively, are equal to each other, i.e. $MP = MQ$, and (2) any point lying inside the angle ABC but not on its bisector will be nearer to one side of the angle than to the other.

Sec. 9. The Equation of a Line in the Plane

Let us now formulate more precisely the definition of the equation of a line* in the plane.

* The line will be frequently called the *curve* irrespective of whether it is rectilinear or not.

Definition. *The equation of a line (curve) on the plane Oxy is defined as an equation which is satisfied by the coordinates x and y of each point of a given line, and is not satisfied by the coordinates of any point not lying on this line.*

Hence, in order to establish that a given equation is the equation of a line K, it is necessary and sufficient to prove that: (1) the coordinates of any point lying on the line K satisfy this equation, and (2) conversely, if the coordinates of a point satisfy this equation, then the point necessarily lies on the line K.

Whence it follows automatically that: (1′) if the coordinates of a point do not satisfy a given equation, then this point does not lie on the line K, and (2′) if a point is not on the line K, then its coordinates do not satisfy the given equation.

If a point $M(x, y)$ moves along the line K, then its coordinates x and y continuously vary but all the time satisfy this equation. Therefore, the coordinates of the point $M(x, y)$ are called the *current* (running, or moving) *coordinates* of a point of the line K.

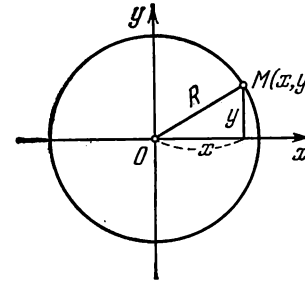


FIG. 17

In the plane Oxy the current coordinates of a point M of a given curve K are usually denoted by x and y, the first of them being the abscissa of the point M, and the second its ordinate. But if it is appropriate, the current coordinates of the point M may be denoted by any letters, for instance, $M(X, Y)$, or $M(\xi, \eta)$, and so on. Thus, for instance, the equations

$$y = 2x \quad \text{and} \quad Y = 2X,$$

where the points $N(x, y)$ and $N(X, Y)$ are situated in the plane Oxy, represent the equation of one and the same straight line contained in this plane.

Let us explain the principal concept of analytical geometry, i.e. the *equation of a line* by giving a number of examples.

Example 1. Form the equation of a circle of radius R with centre at the origin.

We take on the circle (Fig. 17) an arbitrary point $M(x, y)$ and join it to the centre O. By definition we have $OM = R$, i.e.

$$\sqrt{x^2+y^2} = R,$$

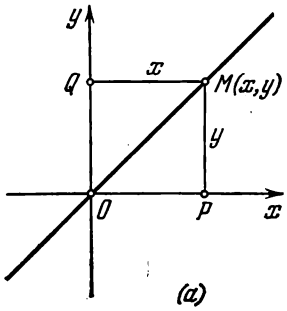
whence

$$x^2 + y^2 = R^2. \tag{9-1}$$

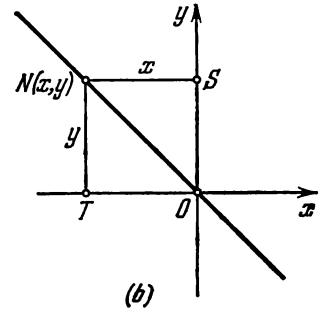
Equation (9-1) relates the coordinates x and y of each point of the given circle. Conversely, if the coordinates of the point $M(x, y)$ satisfy equation (9-1), then, obviously, $OM = R$ and, hence, this point lies on our circle. Thus, (9-1) represents the equation of a circle of radius R with centre at the origin.

Example 2. Form the equations of the bisectors of the quadrants.

We first consider the bisector of the first and third quadrants (Fig. 18, *a*). Let us take on it an arbitrary point $M(x, y)$. If the point M lies in the first quadrant, then its coordinates are positive and equal to each other (by the property of the bisector). Now, if



(a)



(b)

FIG. 18

the point $M(x, y)$ lies in the third quadrant, then its abscissa and ordinate will be both negative, their absolute values being equal to each other. Therefore, the coordinates x and y of this point will also be equal to each other. Consequently, in both cases we have

$$y = x. \tag{9-2}$$

Conversely, if the coordinates x and y of some point $M(x, y)$ satisfy equation (9-2), then this point, obviously, lies on the bisector of the first and third quadrants. Therefore, (9-2) represents the equation of the bisector of the first and third quadrants.

Let us now consider the bisector of the second and fourth quadrants (Fig. 18, *b*). We take on it an arbitrary point $N(x, y)$. Its coordinates x and y are equal to each other by the absolute value and differ in signs irrespective of whether this point is situated in the second or fourth quadrant. Hence, in both cases we have

$$y = -x. \tag{9-3}$$

Conversely, if for some point $N(x, y)$ equation (9-3) is fulfilled, then, obviously, this point lies on the bisector of the second and fourth quadrants. Hence, (9-3) is the equation of the bisector of the second and fourth quadrants.

Example 3. Form the equation of a straight line parallel to the axis of ordinates.

Let the straight line AB be parallel to the y-axis, and the line segment OA be equal to a (Fig. 19, *a*). Then for any point $M(x, y)$ of the line AB its abscissa x is equal to a:

$$x = a. \tag{9-4}$$

Conversely, if the abscissa of some point $M(x, y)$ is equal to a, then this point lies on the straight line AB.

Thus, (9-4) represents the equation of a straight line parallel to the y-axis and situated at a distance equal to the numerical value

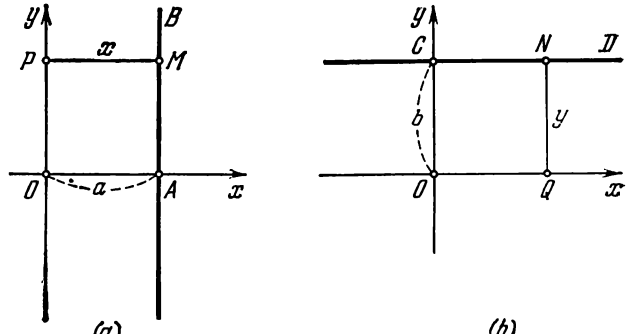


FIG. 19

of a; in this case if the line is situated to the right of the y-axis, then a is positive; and if it is to the left of the y-axis, then a is negative.

In particular, for $a = 0$ we get the equation of the axis of ordinates:

$$x = 0.$$

Example 4. Form the equation of a straight line parallel to the axis of abscissas.

Quite analogously, if the line CD is parallel to the x-axis and $OC = b$ (Fig. 19, *b*), then its equation will be

$$y = b;$$

here if the line CD is situated above the x-axis, then b is positive, and if it is below the x-axis, then b is negative.

In particular, for $b = 0$ we get the equation of the axis of abscissas:

$$y = 0.$$

Example 5. Find the line the distance of whose points from the point B (12, 16) is twice their distance from the point A (3, 4).

If $M(x, y)$ is an arbitrary point of the required line, then, according to the initial condition of the problem, we have

$$2AM = BM. \tag{9-5}$$

To form the equation of this line, we have to express AM and BM in terms of the coordinates x and y of the point M. By virtue of the formula for the distance between two points (see Sec. 3), we have

$$AM = \sqrt{(x-3)^2 + (y-4)^2},$$

$$BM = \sqrt{(x-12)^2 + (y-16)^2},$$

whence, according to (9-5),

$$2\sqrt{(x-3)^2 + (y-4)^2} = \sqrt{(x-12)^2 + (y-16)^2},$$

which is the equation of the required line.

But it is difficult to judge by such a form what line is represented by this equation, therefore we are going to simplify it. Squaring both members and removing the parentheses, we get

$$4x^2 - 24x + 36 + 4y^2 - 32y + 64 = x^2 - 24x + 144 + \\ + y^2 - 32y + 256,$$

and after simple transformations we have an equivalent equation

$$x^2 + y^2 = 100.$$

Comparing the obtained equation with (9-1), we see that the required line is a circle of radius 10 with centre at the origin.

Sec. 10. Constructing a Line on the Basis of Its Equation

If the variables x and y are related by some equation, then the set of points $M(x, y)$ whose coordinates satisfy this equation represents, generally speaking, a line in the plane ("*geometrical image of an equation*").

In particular cases this line may degenerate into one or several points. There are also cases when to an equation there corresponds no set of points.

For instance, to the equation

$$(x - 1)^2 + (y - 2)^2 = 0$$

there corresponds a single point (1, 2), since this equation is satisfied by a single pair of values: $x = 1$ and $y = 2$.

To the equation

$$x^2 + y^2 = -1$$

there corresponds no set of points, since this equation is satisfied by none of the real values of x and y.

Knowing the equation of a line, we can plot this line point by point.

Example. Construct the line represented by the equation

$$y = x^2 \tag{10-1}$$

(we usually say: construct the line $y = x^2$).

Assigning numerical values to the abscissa x in the equation (10-1) and calculating the corresponding values of the ordinate y, we obtain the following table:

x	...	−3	−2	−1	0	1	2	3	...
y	...	9	4	1	0	1	4	9	...

Plotting the corresponding points on a plane, we see that the configuration of these points determines the contour of some line; the more points are plotted, the more distinctly its contour is seen. Joining the constructed points with a curve which takes into account the positions of the intermediate points*, we get the line defined by the given equation (10-1). This line is called the *parabola* (Fig. 20).

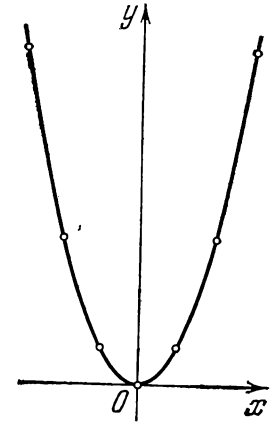


FIG. 20

Sec. 11. Some Elementary Problems

If the equation of a line is known, then we can easily solve elementary problems concerning the arrangement of this line in the plane.

Problem 1. Given: the equation of line K and coordinates of the point $M(a, b)$. Required: determine whether the point M lies on the line K or it does not. In other words, it is required to find out whether or not the line K passes through the point M.

On the basis of the concept of the equation of a line, we obtain the following rule: *To find out whether a point M lies on a given line K, we have to substitute the coordinates of our point into the equation of*

* For more detail see Chapter 11.

this line. If the equation is satisfied (i.e. the substitution results in an equality), *then the point lies on the line, otherwise the given point does not lie on the line.*

In a particular case the line passes through the origin if and only if the equation of the line is satisfied for $x = 0$ and $y = 0$.

Example 1. Given: a circle

$$x^2 + y^2 = 25. \tag{11-1}$$

Required: find out whether the points M $(-3, 4)$ and N $(4, -2)$ lie on this circle.

Substituting the coordinates of the point M into equation (11-1), we get an identity

$$(-3)^2 + 4^2 = 25.$$

Hence, the point M lies on the given circle.

Analogously, substituting the coordinates of the point N into the equation (11-1), we shall have

$$4^2 + (-2)^2 \neq 25.$$

Consequently, the point N does not lie on the given circle.

Problem 2. Find the point of intersection of two lines specified by their equations.

The point of intersection is situated simultaneously on both lines. Consequently, the coordinates of this point satisfy the equations of both lines. Hence, we obtain the rule: *To find the coordinates of the point of intersection of two lines, it is sufficient to jointly solve the system of their equations.* If this system has no real solutions, then it means that the lines do not intersect.

Example 2. Find the points of intersection of a parabola $y = x^2$ and a straight line $y = 4$.

Solving the system

$$\left.\begin{aligned} y &= x^2, \\ y &= 4, \end{aligned}\right\}$$

we obtain two points of intersection: A $(-2, 4)$ and B $(2, 4)$.

Problem 3. Find the points of intersection of a given line with the coordinate axes.

This problem is a particular case of Problem 2.

Bearing in mind that the equation of the x-axis is $y = 0$, we get the rule: *To find the abscissas of the points of intersection of a given line with the x-axis, we have to put* $y = 0$ *in the equation of this line and to solve the obtained equation with respect to* x.

Analogously, since the equation of the y-axis is $x = 0$, we get

the required rule: *To find the ordinates of the points of intersection of a given line with the y-axis, we have to put* $x = 0$ *in the equation of this line and to solve the obtained equation with respect to* y.

Example 3. Find the points of intersection of the circle

$$x^2 + y^2 = 1 \qquad (11\text{-}2)$$

with the coordinate axes.

Putting $y = 0$ in the above equation, we get $x^2 = 1$, i.e. $x_1 = -1$ and $x_2 = 1$. Whence we find two points of intersection of the given circle with the x-axis (Fig. 21): A $(-1, 0)$ and B $(1, 0)$.

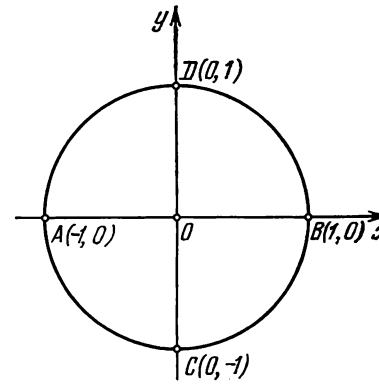


FIG. 21

Similarly, putting $x = 0$ in equation (11-2), we get $y^2 = 1$, i.e. $y_1 = -1$ and $y_2 = 1$. Consequently, there are two points of intersection of the given circle with the y-axis (Fig. 21): C $(0, -1)$ and D $(0, 1)$.

Sec. 12. Two Basic Problems of Plane Analytical Geometry

Summarizing the contents of this chapter, we may say that to any line in the plane there corresponds some equation relating the current coordinates (x, y) of a point belonging to this line. Conversely, to any equation relating x and y, where x and y are the coordinates of a point in the plane, there corresponds, generally speaking, a line whose properties are fully defined by a given equation.

Naturally, there arise two fundamental problems of plane analytical geometry:

(1) *Given: a line regarded as the set of points. Required: form the equation of this line.*

(2) *Given: the equation of a line. Required: study the geometrical properties (the shape and position) of the line by its equation.*

Sec. 13. Algebraic Lines

Definition. *A plane line is called an algebraic line (or curve) of degree* n $(n = 1, 2, \ldots)$, *if it is defined by an equation of the* n*th degree with respect to the current rectangular coordinates.*

For instance, the lines

$$x + y - 1 = 0, \quad x^2 + y^2 = 1, \quad x^3 + y^3 - 3xy = 0$$

are curves of the first, second, and third degree, respectively.

The general form of first-degree curves is

$$Ax + By + C = 0,$$

where the coefficients A and B are not both zero, i.e. $A^2 + B^2 \neq 0$. As will be proved below (see Chapter 3), all curves of degree one are straight lines.

The general form of second-degree curves is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

in which the coefficients A, B, and C are not all zero, i.e. $A^2 + B^2 + C^2 \neq 0$.

It is worth noting that not to any second-degree equation there corresponds a real curve. For instance, no curve in the plane Oxy corresponds to the equation $x^2 + 2xy + y^2 + 1 = 0$, since, obviously, there are no real numbers x and y satisfying this equation.

In the subsequent chapters we shall study in detail the curve of the first degree (the straight line) and consider the main representatives of the second-degree curves (the circle, ellipse, hyperbola, and parabola).

The equation of a curve of the nth degree may be written in the following form:

$$\sum_{\substack{p,\ q=0,\\ p+q\leqslant n}}^{n} a_{pq}x^p y^q = 0, \tag{13-1}$$

where at least one of senior coefficients a_{pq}, i.e. such that $p + q = n$, is nonzero ($\sum$ is the symbol for summation).

Let us mention here the following important property: *the degree of curve* (13-1) *is independent of the choice of a rectangular coordinate system.*

Indeed, taking another system of rectangular coordinates $O'x'y'$, by the transition formulas (see Sec. 2), we have

$$\left.\begin{aligned} x &= a_1x' + b_1y' + c_1, \\ y &= a_2x' + b_2y' + c_2, \end{aligned}\right\} \tag{13-2}$$

where a_i, b_i, c_i $(i = 1, 2)$ are constant coefficients.

Hence, the equation of curve (13-1) in the new coordinates $O'x'y'$ will have the form

$$\sum_{\substack{p',\ q'=0\\ p'+q'\leqslant n'}}^{n'} a'_{p'q'}x'^{p'}y'^{q'} = 0, \tag{13-3}$$

where n' is the degree of the transformed curve. It is obvious that $n' \leqslant n$.

Analogously, proceeding from equation (13-3) and returning back to the coordinates x, y, we get (13-1) in which $n \leqslant n'$. Hence, $n' = n$.

Exercises

1. Form the equation of the line the distance of whose points from the x-axis is twice the distance from the y-axis.
2. Form the equation of the line whose points are equidistant from two given points: A (2, 1) and B (—3, 0).
3. What curve is described by the centroid of the triangle ABC with two fixed vertices A (6, 0) and B (—6, 0) if the third vertex C (x_3, y_3) describes a circle $x_3^2 + y_3^2 = 36$?
4. What geometrical images correspond to the following equations: (a) $xy = 0$; (b) $x^2 + y^2 = 0$; (c) $x^2 - 1 = 0$; (d) $y^2 - 3y + 2 = 0$; (e) $x^2 - xy = 0$?
5. Plot the curves specified by the equations: (a) $y = 2 - x$; (b) $y = 2x - x^2$; (c) $y = \pm \sqrt{100 - x^2}$; (d) $y = \pm \frac{3}{5} \sqrt{100 - x^2}$.
6. Which of the given points A (0, 0), B (1, 1), C (1, —1), D (—1, —1), E (1, 2) lie on the curve $y = x^2$ and which do not?
7. Find the points of intersection of the curve $y = 2 + x - x^2$ with the coordinate axes.
8. Find the points of intersection of the circle $x^2 + y^2 = 8$ and the straight line $x - y = 0$.

Chapter 3

The Straight Line

Sec. 14. The Equation of a Straight Line

Let PQ be a straight line in the plane Oxy (Fig. 22). Through an arbitrary point $M_0\,(x_0,\ y_0)$ on this line (conventionally called the "initial point") we draw a straight line M_0x' parallel to the x-axis and in the same direction. Then the least non-negative angle $\varphi = \angle\, QM_0x'$ $(0 \leqslant \varphi < \pi)$ formed by the half-line M_0Q lying above the axis M_0x' or coinciding with it with the axis M_0x' is called the *angle* between the given straight line and the x-axis. Obviously,

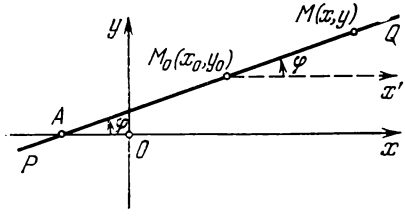


FIG. 22

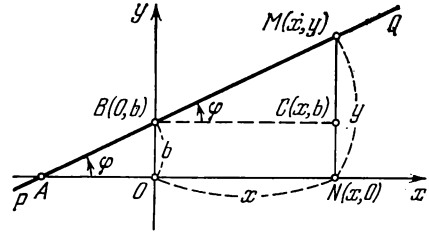


FIG. 23

this angle is independent of the choice of the point M_0. If the straight line intersects the x-axis at some point $A\ (a,\ 0)$, then φ is a usual angle between directed straight lines. If PQ is parallel to Ox, then, obviously, $\varphi = 0$. The initial point M_0 of a straight line and the angle φ (the "direction of the line") uniquely determine the position of this line in the plane.

(1) Let us first consider the case $0 \leqslant \varphi < \frac{\pi}{2}$. In this case the straight line PQ intersects the y-axis at point $B\ (0,\ b)$ which may be taken for the initial one.

The ordinate $y = NM$ of a moving point $M\ (x,\ y)$ of a straight line (Fig. 23) consists of two parts:

$$y = NC + CM \tag{14-1}$$

of which the first part is constant and the second is variable. Introducing the slope

$$\tan \varphi = k,$$

from Fig. 23 we shall have

$$NC = b \quad \text{and} \quad CM = BC \tan \varphi = kx \tag{14-2}$$

for $x \geqslant 0$.

Thus,

$$y = b + kx \tag{14-3}$$

for $x \geqslant 0$.

It is easy to check that formula (14-3) also holds for $x < 0$.

We have proved that the coordinates of any point $M(x, y)$ of the straight line PQ satisfy equation (14-3). It is easy to prove the converse: if the coordinates of an arbitrary point $M_1(x_1, y_1)$ satisfy equation (14-3), then the point M_1 *necessarily* lies on the straight line PQ. Consequently, (14-3) represents the equation of the straight line PQ (so-called *slope-intercept form of the equation of a straight line*). The constant quantities b and k (*parameters*) have the following meaning: $b = OB$ is the *initial intercept* (or more precisely, the *initial ordinate*), and $k = \tan \varphi$ is the *slope*. It is worth noting that if the point B is situated above the x-axis, then $b > 0$, and if below it, then $b < 0$. For $b = 0$ the line passes through the origin, and the equation of such a straight line is

$$y = kx. \tag{14-4}$$

For $k = 0$ we get the equation of a straight line parallel to the x-axis:

$$y = b.$$

(2) If $\frac{\pi}{2} < \varphi < \pi$, then, reasoning in an analogous manner, we also arrive at equation (14-3).

(3) If $\varphi = \frac{\pi}{2}$, i.e. the line AB is perpendicular to the axis Ox, then its equation is (see Chapter 2)

$$x = a, \tag{14-5}$$

where a is the abscissa of the *trace* of this line on the x-axis (i.e. its point of intersection with the axis Ox).

Note. As particular cases, we get the equations of the coordinate axes:

$$y = 0 \ (x\text{-axis}) \quad \text{and} \quad x = 0 \ (y\text{-axis}). \tag{14-6}$$

A straight line can be readily constructed from its equation.

Example. Construct the straight line given by the equation

$$y = \frac{3}{2}x - 4.$$

As is known, two points completely determine the position of a straight line. Therefore, it suffices to find two points through which our line passes. In the given equation $b = -4$; hence, the straight line passes through the point $B\,(0, -4)$. On the other hand, the coordinates x and y of any point lying on our straight line are related by the given equation. Therefore, given the abscissa of some point lying on the line, we find its ordinate from its equation. For instance, putting $x = 2$, we find from the given equation: $y = -1$. Hence, our straight line passes through the points $A\,(2, -1)$ and $B\,(0, -4)$. Constructing these points and drawing through them a straight line (Fig. 24), we get the required straight line.

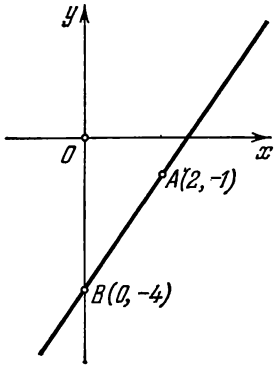


FIG. 24

As is clear from what was said, for an arbitrary straight line in the plane it is possible to form its equation; conversely, knowing the equation of a straight line, we can construct this line. Thus, the equation of a straight line completely characterizes its position in the plane.

From formulas (14-3) and (14-5) it is seen that the equation of a straight line is a *first-degree* equation with respect to the current coordinates x and y. The converse is also true.

Theorem. *Any non-degenerate equation of degree one*

$$Ax + By + C = 0 \quad (A^2 + B^2 \neq 0) \tag{14-7}$$

represents the equation of a straight line in the plane Oxy (the general equation of the straight line).

Proof. (1) Let first $B \neq 0$. Then equation (14-7) may be rewritten in the form

$$y = -\frac{A}{B}x - \frac{C}{B}. \tag{14-8}$$

Comparing the above equation with (14-3), we see that this is the equation of the straight line with the slope

$$k = -\frac{A}{B}$$

and the initial ordinate

$$b = -\frac{C}{B}.$$

(2) Let now $B = 0$; then $A \neq 0$. We have

$$Ax + C = 0$$

and

$$x = -\frac{C}{A}. \tag{14-9}$$

Equation (14-9) represents the equation of a straight line which is parallel to the y-axis and intercepts a segment $a = -\frac{C}{A}$.

Since all possible cases are exhausted, the theorem has been proved.

Sec. 15. The Angle Between Two Straight Lines

Consider two straight lines (not parallel to the y-axis) given by their equations in the slope-intercept form (Fig. 25):

$$y = kx + b, \quad \text{where} \quad k = \tan \varphi, \tag{15-1}$$

and

$$y = k'x + b', \quad \text{where} \quad k' = \tan \varphi'. \tag{15-2}$$

It is required to determine the angle θ between them. To be more precise, by θ we shall understand the least angle measured anticlockwise through which the second line is rotated with respect to the first ($0 \leqslant \theta < \pi$).

This angle θ (Fig. 25) is equal to the angle ACB of the triangle ABC. Further, as is known from elementary geometry, an exterior angle of a triangle is equal to the sum of interior angles not adjacent to it. Therefore,

$$\varphi' = \varphi + \theta,$$

or

$$\theta = \varphi' - \varphi;$$

hence, applying the known trigonometric formula, we get

$$\tan \theta = \tan (\varphi' - \varphi) = \frac{\tan \varphi' - \tan \varphi}{1 + \tan \varphi \cdot \tan \varphi'}.$$

Substituting $\tan \varphi$ and $\tan \varphi'$ by k and k', respectively, we shall finally have

$$\tan \theta = \frac{k' - k}{1 + kk'}. \tag{15-3}$$

This formula expresses the tangent of the angle between two straight lines in terms of the slopes of these lines.

Let us now consider the conditions of parallelism and perpendicularity of two straight lines.

If the straight lines (15-1) and (15-2) are parallel, then $\varphi' = \varphi$ and, consequently,

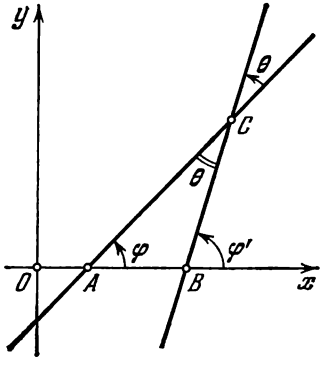


FIG. 25

$$k' = k. \tag{15-4}$$

Conversely, if the above condition is fulfilled, then, taking into account that φ' and φ are contained within the limits from 0 to π, we get

$$\varphi' = \varphi, \tag{15-5}$$

and, hence, our lines are either parallel or coincide (parallelism in a broad sense).

Rule 1. *Straight lines in the plane are parallel (in a broad sense) if and only if their slopes are equal to each other.*

If the lines are mutually perpendicular, then $\theta = \frac{\pi}{2}$ and, hence,

$$\cot\theta = \frac{1}{\tan\theta} = \frac{1+kk'}{k'-k} = 0;$$

whence $1 + kk' = 0$ and

$$k' = -\frac{1}{k}. \tag{15-6}$$

The converse is also true.

Rule 2. *Two straight lines in the plane are perpendicular if and only if their slopes are reciprocal quantities with opposite signs*.*

Let now the equations of straight lines be given in the general form:

$$Ax + By + C = 0 \tag{15-7}$$

and

$$A'x + B'y + C' = 0. \tag{15-8}$$

Hence, assuming that $B \neq 0$ and $B' \neq 0$, we get

$$y = -\frac{A}{B}x - \frac{C}{B} \tag{15-7'}$$

* For straight lines parallel to x- and y-axes we conventionally put $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

and

$$y = -\frac{A'}{B'}x - \frac{C'}{B'}. \qquad (15\text{-}8')$$

Consequently, the slopes of these lines are

$$k = -\frac{A}{B}, \quad k' = -\frac{A'}{B'}. \qquad (15\text{-}9)$$

Using formula (15-3), after simple transformations we find the tangent of the angle between these lines:

$$\tan\theta = \frac{AB' - A'B}{AA' + BB'}. \qquad (15\text{-}10)$$

Whence we obtain: (1) the *parallelism condition* for straight lines ($\theta = 0$)

$$\frac{A'}{A} = \frac{B'}{B} \qquad (15\text{-}11)$$

and (2) the *perpendicularity condition* for straight lines $\left(\theta = \frac{\pi}{2}\right)$

$$AA' + BB' = 0. \qquad (15\text{-}12)$$

It should be noted, in particular, that the straight lines

$$Ax + By + C = 0$$

and

$$Bx - Ay + C_1 = 0$$

are mutually perpendicular.

Example. Determine the angle between the straight lines $y = x$ and $y = 1.001x + 10$.

Here the slopes of the lines are: $k = 1$ and $k' = 1.001$.

By formula (15-3) we get

$$\tan\theta = \frac{1.001 - 1}{1 + 1 \times 1.001} = \frac{0.001}{2.001} \approx 0.0005 = \frac{1}{2000}.$$

Since for small angles θ the approximate equality $\theta \approx \tan\theta$ is valid, we have

$$\theta \approx \frac{1}{2000}\text{ rad} \approx \frac{1}{2000} \cdot 57°18' = \frac{3438'}{2000} \approx 1.7'.$$

Sec. 16. The Equation of a Straight Line Passing Through a Given Point in a Given Direction

Let the straight line PM form an angle φ with the positive direction of the x-axis (Fig. 26) and pass through a given point $P(x_1, y_1)$. Let us derive the equation of this line assuming first that the line is not parallel to the y-axis.

As we know, in this case the equation of a straight line has the form

$$y = kx + b, \tag{16-1}$$

where $k = \tan \varphi$ is the slope of the line, and b is the length of the segment intercepted by the line on the y-axis. Since the point $P(x_1, y_1)$ lies on the straight line PM, its coordinates x_1 and y_1 must satisfy the above equation, i.e.

$$y_1 = kx_1 + b. \tag{16-2}$$

Subtracting equality (16-2) from equality (16-1), we get

$$y - y_1 = k(x - x_1). \tag{16-3}$$

This is just the equation of the required line.

If a straight line passing through the point $P(x_1, y_1)$ is parallel to the y-axis, then its equation will, obviously, be

$$x = x_1. \tag{16-4}$$

If k is a given number, then equation (16-3) represents a quite definite straight line. But if k is a varying parameter, then this

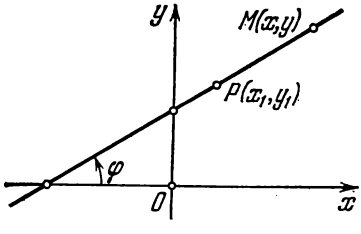


FIG. 26

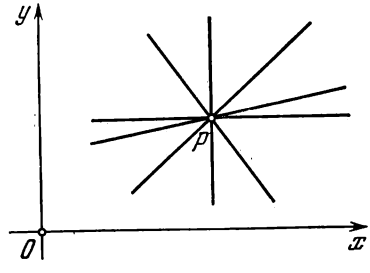


FIG. 27

equation will determine a *pencil of straight lines* passing through the point $P(x_1, y_1)$ (see Fig. 27). In this case k is called the *parameter of the pencil.*

Example 1. Write the equation of the straight line passing through the point P (3, 2) parallel to the line:

$$y = \frac{4}{3}x - 7.$$

Since the required straight line is parallel to the given line, its slope $k = \frac{4}{3}$. Consequently, on the basis of formula (16-3), the

equation of this straight line has the form

$$y-2=\frac{4}{3}(x-3)$$

or

$$y=\frac{4}{3}x-2.$$

Example 2. Write the equation of the straight line passing through the point P (4, 5) perpendicular to the line

$$y=-\frac{2}{3}x+7.$$

Since the required line is perpendicular to the straight line having the slope $k=-\frac{2}{3}$, its slope is

$$k'=-\frac{1}{k}=\frac{3}{2}.$$

Consequently, on the basis of (16-3), the equation of this straight line is

$$y-5=\frac{3}{2}(x-4)$$

or finally,

$$y=\frac{3}{2}x-1.$$

Sec. 17. The Equation of a Straight Line Passing Through Two Points (Two-Point Form)

As is known, through two non-coincident points we can draw only one straight line. Let us find the equation of the line passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

We first suppose that $x_1 \neq x_2$, i.e. the line PQ is not parallel to the y-axis. Since PQ passes through the point $P(x_1, y_1)$, its equation has the form (see Sec. 16)

$$y-y_1=k(x-x_1), \tag{17-1}$$

where k is the unknown slope of this straight line. But since our line also passes through the point $Q(x_2, y_2)$, the coordinates x_2 and y_2 of this latter point must satisfy equation (17-1). Hence

$$y_2-y_1=k(x_2-x_1)$$

and, consequently, for $x_2 \neq x_1$ we have

$$k=\frac{y_2-y_1}{x_2-x_1}. \tag{17-2}$$

Substituting the above expression for the slope k into (17-1), we get the equation of the line PQ:

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1). \tag{17-3}$$

This equation for $y_1 \neq y_2$ can also be written in the form of a proportion

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}. \tag{17-3'}$$

If $x_1 = x_2$, i.e. if the straight line passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is parallel to the y-axis, then the equation of this line will, obviously, be

$$x = x_1.$$

Example. Write the equation of the straight line passing through the points $P(4, -2)$ and $Q(3, -1)$.

By virtue of (17-3), we have

$$\frac{x-4}{3-4}=\frac{y+2}{-1+2}, \quad \text{or} \quad y=-x+2.$$

Sec. 18. The Intercept Form of the Equation of a Straight Line

We are going now to derive the equation of a straight line whose position in the plane is specified by non-zero segments intercepted by this line on the coordinate axes. Suppose, for instance, that the straight line AB intercepts a segment $OA = a$ on the x-axis, and a segment $OB = b$ on the y-axis (Fig. 28). As is clear, the position of the line is thus completely determined.

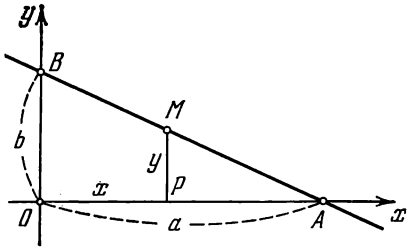


FIG. 28

For deriving the equation of the straight line AB, let us note that this line passes through the points $A(a, 0)$ and $B(0, b)$. Therefore, its equation is easily obtained from (17-3') if we put $x_1 = a$, $y_1 = 0$, and $x_2 = 0$, $y_2 = b$. We have

$$\frac{x-a}{0-a}=\frac{y-0}{b-0}.$$

Whence

$$\frac{y}{b} = -\frac{x}{a} + 1$$

and finally

$$\frac{x}{a} + \frac{y}{b} = 1. \tag{18-1}$$

Equation (18-1) is the *intercept form of the equation of a straight line*. Here, as usual, x and y are the coordinates of an arbitrary point $M(x, y)$ lying on the line AB (Fig. 28).

Example. Write the equation of the straight line AB intercepting a segment $OA = 5$ on the x-axis and a segment $OB = -4$ on the y-axis.

Putting $a = 5$ and $b = -4$ in (18-1), we get

$$\frac{x}{5} + \frac{y}{-4} = 1$$

or

$$\frac{x}{5} - \frac{y}{4} = 1.$$

Note. The equation of a straight line passing through the origin or parallel to one of the coordinate axes cannot be written in the intercept form.

Sec. 19. The Point of Intersection of Two Straight Lines

Suppose we have two straight lines

$$Ax + By + C = 0 \tag{19-1}$$

and

$$A'x + B'y + C' = 0. \tag{19-2}$$

The point of intersection of these lines lies both on the first and on the second lines. Therefore, the coordinates of the point of intersection must satisfy the equations of both lines. Consequently, *to find the coordinates of the point of intersection of two given straight lines, it is sufficient to solve jointly the system of equations of these lines.*

Successively eliminating the unknowns y and x from equations (19-1) and (19-2), we have

$$(AB' - A'B)\,x + (CB' - C'B) = 0 \tag{19-3}$$

and

$$(AB' - A'B)\,y + (AC' - A'C) = 0. \tag{19-4}$$

Hence, if $AB' - A'B \neq 0$, then for the coordinates of the point of intersection of the straight lines we get the following expressions:

$$x = -\frac{CB' - C'B}{AB' - A'B}, \quad y = -\frac{AC' - A'C}{AB' - A'B}, \tag{19-5}$$

or, introducing second-order determinants (see Sec. 5, Chapter 1), we have

$$x = -\frac{\begin{vmatrix} C & B \\ C' & B' \end{vmatrix}}{\begin{vmatrix} A & B \\ A' & B' \end{vmatrix}}, \quad y = -\frac{\begin{vmatrix} A & C \\ A' & C' \end{vmatrix}}{\begin{vmatrix} A & B \\ A' & B' \end{vmatrix}}. \tag{19-6}$$

For straight lines (19-1) and (19-2) the following three cases are possible:

(1) $AB' - A'B \neq 0$, i.e.

$$\frac{A'}{A} \neq \frac{B'}{B}.$$

According to Sec. 15, the lines are not parallel. The coordinates of their single point of intersection are determined from formulas (19-6).

(2) $AB' - A'B = 0$, $CB' - C'B \neq 0$ or $AC' - A'C \neq 0$, i.e.

$$\frac{A'}{A} = \frac{B'}{B} \neq \frac{C'}{C}.$$

The lines are parallel (see Sec. 15) and, hence, have no points of intersection. Analytically it follows from the fact that at least one of equations (19-3) or (19-4) is contradictory, which means that the system (19-1) and (19-2) is incompatible.

(3) $AB' - A'B = 0$, $CB' - C'B = 0$, $AC' - A'C = 0$, i.e.

$$\frac{A'}{A} = \frac{B'}{B} = \frac{C'}{C}$$

Lines (19-1) and (19-2) merge and, thus, there is an infinite number of points of intersection. In this case the left member of their equations differ only in a constant factor and, hence, the system of these equations allows an infinite number of solutions.

Example. Solving the system of equations of the straight lines

$$3x + 4y - 10 = 0$$

and

$$2x + 5y - 9 = 0,$$

we get: $x = 2$ and $y = 1$. Hence, the straight lines intersect at the point N (2, 1).

Sec. 20. The Distance from a Point to a Straight Line

Let us consider the straight line KL given by the general equation

$$Ax + By + C = 0 \tag{20-1}$$

and a point $M(x_1, y_1)$. The *distance* of the point M from the line KL (or of the line KL from the point M) is understood as the length of the perpendicular $d = MN$ (MN is perpendicular to KL) dropped from the point M onto the line KL (Fig. 29).

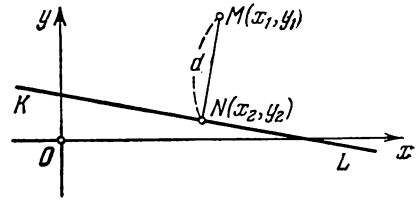


FIG. 29

The equation of the perpendicular MN can be written in the form (see Sec. 15)

$$B(x - x_1) - A(y - y_1) = 0. \tag{20-2}$$

Hence, for the foot of the perpendicular $N(x_2, y_2)$ we shall have

$$B(x_2 - x_1) - A(y_2 - y_1) = 0, \tag{20-3}$$

and, consequently,

$$\frac{x_2 - x_1}{A} = \frac{y_2 - y_1}{B} = t, \tag{20-4}$$

where t is a proportionality factor. Therefore

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{A^2 + B^2}\,|t|. \tag{20-5}$$

On the other hand, taking into account that the point $N(x_2, y_2)$ lies on the straight line KL, and that from (20-4) we have

$$x_2 = x_1 + At, \quad y_2 = y_1 + Bt,$$

we get

$$Ax_2 + By_2 + C = A(x_1 + At) + B(y_1 + Bt) + C = \\ = (Ax_1 + By_1 + C) + t(A^2 + B^2) = 0.$$

Hence,

$$t = -\frac{Ax_1 + By_1 + C}{A^2 + B^2}. \tag{20-6}$$

Thus, by virtue of (20-5), we have

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}. \tag{20-7}$$

In particular, putting $x_1 = 0$, $y_1 = 0$, we get the distance of the straight line from the origin

$$d_0 = \frac{|C|}{\sqrt{A^2+B^2}}. \tag{20-8}$$

Note. Dividing both members of (20-1) by $\sqrt{A^2+B^2}$, we obtain the equation

$$\frac{Ax+By+C}{\sqrt{A^2+B^2}} = 0, \tag{20-9}$$

whose constant term $\frac{C}{\sqrt{A^2+B^2}}$ is numerically equal to the distance of the line from the origin. Such equation is called *normalized.*

From formula (20-7) we get the **rule**: *To determine the distance from a point to a straight line, we have to substitute the coordinates of the given point into the left member of the normalized equation of this line and take the absolute value of the result obtained.*

Example. Determine the distance of the point M (—2, 7) from the straight line

$$24x + 7y - 2 = 0.$$

Normalizing the equation of this line, we shall have

$$\frac{24x+7y-2}{\sqrt{24^2+7^2}} = 0,$$

or

$$\frac{24x+7y-2}{25} = 0.$$

Hence, the required distance is

$$d = \frac{|24 \times (-2)+7 \times 7-2|}{25} = \frac{1}{25} = 0.04.$$

Exercises

1. Construct the straight lines given by the equations:
(a) $y = 2x - 1$; (b) $2x - 3y - 6 = 0$.
2. The bases of an isosceles trapezoid are equal to 10 and 6 and the angle at the base is equal to 60°. Write the equations of the sides of this trapezoid taking the longer base and the axis of symmetry of the trapezoid for the coordinate axes.
3. Write the equation of the straight line passing through the point M (3, 4) and making an angle of 45° with the straight line $y = 2x + 1$.

4. Write the equation of the straight line parallel to the lines $3x + 2y - 6 = 0$, $6x + 4y - 3 = 0$ and equidistant from them.

5. Given a line segment AB with the end-points A $(-3, 2)$ and B $(1, -1)$. Write the equation of the straight line joining the mid-point of the segment to the origin.

6. Given a triangle ABC with the vertices A $(4, 2)$, B $(-2, 4)$, and C $(-1, -4)$. Write the equation of the median passing through the vertex C, and find the length of this median.

7. Given a triangle ABC with the vertices A $(5, 3)$, B $(-3, 4)$, and C $(-2, -5)$. Write the equation of the altitude passing through the vertex B and find its length.

8. Given a triangle ABC with the vertices A $(6, 4)$, B $(-3, 5)$, and C $(-2, -6)$. Write the equation of the straight line passing through the vertex A parallel to the median passing through the vertex B.

9. Through the point $(5, 2)$ draw a straight line intercepting equal segments on the coordinate axes.

10. In section a coal layer has the thickness $y_1 = 5$ m for $x_1 = 100$ m and $y_2 = 15$ m for $x_2 = 200$ m. Assuming that the layer has the shape of a wedge, find the law of change of its thickness y depending on distance x. What is its thickness for $x = 300$ m? At what point of the section is the thickness of the layer $y = 10$ m?

11. Find the point of intersection of the straight lines:

(a) $5x - 7y - 20 = 0$ and $7x - 10y + 15 = 0$;

(b) $2x + 3y - 7 = 0$ and $4x + 6y + 11 = 0$;

(c) $2x - y = 0$ and $x - 0.5y = 0$.

12. Write the equation of the straight line passing through the point of intersection of the lines

$$3x + 4y - 7 = 0, \quad 5x + 3y - 8 = 0$$

and through the origin.

13. Find the projection of the point M $(1, 2)$ on the straight line $5x + 2y + 20 = 0$.

14. One straight line passes through the origin and is inclined to the x-axis at an angle α; the other line passes through the point A $(a, 0)$ and is inclined to the x-axis at an angle β $(\beta \neq \alpha)$. Find the point of intersection of these lines.

15. Given a triangle with the vertices A $(-2, 1)$, B $(2, -1)$, and C $(4, 3)$. Determine the coordinates of the point of intersection of the medians of this triangle.

16. The sides of a triangle ABC have the following equations: $x + 7y - 11 = 0$ (AB); $2x + y + 4 = 0$ (BC), $3x - 5y - 7 = 0$ (CA). Compute the area of the triangle ABC.

17. Find the distances of the points O (0, 0), A (1, 2), and B (−2, 1) from the straight line $3x - 4y + 10 = 0$.

18. Given a triangle with the vertices A (1, 1), B (−2, 5), and C (−4, −3). Find the altitude of the triangle dropped from the vertex C onto the side AB.

19. Find the length of the line segment perpendicular to the straight lines $3x + 4y - 10 = 0$ and $3x + 4y - 45 = 0$ and enclosed between these lines.

20. Write the equations of the straight lines parallel to the line $8x - 6y + 5 = 0$ and passing at distances of 2 from it.

21. Find the equations of the straight lines passing through the origin at distances equal to $\frac{2}{5}$ from the point A (2, 1).

Chapter 4

Second-Order Lines

Sec. 21. The Circle

Let us derive the equation of a circle (Fig. 30) of radius R with centre at $C(x_0, y_0)$. For an arbitrary point $M(x, y)$ of the circle the following equation is fulfilled

$$MC = R. \tag{21-1}$$

Hence, remembering the formula for the distance between two points (see Sec. 3, Chapter 1), we have

$$\sqrt{(x-x_0)^2+(y-y_0)^2} = R. \tag{21-2}$$

Since both members of (21-2) are positive, on squaring, we get an equivalent equation

$$(x - x_0)^2 + (y - y_0)^2 = R^2. \tag{21-3}$$

Thus, the coordinates of any point $M(x, y)$ of the given circle satisfy (21-3). The converse statement is also true.

And so, (21-3) is the equation of a circle of radius R with centre at the point $C(x_0, y_0)$. This equation is called the *normal form of the equation of a circle.*

In particular, putting $x_0 = 0$ and $y_0 = 0$, we get the equation of a circle with centre at the origin

$$x^2 + y^2 = R^2. \tag{21-4}$$

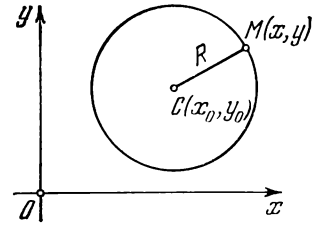


FIG. 30

Equation (21-3) can be easily transformed to the form

$$x^2 + y^2 + \alpha x + \beta y + \gamma = 0, \tag{21-5}$$

where

$$\alpha = -2x_0, \quad \beta = -2y_0,$$

$$\gamma = x_0^2 + y_0^2 - R^2.$$

Hence, the circle is a *second-order curve (or a curve of degree two)* (See Sec. 13, Chapter 2).

Comparing (21-5) with the general equation of a second-order curve

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{21-6}$$

we see that in (21-5) $B = 0$ and, besides, $A = 1$, $C = 1$, i.e. $A = C$.

Conversely, let us put in (21-6) $B = 0$ and $A = C \neq 0$:

$$Ax^2 + Ay^2 + Dx + Ey + F = 0. \tag{21-7}$$

Dividing (21-7) termwise by $A \neq 0$ and putting

$$\frac{D}{A} = \alpha, \quad \frac{E}{A} = \beta, \quad \frac{F}{A} = \gamma, \tag{21-8}$$

we come to (21-5).

Equation (21-7) is called the *general equation of the circle.*

It should be noted, however, that not every equation (21-7) is the equation of a real circle. We can easily show that equation (21-7) defines a real curve (circle) only for $\frac{\alpha^2 + \beta^2}{4} - \gamma \geqslant 0$, where α, β, γ are expressed by equalities (21-8).

Thus: *a real curve of degree two is a circle if and only if* (1) *the coefficients of the squares of the current coordinates are equal to each other, and* (2) *there is no term containing the product of the current coordinates.*

Sec. 22. Central Second-Order Curves (Conics)

Consider the equation of a second-order curve

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \tag{22-1}$$

($A \neq 0$, $C \neq 0$) having no term containing the product of thl coordinates x and y ($B = 0$)*. Completing the squares, we shale have

$$A\left(x + \frac{D}{2A}\right)^2 + C\left(y + \frac{E}{2C}\right)^2 = \frac{D^2}{4A} + \frac{E^2}{4C} - F. \tag{22-2}$$

Whence, putting

$$x_0 = -\frac{D}{2A}, \quad y_0 = -\frac{E}{2C} \tag{22-3}$$

and

$$\Delta = \frac{D^2}{4A} + \frac{E^2}{4C} - F, \tag{22-4}$$

we get

$$A(x - x_0)^2 + C(y - y_0)^2 = \Delta. \tag{22-5}$$

* In our brief course we shall confine ourselves only to this case.

The point $O'(x_0, y_0)$ represents the centre of symmetry of curve (22-5) (*the centre of the curve*). Indeed, if a point $M_1(x_1, y_1)$ lies on curve (22-5), then point $M_2(x_2, y_2)$ symmetrical to it about O', where $x_2 = 2x_0 - x_1$, $y_2 = 2y_0 - y_1$, also lies on curve (22-5) (Fig. 31).

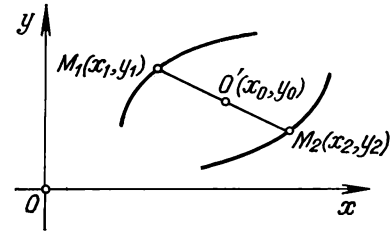


FIG. 31

The straight lines $y = y_0$ and $x = x_0$ parallel to the coordinate axes Ox and Oy, respectively, are the *axes of symmetry of curve* (22-5) (*the axes of the curve*). Indeed, if a point $M(x_0, y_0 - h)$ lies on curve (22-5), then point $M'(x_0, y_0 + h)$ symmetrical to it about the line $y = y_0$ also lies on this curve. The line $x = x_0$ possesses analogous properties.

For the sake of simplicity, in our further investigations we shall assume that the centre of the curve is situated at the origin, i.e. $x_0 = 0$, $y_0 = 0$. Then the equation of the curve will take the form

$$Ax^2 + Cy^2 = \Delta. \tag{22-6}$$

Definition 1. *Second-order curve* (22-6) *is called the ellipse* (*or more precisely, belongs to the elliptic type*) *if the coefficients A and C have the same sign*, i.e.

$$AC > 0. \tag{22-7}$$

For the sake of definiteness, we shall assume that $A > 0$ and $C > 0$ (since, otherwise, the signs of the terms of equation (22-6) may be reversed).

Three cases are possible here: (1) $\Delta > 0$, (2) $\Delta = 0$, and (3) $\Delta < 0$.

In the first case ($\Delta > 0$) we have a *real ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{22-8}$$

where the numbers

$$a = \sqrt{\frac{\Delta}{A}}, \qquad b = \sqrt{\frac{\Delta}{C}} \tag{22-9}$$

are called *semi-axes* of the ellipse. We usually put $0 < b \leqslant a$ (this can always be obtained by a proper choice of the x- and y-axes). Equation (22-8) is called the *canonical equation of the ellipse* with the semi-axes a and b (Fig. 32). The points $A(a, 0)$, $B(0, b)$, $A'(-a, 0)$, $B'(0, -b)$ are called the *vertices of the ellipse* and the line segments $A'A = 2a$ and $B'B = 2b$ its *axes*. It should be noted that from equation (22-8) we have $|x| \leqslant a$, $|y| \leqslant b$.

If $a = b$, we get a circle

$$x^2 + y^2 = a^2.$$

In the second case ($\Delta = 0$) curve (22-6) represents a point O (0, 0) (*a degenerate ellipse*).

Finally, in the third case ($\Delta < 0$) the curve has no real points; it is conventionally called an *imaginary ellipse.*

Definition 2. *Second-order curve* (22-6) *is called the hyperbola* (*more*

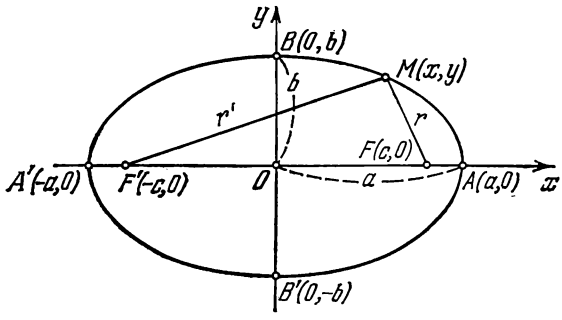


FIG. 32

precisely, a curve of the hyperbolic type) *if the coefficients A and C have opposite signs*, i.e.

$$AC < 0. \tag{22-10}$$

For definiteness, let us put $A > 0$, then $C < 0$ Three cases are possible here: (1) $\Delta > 0$, (2) $\Delta = 0$, (3) $\Delta < 0$.

In the first case ($\Delta > 0$) we have a *hyperbola with the canonical equation*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{22-11}$$

where $a = \sqrt{\frac{\Delta}{A^2}}$ (*real or transverse semi-axis*) and $b = \sqrt{\frac{\Delta}{-C}}$ (*imaginary or conjugate semi-axis*) (Fig. 33). The points A (a, 0) and A' ($-a$, 0) are called the *vertices of the hyperbola.* It should be noted that $|x| \geqslant a$.

In the second case ($\Delta = 0$) we get a pair of intersecting straight lines (*a degenerate hyperbola*)

$$(\sqrt{A}x - \sqrt{-C}y)(\sqrt{A}x + \sqrt{-C}y) = 0.$$

Finally, in the third case ($\Delta < 0$), we get a hyperbola

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = -1 \tag{22-12}$$

with the semi-axes $a' = \sqrt{\frac{-\Delta}{A}}$ and $b' = \sqrt{\frac{\Delta}{C}}$. If $a' = a$ and $b' = b$, then hyperbola (22-12) is said to be *conjugate* to hyperbola

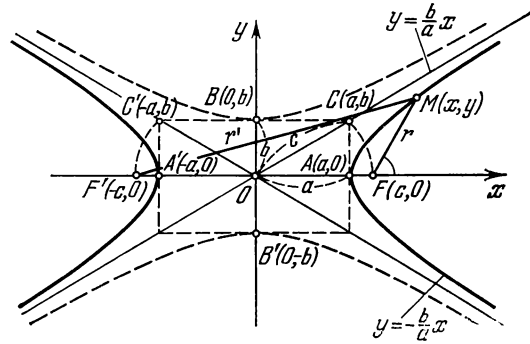


FIG. 33

(22-11), its vertices being $B(0, b)$ and $B'(0, -b)$ (Fig. 33).

The line segment $A'A = 2a$ is called the *real* (or transverse) *axis*, and the segment $B'B = 2b$ the *imaginary* (or conjugate) *axis* of hyperbola (22-11).

Example. Determine the shape and arrangement of the curve

$$x^2 + 2y^2 - 2x + 3y = 0. \tag{22-13}$$

Completing the terms containing x and y to perfect squares, we have

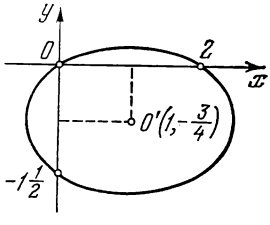


FIG. 34

$$(x-1)^2 + 2\left(y+\frac{3}{4}\right)^2 = 1 + \frac{9}{8}.$$

Hence

$$\frac{(x-1)^2}{\frac{17}{8}} + \frac{\left(y+\frac{3}{4}\right)^2}{\frac{17}{16}} = 1.$$

Consequently, curve (22-13) represents an ellipse with the semi-axes $a = \sqrt{\frac{17}{8}} \approx 1.46$ and $b = \sqrt{\frac{17}{16}} \approx 1.03$, whose centre is situated at point $O'\left(1, -\frac{3}{4}\right)$ (Fig. 34).

Sec. 23. Focal Properties of Central Curves of the Second Order

The points $F(c, 0)$ and $F'(-c, 0)$ where

$$c = \sqrt{a^2 \mp b^2}, \tag{23-1}$$

are called the *foci* of the ellipse specified by canonical equation (22-8), Fig. 32 (the minus sign) and the hyperbola given by canonical equation (22-11), Fig. 33 (the plus sign), respectively.

The ratio

$$\varepsilon = \frac{c}{a} \tag{23-2}$$

is called the *eccentricity* of a conic.

From formula (23-1) we have: for the ellipse $0 \leqslant \varepsilon < 1$, for the hyperbola $1 < \varepsilon < +\infty$. It should be noted that for the circle $\varepsilon = 0$.

Let $r = MF$ and $r' = MF'$ be the distances of a point M on a conic from its foci (the so-called *focal radii of M*). We have

$$r = \sqrt{(x-c)^2 + y^2} \tag{23-3}$$

and

$$r' = \sqrt{(x+c)^2 + y^2}. \tag{23-4}$$

Since

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

where the plus sign corresponds to the ellipse and the minus sign to the hyperbola, we have

$$y^2 = \pm b^2 \left(1 - \frac{x^2}{a^2}\right),$$

and, consequently, taking into account (23-1), we get

$$r = \sqrt{x^2 - 2cx + c^2 \pm b^2 \left(1 - \frac{x^2}{a^2}\right)} =$$

$$= \sqrt{\left(1 \mp \frac{b^2}{a^2}\right) x^2 - 2cx + (c^2 \pm b^2)} =$$

$$= \sqrt{\frac{c^2}{a^2} x^2 - 2cx + a^2} = \left|\frac{c}{a} x - a\right| = |\varepsilon x - a| \tag{23-5}$$

and, analogously,

$$r' = \sqrt{x^2 + 2cx + c^2 \pm b^2 \left(1 - \frac{x^2}{a^2}\right)} = |\varepsilon x + a|. \tag{23-6}$$

If the curve is an ellipse, then $0 \leqslant \varepsilon < 1$, $|x| \leqslant a$ and therefore

$r = a - \varepsilon x$, $\quad r' = a + \varepsilon x$;

whence

$$r + r' = 2a, \tag{23-7}$$

and for any r and r' satisfying equality (23-7) there exists a point of a given ellipse.

Thus, *for any point of the ellipse the sum of its focal radii is constant.* This property is frequently taken for the definition of the ellipse (*the characteristic property of the ellipse*).

For the hyperbola we have: $\varepsilon > 1$, $|x| \geqslant a$. Therefore

$$r = \pm(\varepsilon x - a), \quad r' = \pm(\varepsilon x + a),$$

where the plus sign corresponds to the right-hand branch of the hyperbola ($x > 0$), and the minus sign to its left-hand branch ($x < 0$). Hence

$$r' - r = \pm 2a. \tag{23-8}$$

Thus, *for any point of the hyperbola the absolute value of the difference between its focal radii is constant* (*the characteristic property of the hyperbola*).

Sec. 24. The Ellipse as a Uniformly Compressed Circle

Consider a circle of radius a. Let us choose a rectangular coordinate system Oxy and place its origin, for simplicity, at the centre of the circle O (0, 0). For the sake of convenience, we denote the current coordinates of a point M of the circle by X and Y. Then the equation of the circle will have the form (see Sec. 21)

$$X^2 + Y^2 = a^2. \tag{24-1}$$

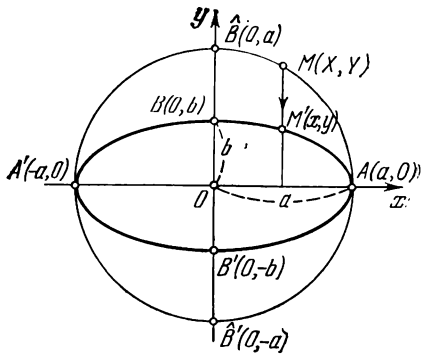


FIG. 35

Let us now uniformly compress circle (24-1) toward one of its diameters which, without loss of generality of reasoning, may be regarded as vertical, i.e. directed along the y-axis. Let k be the coefficient of compression of the circle in the chosen direction, i.e. k is the ratio of the length of the transformed vertical segment to its initial length. It should be noted that for $0 \leqslant k < 1$ we have uniform compression, and for $k > 1$ uniform stretching of the circle.

Suppose that, as a result of our deformation, the point M (X, Y) of the circle goes into some point M' (x, y) of the transformed curve (Fig. 35). Since the points M and M' lie on one and the same vertical

line, we have

$$x = X, \quad y = kY. \tag{24-2}$$

Hence for $k \neq 0$* we get

$$X = x, \quad Y = \frac{y}{k}. \tag{24-3}$$

Substituting these expressions in (24-1), we find

$$x^2 + \frac{y^2}{k^2} = a^2,$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{24-4}$$

where $b = ka$, i.e. the transformed point $M'(x, y)$ is situated on the ellipse with the semi-axes a and b.

Conversely, if the point $M'(x, y)$ belongs to ellipse (24-4), then point $M(X, Y)$ corresponding to it by (24-2) lies on circle (24-1).

Thus, *uniform compression of a circle along one of its diameters results in an ellipse.*

Sec. 25. The Asymptotes of a Hyperbola

Consider the hyperbola (Fig. 33)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{25-1}$$

Solving equation (25-1) with respect to y, we get

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2} \tag{25-2}$$

or

$$y = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}}. \tag{25-3}$$

If $|x|$ increases without bound, then $\sqrt{1 - \frac{a^2}{x^2}} \approx 1$, and consequently, in a sense, there takes place an approximate equality

$$y \approx \pm \frac{b}{a} x.$$

---

* If $k = 0$, then we get a line segment $-a \leqslant x \leqslant a$, $y = 0$ which may be regarded as a degenerate ellipse.

Let us show that the branches of hyperbola (25-1) arbitrarily approach the straight lines (Fig. 33)

$$y = \pm \frac{b}{a} x, \tag{25-4}$$

which are termed the *asymptotes of the hyperbola**. Indeed, for instance, for $x > 0$ we take the plus signs in formulas (25-2) and (25-4). Let us now consider the corresponding points $M(x, y)$ of hyperbola (25-2) and $N(x, Y)$ of straight line (25-4) which have one and the same abscissa x. Then

$$Y - y = \frac{b}{a} x - \frac{b}{a} \sqrt{x^2 - a^2} =$$

$$= \frac{b}{a} \cdot \frac{(x - \sqrt{x^2 - a^2})(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}} = \frac{ab}{x + \sqrt{x^2 - a^2}} \to 0$$

as $x \to +\infty$.

Analogous consideration is given to three more cases: (1) the minus signs in (25-2) and (25-4) as $x \to +\infty$; (2) plus in (25-2) and minus in (25-4) as $x \to -\infty$; and, finally, minus in (25-2) and plus in (25-4) as $x \to -\infty$.

It should be noted that the conjugate hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1, \tag{25-5}$$

as it is easily checked, has common asymptotes with hyperbola (25-1).

In the case of an *equilateral hyperbola* ($a = b$)

$$x^2 - y^2 = a^2$$

its asymptotes $y = \pm x$ are mutually perpendicular.

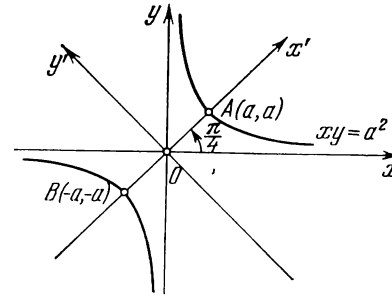


FIG. 36

Sec. 26. The Graph of Inverse Proportionality

Consider the curve

$$xy = a^2 \ (a > 0) \tag{26-1}$$

(Fig. 36).

Choosing the bisectors of the quadrants for new coordinate axes Ox' and Oy', and taking into consideration that the angle of rotation $\alpha = \frac{\pi}{4}$, we have (see Sec. 2, Chapter 1)

$$x = x' \cos \frac{\pi}{4} - y' \sin \frac{\pi}{4} = \frac{x' - y'}{\sqrt{2}},$$

$$y = x' \sin \frac{\pi}{4} + y' \cos \frac{\pi}{4} = \frac{x' + y'}{\sqrt{2}}.$$

* 'Asymptote is from the Greek meaning 'not meeting'.

Hence, by (26-1), we get

$$\frac{x'^2-y'^2}{2}=a^2,$$

i.e.

$$x'^2 - y'^2 = 2a^2. \tag{26-2}$$

The graph of inverse proportionality (26-1) *is thus an equilateral hyperbola.*

Sec. 27. Noncentral Quadric Curves

A quadric curve is said to be *noncentral* if it has either no centre of symmetry, or an infinite number of centres of symmetry (i.e. has no single centre).

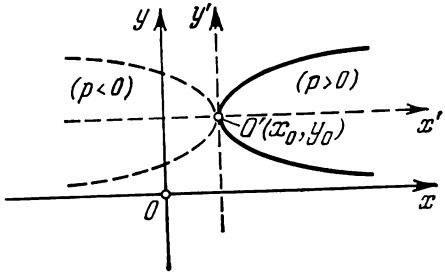


FIG. 37

Consider the quadric curve

$$Ax^2 + Cy^2 + Dx + Ey + F = 0, \tag{27-1}$$

where $AC = 0$ and $A^2 + C^2 \neq 0$. For definiteness we shall assume that

$$A = 0, \quad C \neq 0. \tag{27-2}$$

Besides, suppose that $D \neq 0$, otherwise we would have a pair of parallel straight lines.

Completing the terms containing y to a perfect square in (27-1), we have

$$C\left(y+\frac{E}{2C}\right)^2 = -Dx-F+\frac{E^2}{4C},$$

or, putting,

$$x_0 = -\frac{F}{D}+\frac{E^2}{4DC}, \quad y_0 = -\frac{E}{2C}, \quad 2p = -\frac{D}{C}, \tag{27-3}$$

we get

$$(y - y_0)^2 = 2p\,(x - x_0). \tag{27-4}$$

Curve (27-4) is called the *parabola* (Fig. 37); point O' (x_0, y_0) its *vertex*, and the number p its *parameter*. We can easily get convinced that the straight line $y = y_0$ is the *axis of symmetry of the parabola* (*the axis of the parabola*). Parabola (27-4) has no centre of symmetry.

If the vertex of a parabola is at the origin and the x-axis serves as its axis, then we obtain the so-called *canonical equation of the*

parabola

$$y^2 = 2px, \tag{27-5}$$

the parameter p being here usually regarded as positive (this can be achieved by choosing a proper direction of the x-axis; Fig. 38, *a*).

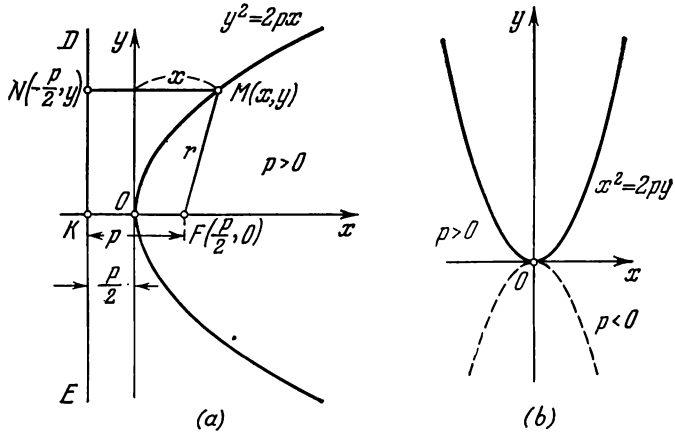


FIG. 38

It should be noted that if we switch the roles of the x- and y-axes, then the canonical equation of the parabola will take the form

$$x^2 = 2py. \tag{27-6}$$

This is the equation of a parabola with a *vertical axis* (Fig. 38, *b*).

Sec. 28. The Focal Property of the Parabola

Consider the parabola (Fig. 38, *a*)

$$y^2 = 2px \qquad (p > 0). \tag{28-1}$$

The point $F\left(\frac{p}{2}, 0\right)$ is called its *focus* and the straight line $x = -\frac{p}{2}$ its *directrix*.

For the point $M(x, y)$ its *focal radius* $r = MF$ is equal to

$$r = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = \sqrt{x^2 - px + \frac{p^2}{4} + 2px} =$$

$$= \sqrt{x^2 + px + \frac{p^2}{4}} = x + \frac{p}{2}. \tag{28-2}$$

On the other hand, the distance of this point from the directrix

is equal to

$$MN = x + \frac{p}{2} = r.$$

Thus, *the parabola represents the set of all points in the plane equidistant from a given point (the focus) and a given straight line (the directrix). This is the characteristic property of the parabola.*

Example. Determine the coordinates of the focus and the equation of the directrix of the parabola

$$y = x^2.$$

Comparing this equation with (27-6), we get

$$2p = 1;$$

hence $p = \frac{1}{2}$. Consequently, the focus has the coordinates $\left(0, \frac{1}{4}\right)$; and the equation of the directrix is $y = -\frac{1}{4}$.

Sec. 29. The Graph of a Quadratic Trinomial

Consider the quadratic trinomial

$$y = Ax^2 + Bx + C \quad (A \neq 0). \tag{29-1}$$

Whence

$$y = A\left(x^2 + \frac{B}{A}x + \frac{C}{A}\right). \tag{29-2}$$

Completing the expression in the parentheses to a perfect square, we get

$$y = A\left[\left(x + \frac{B}{2A}\right)^2 + \left(\frac{C}{A} - \frac{B^2}{4A^2}\right)\right]$$

or

$$y - \frac{4AC - B^2}{4A} = A\left(x + \frac{B}{2A}\right)^2. \tag{29-3}$$

If we put

$$x_0 = -\frac{B}{2A}, \quad y_0 = \frac{4AC - B^2}{4A}, \tag{29-4}$$

then from (29-3) we shall get

$$y - y_0 = A(x - x_0)^2. \tag{29-5}$$

Translating the coordinate system

$$x' = x - x_0, \quad y' = y - y_0,$$

we finally have

$$y' = Ax'^2. \qquad (29\text{-}6)$$

Expression (29-6) (see formula (27-6)) represents the canonical equation of a parabola with a vertical axis and with the vertex at point $O'(x_0, y_0)$, the parameter p being equal to $\frac{1}{2A}$. Thus, *the graph of a quadratic trinomial is a parabola with the vertex at point* $O'(x_0, y_0)$ *whose axis is parallel to the y-axis (a parabola with a shifted vertical axis*; Fig. 39).

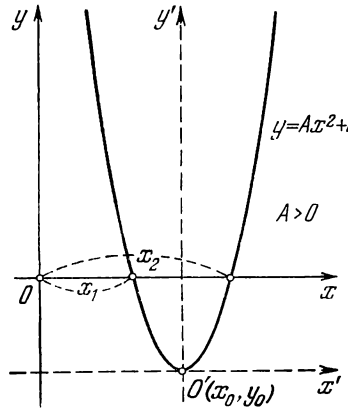


FIG. 39

Let us note here that the abscissas x_1 and x_2 of the points of intersection of parabola (29-1) with the x-axis are the roots of the quadratic equation

$$Ax^2 + Bx + C = 0. \qquad (29\text{-}7)$$

This property underlies the graphical method for solving quadratic equation (29-7).

Example. Reduce the equation

$$y = x^2 - 4x + 3$$

to the canonical form and construct the corresponding parabola.

Transposing the constant term to the left-hand member of the equation and completing the right-hand member to a perfect square, we shall have

$$y - 3 + 4 = x^2 - 4x + 4$$

or

$$y + 1 = (x - 2)^2.$$

Putting

$$x - 2 = x', \quad y + 1 = y',$$

we get

$$y' = x'^2.$$

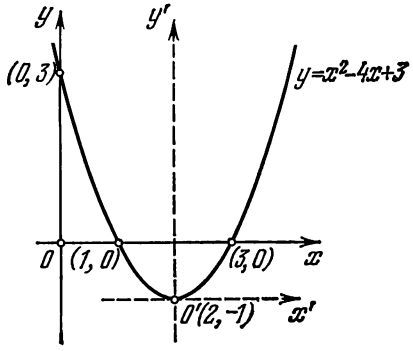


FIG. 40

Hence, the given equation is the equation of the parabola with the vertex at the point $O'(2, -1)$ and the axis of symmetry $O'y'$ parallel to the y-axis (Fig. 40).

Exercises

1. (a) Find the coordinates of the centre C and radius R of the circle

$$x^2 + y^2 + 8x - 9 = 0.$$

(b) Write the equation of the circle passing through the origin with centre at the point C (1, 0).

(c) Write the equation of the circle contacting the coordinate axes if its centre lies at the point $C\left(\frac{1}{2}, \frac{1}{2}\right)$.

(d) Write the equation of the circle whose diameter is a line segment with the end-points A (—1, 2) and B (5, 6).

2. Write the equation of the straight line passing through the centres of the circles

$$x^2 + y^2 - 6x - 8y - 3 = 0 \quad \text{and} \quad x^2 + y^2 + x - 3y - 1 = 0.$$

Determine the distance between the centres of the circles.

3. Find the equation of the common chord of the circles

$$x^2 + y^2 - 4x - 2y - 13 = 0 \text{ and } x^2 + y^2 - 2x - 4y - 15 = 0.$$

4. Find the semi-axes, coordinates of the foci and eccentricity of the ellipse

$$x^2 + 2y^2 = 8.$$

Construct this ellipse.

5. (a) Write the canonical equation of the ellipse the length of whose minor axis is equal to 6 and focal distance to 8.

(b) Write the canonical equation of the ellipse if it is known that the distance between the end-points of its axes is equal to 5 and the sum of the lengths of its semi-axes is equal to 7.

(c) Write the canonical equation of the ellipse if the distances of its focus from the end-points of its major axis are equal to 2 and 18.

6. Find the length of the diameter* of the ellipse

$$5x^2 + 7y^2 = 24$$

bisecting the angle between the coordinate axes.

7. (a) Find the semi-axes, coordinates of the foci and eccentricity of the hyperbola

$$9x^2 - 16y^2 = 36.$$

(b) Write the canonical equation of the hyperbola if the length of its real (transverse) axis is equal to 8 and the focal distance to 10.

Construct these hyperbolas.

---

* That is the chord passing through the centre of the ellipse.

8. Find the length of the diameter of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ perpendicular to the asymptote of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ passing through the first and third quadrants.

9. Find the eccentricity of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ and of the hyperbola conjugate to it.

10. Find the distance between the focus F_1 of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$ and the focus F_2 of the conjugate hyperbola.

11. Find the equation of the straight lines passing through the foci of the hyperbola $\frac{x^2}{36}-\frac{y^2}{64}=1$ and of the hyperbola conjugate to it.

12. The asymptotes of a hyperbola have the equations

$$y=\frac{4}{3}x \quad \text{and} \quad y=-\frac{4}{3}x.$$

Find the eccentricity of the hyperbola if its real axis coincides with the x-axis.

13. Write the canonical equation of a parabola if the distance from its focus to the directrix is equal to 10.

14. Determine the coordinates of the focus and the equation of the directrix of the parabola $y = 0.25x^2$.

15. The cross-section of a searchlight mirror has the shape of a parabola. Determine the position of the focus if the diameter of the mirror is equal to 60 cm and its depth to 30 cm.

16. Given a parabola $y^2 = 12x$. Find the length of its chord passing through the point M (8, 0) and inclined to the parabola axis at an angle of 60°.

17. Write the equation of the line whose points are equidistant from the point A (0, 2) and x-axis.

18. Reduce the equation of the parabola $y = 2x^2 - 8x + 5$ to the canonical form and determine the coordinates of its vertex.

19. Reduce the equation of the parabola $y = -3 + 4x - x^2$ to the canonical form and determine the coordinates of its vertex.

20. Reduce the equation of the parabola $x = y^2 - y + 2$ to the canonical form and determine the coordinates of its vertex.

21. The arch of a railway bridge whose span $l = 60$ m and height $h = 12$ m has the shape of a parabola. Determine the height h_1 of the side uprights of the arch situated at a distance of 15 m from its ends.

Chapter 5

Polar Coordinates. Parametric Equations of a Line

Sec. 30. Polar Coordinates

The main idea of the method of coordinates consists in that the position of a point in the plane is uniquely defined with the aid of two numbers. The concrete geometrical meaning of these numbers yields this or that coordinate system. In the previous sections we used only the rectangular coordinate system. Now we are going to consider the *polar coordinate system* which is the most important after the rectangular coordinates.

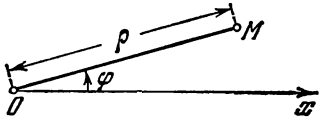


FIG. 41

In a plane (Fig. 41) take an arbitrary point O which will be called the *pole* and draw a directed ray Ox called the *polar axis*.

Let M be an arbitrary point in the plane. We then join the point M to the pole O. The length of the line segment $OM = \rho$, i.e. the distance of the point M from the pole is termed the *radius vector* of the point M, and the angle $\varphi = \angle xOM$ measured from the polar axis to the segment OM anticlockwise is called the *polar angle*. The numbers ρ and φ are said to be the *polar coordinates* of the point M.

The point M with the polar coordinates ρ and φ is denoted in the following way: $M(\rho, \varphi)$, the radius vector ρ preceding the polar angle φ.

As to the values attained by the polar coordinates, it is, obviously, sufficient to consider the values of ρ from 0 to $+\infty$ ($0 \leqslant \rho < +\infty$) and those of φ from 0 to 2π ($0 \leqslant \varphi < 2\pi$), where the angle φ, as we agreed upon, is measured from the polar axis anticlockwise. But sometimes we have to consider angles exceeding 2π, as also negative angles, i.e. angles reckoned clockwise.

Sec. 31. Relationship Between Rectangular and Polar Coordinates

Let us establish the relations between polar and rectangular coordinates.

Suppose that the pole of the polar coordinate system coincides with the origin of the rectangular coordinate system Oxy, and the polar axis coincides with the positive semi-axis Ox (Fig. 42). Then for an arbitrary point M we have

$$OA = x, \quad AM = y, \quad OM = \rho, \quad \angle xOM = \varphi.$$

Putting φ to be an acute angle, from the right-angled triangle AOM we find

$$OA = OM \cos\varphi, \quad AM = OM \sin\varphi,$$

or

$$\left.\begin{aligned} x &= \rho\cos\varphi, \\ y &= \rho\sin\varphi. \end{aligned}\right\}$$

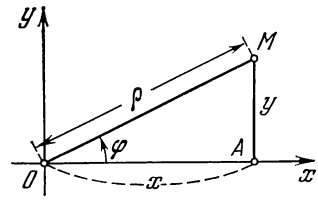


FIG. 42

The obtained formulas are true for any angle φ. In such a way the rectangular coordinates of the point M are expressed in terms of its polar coordinates. On the other hand, from the same triangle AOM we get

$$OM = \sqrt{OA^2 + AM^2}, \qquad \tan\varphi = \frac{AM}{OA},$$

or

$$\left.\begin{aligned} \rho &= \sqrt{x^2 + y^2}, \\ \tan\varphi &= \frac{y}{x}. \end{aligned}\right\}$$

This is how the polar coordinates of a point are expressed in terms of its rectangular coordinates.

It should be noted that when determining the polar angle by $\tan\varphi$, the signs of the coordinates x and y must be taken into account.

In the preceding sections we saw that lines can be specified by equations relating their rectangular coordinates. We are going now to give a simple example to show that it is also possible to define lines by polar equations.

Example. Consider the curve

$$\rho = a\varphi,$$

where a is a certain positive number. This curve is called the *spiral of Archimedes**. To construct it, we tabulate the corresponding values of φ and ρ:

φ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π	$\frac{5}{2}\pi$	...
ρ	0	$\frac{\pi}{4}a$	$\frac{\pi}{2}a$	πa	$\frac{3}{2}\pi a$	$2\pi a$	$\frac{5}{2}\pi a$	...

Using this table, we plot the points and join them with a smooth curve, defining (if necessary) more exactly the positions of intermediate points (Fig. 43).

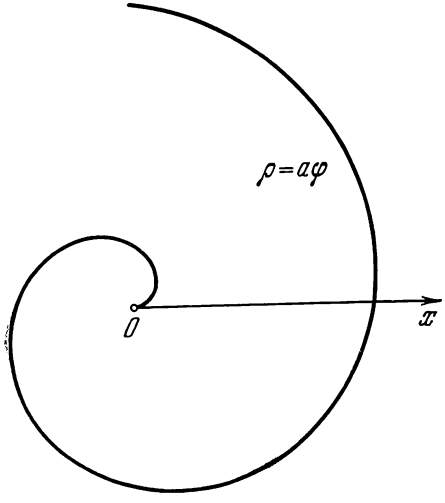


FIG. 43

Sec. 32. Parametric Equations of a Line

It is sometimes more convenient, instead of an equation of a line relating the rectangular coordinates x and y, to deal with the so-called *parametric equations* of the line representing the expressions for the current coordinates x and y as functions of a variable quantity t (called a *parameter*)**. Parametric equations play an important role, for instance, in mechanics where the coordinates x and y of a moving point $M(x, y)$ are considered as functions of time (*equations of motion*).

Example 1. Form the parametric equations of a circle.

Let $M(x, y)$ be an arbitrary point of a circle of radius R with centre at the origin (Fig. 44). In the right-angled triangle AOM we denote the angle xOM by t. Then, obviously, we shall have

$$OA = OM \cos t, \quad AM = OM \sin t,$$

* In honour of the great Greek scholar Archimedes (third century B.C.) who first studied that curve.

** The term "parameter" is used in yet another sense to denote a quantity which for a given curve is invariable but changes when moving from one curve of a given type to another. For example, the quantity p in the equation of the parabola $y^2 = 2px$ is constant for the given parabola but changes when we pass to another parabola.

or

$$\left.\begin{aligned} x &= R\cos t, \\ y &= R\sin t. \end{aligned}\right\} \qquad (32\text{-}1)$$

These are *parametric equations of the circle.*

In order to get an ordinary equation of the circle, we have to eliminate the parameter t. To this end we square (32-1) and add them

$$x^2 + y^2 = R^2(\cos^2 t + \sin^2 t) = R^2.$$

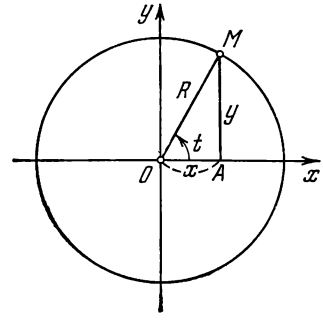


FIG. 44

Example 2. Derive the parametric equations of the ellipse.

An ellipse with the semi-axes a and b may be regarded as a circle uniformly compressed along the vertical diameter, of radius a, where the coefficient of compression $k = \frac{b}{a}$ (see Sec. 24, Chapter 4). Let $M(x, y)$ be a point on the ellipse and $N(X, Y)$ the corresponding point on the circle (Fig. 45), where

$$x = X, \quad y = \frac{b}{a}Y. \qquad (32\text{-}2)$$

We take the angle formed by the radius ON of the circle with the positive direction of the axis Ox for the parameter t: $t = \angle NOx$. Using (32-2), we have

$$x = X = a\cos t,$$

$$y = \frac{b}{a}Y = \frac{b}{a}\cdot a\sin t = b\sin t.$$

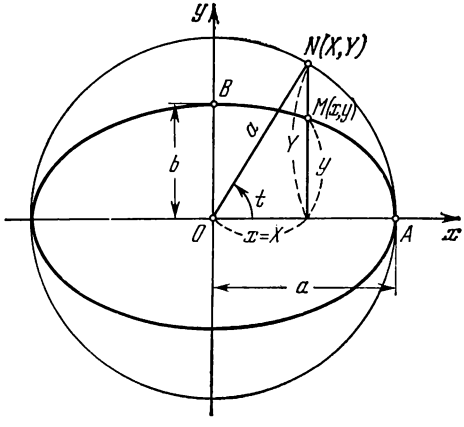


FIG. 45

Thus, the *parametric equations of the ellipse* with the semi-axes a and b are

$$\left.\begin{aligned} x &= a\cos t, \\ y &= b\sin t. \end{aligned}\right\} \qquad (32\text{-}3)$$

Eliminating the parameter t from (32-3), we get the canonical equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Having parametric equations of a line, we are able to plot it point by point.

Example 3. Construct the curve

$$\left.\begin{aligned} x &= t^2, \\ y &= 2t. \end{aligned}\right\} \qquad (32\text{-}4)$$

Tabulating the values, we have

t	...	-2	-1	0	1	2	...
x	...	4	1	0	1	4	...
y	...	-4	-2	0	2	4	...

Plotting the points according to their coordinates (x, y) on the plane Oxy and joining them with a line, we get the required curve (Fig. 46). This curve is a parabola. Indeed, eliminating the parameter t from equations (32-4), we get $y^2 = 4x$, i.e. the canonical equation of the parabola.

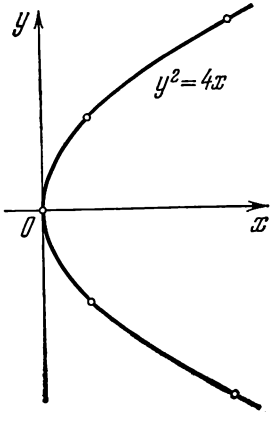


FIG. 46

Sec. 33. Parametric Equations of the Cycloid

Definition. *The cycloid is a curve described by a point M on the circumference of a circle rolling (without sliding) along a straight line* (Fig. 47).

Let us derive the parametric equations of the cycloid, taking the straight line for the x-axis, assuming that the radius of the rolling circle is equal to a, and that in its initial position the moving point M coincides with the origin. We take for the parameter t the angle (in radians) of rotation of the moving radius MC of the circle relative to the vertical radius KC, where K is the point of contact of the circle with the x-axis (Fig. 47). Since the circle rolls without sliding, obviously, we have

$$OK = \overset{\frown}{MK} = at.$$

Hence, on the basis of Fig. 47 we get the following expressions for the coordinates of a moving point M:

$$x = OP = OK - PK = OK - MQ = at - a \sin t = \\ = a(t - \sin t)$$

and

$$y = PM = KC - QC = a - a \cos t = a(1 - \cos t).$$

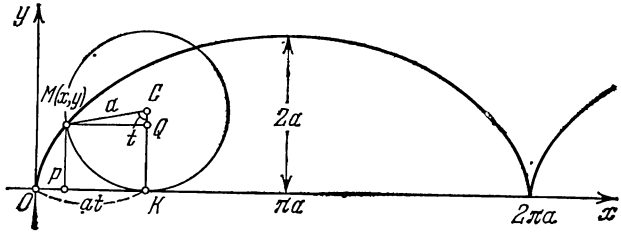


FIG. 47

Thus, the parametric equations of the cycloid are

$$\left.\begin{aligned} x &= a(t - \sin t), \\ y &= a(1 - \cos t). \end{aligned}\right\}$$

Exercises

1. Construct the points given by their polar coordinates: $A(5, 0)$, $B\left(2, \frac{\pi}{4}\right)$, $C\left(3, \frac{\pi}{2}\right)$, $D(1, \pi)$, $E\left(2, \frac{5}{3}\pi\right)$.

2. What rectangular coordinates have the points specified by their polar coordinates: $A(5, 0)$, $B\left(6, \frac{\pi}{4}\right)$, $C\left(2, \frac{\pi}{2}\right)$, $D\left(4, \frac{5\pi}{4}\right)$?

3. Construct (point by point) the logarithmic spiral $\rho = 2^{\varphi/\pi}$.

4. Write in polar coordinates the equations of the following lines (a) $x = 1$; (b) $y = -2$; (c) $y = x$; (d) $y = 2x$; (e) $x + y = \sqrt{2}$; (f) $x^2 + y^2 = 25$.

5. Write the equation of the straight line $x \cos \alpha + y \sin \alpha - p = 0$ in polar coordinates.

6. The equation of a curve in polar coordinates has the form

$$\rho = a \cos \varphi.$$

Write the equation of this curve in rectangular coordinates and find out what curve is this.

7. A line is given by the polar equation

$$\rho = \frac{1}{1 - \cos \varphi}.$$

Write the equation of this line in rectangular coordinates.

8. A line is represented parametrically by the equations

$$x = a \sin t, \quad y = b \cos t.$$

Find its equation in rectangular coordinates.

9. A line is specified by the parametric equations

$$x = \frac{a}{2}\left(t + \frac{1}{t}\right), \quad y = \frac{b}{2}\left(t - \frac{1}{t}\right).$$

Find its equation in rectangular coordinates.

10. A line is given by the parametric equations

$$x = t^2, \quad y = 4t.$$

Find its equation in rectangular coordinates.

11. A point moves in the plane Oxy occupying at time t measured from the initial moment $t = 0$ the position M $(200 - t,\ 100 - t)$. At what time will the point meet the straight line

$$8x - 6y + 10 = 0$$

and what will its coordinates be at that moment?

Chapter 6

Functions

Sec. 34. Constants and Variables

In the study of natural phenomena we all the time deal with constant quantities and variable quantities, or *constants* and *variables*.

Definition. *A constant is a quantity whose numerical values remain fixed, or in other words, a quantity which has one and only one value (either in general, or within the framework of a given problem; in the latter case a constant quantity is called a parameter).*

A variable is a quantity which can take on different values within the framework of a given problem.

Let us give a couple of examples of variables and constants.

Example 1. The diameter and length of the circumference of a circle can attain different values depending on circumstances and, consequently, are, generally speaking, variables, whereas the ratio of the length of the circumference to its diameter always preserves one and the same value and, hence, is a constant called the number π ($\pi = 3.14159...$).

Example 2. The volume v and pressure p of a certain mass of gas are variables, while, as is known from physics, at a constant temperature the product vp is a constant. But with a change in temperature the product vp, generally speaking, varies.

It should be noted that, for the sake of generality, it is frequently advisable to regard a constant as a variable that can assume only one value.

Sec. 35. The Concept of Function

When studying some phenomenon we usually deal with a collection of variables which are related so that the values of some quantities (*independent variables*) completely determine the values of other quantities (*dependent variables*, or *functions*).

For instance, when studying gas we are interested in its volume v, temperature t, pressure p. According to Mendeleev-Clapeyron law, given the volume and temperature of a gas we can uniquely determine its pressure. Consequently, the quantities v and t may be regarded as independent variables, and p as a dependent variable (or function).

Now we are going to present the definition of the *concept of function* which is the principal concept of higher mathematics. First we shall confine ourselves to the case of two variables.

Definition. *A variable y is said to be a function (single-valued) of a variable x if they are related so that to each value attained by x (permissible values) there corresponds a unique, quite definite value of y*.*

This definition was first roughly formulated by N. I. Lobachevsky, the great Russian mathematician.

Here, the variable x is called the *argument* or *independent variable*, and y is sometimes termed the *dependent variable*. The variables x and y are then said to be *in a functional relationship*.

The collection of all values assumed by the variable x for which the function y is defined is called the *domain of definition*, or the *domain of existence*, or simply the *domain of this function*.

The domain of definition of a function represents most frequently either an *open interval* (a, b) (Fig. 48, *a*), i.e. the set of all numbers x satisfying the inequality

$$a < x < b$$

(here the values $x = a$ and $x = b$ are excluded!), or a *closed interval* $[a, b]$ (Fig. 48, *b*), i.e. the set of all numbers x satisfying the inequality

$$a \leqslant x \leqslant b$$

(here the values $x = a$ and $x = b$ are included!). In some cases the domain of definition of a function is a *half-open interval* closed at

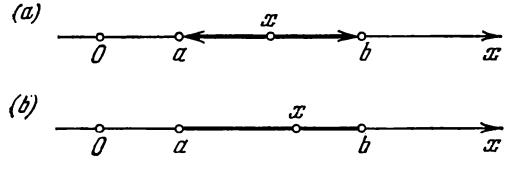


FIG. 48

the left, $[a, b)$, or at the right, $(a, b]$, i.e. the set of numbers x defined by the conditions $a \leqslant x < b$ or $a < x \leqslant b$, respectively. The set of points representing an open interval, or a closed interval, or a half-open interval will be called the *interval* and designated as $\langle a, b \rangle$.

* Here and henceforth we assume that the quantities and numbers we deal with attain only real values, unless otherwise stated.

Besides finite intervals discussed above we often deal with infinite intervals: $(-\infty, a) = \{x \mid x < a\}$, i.e. the set of all numbers less than a; $(b, +\infty) = \{x \mid x > b\}$, i.e. the set of all numbers greater than b; $(-\infty, +\infty)$, i.e. the set of all real numbers (see Sec. 44, Chapter 7). The intervals $(-\infty, a]$ and $[b, +\infty)$ have an analogous meaning.

To express the fact that a variable y is a function of a variable x, we use the notation

$$y = f(x), \tag{35-1}$$

where the letter f is called the *function symbol* (read as "y is equal to f of x"). Instead of f, any other letter (say, g, h, F, φ, etc.) can be used to denote functional relationship (35-1). And it goes without saying that different functions must be designated by different letters within the framework of one and the same problem.

The particular value of the function $f(x)$ for $x = a$ is written in the following way: $f(a)$. For instance, if

$$f(x) = x(1 - x),$$

then

$$f(0) = 0, \quad f(1) = 0, \quad f(2) = -2 \text{ and so on.}$$

Here are a few examples explaining the concept of function.

Example 1. From the formula for the area of a circle

$$S = \pi R^2$$

it follows that to each permissible (i.e. positive) value of the radius R there corresponds a definite value of the area S. Consequently, S is a function of R defined within the infinite interval $0 < R < +\infty$.

Example 2. According to the Boyle-Mariotte law, at a constant temperature we have

$$vp = C,$$

where v is the volume of a definite mass of gas, p its pressure, and C some constant quantity. Whence

$$v = \frac{C}{p}.$$

Consequently, to each value of the pressure p there corresponds a definite volume v of gas. We may say that the volume of gas v is a function of pressure p. From physical considerations it follows that the domain of definition of this function is an infinite interval

$$0 < p < +\infty.$$

Example 3. Find the domain of the function

$$y = \sqrt{4 - x^2}. \tag{35-2}$$

This function has sense if

$$4 - x^2 \geqslant 0.$$

Hence

$$x^2 \leqslant 4 \quad \text{or} \quad |x| \leqslant 2.$$

Consequently, the domain of definition of the function is the segment

$$-2 \leqslant x \leqslant 2.$$

To visualize the behaviour of a function, we construct its *graph* considering the independent variable x and function y as rectangular coordinates of some point M in the plane Oxy.

Definition. *The graph of a function $y = f(x)$ is the set of all points $M(x, y)$ of the plane Oxy whose coordinates are related by the given functional relationship.*

In other words, the graph of a function is a line whose equation is the equality defining the function.

For instance, for function (35-2) we have

$$x^2 + y^2 = 4, \qquad y \geqslant 0.$$

Obviously, its graph is the upper semi-circle of radius $R = 2$ with centre at the origin (Fig. 49). From the figure it becomes clear that the domain of definition of the function represents a segment $[-2, 2]$.

It should be noted here that after the graph of the function $y = f(x)$ is constructed we can approximate the roots of the equation $f(x) = 0$ as the abscissas of the points of intersection of the graph with the x-axis.

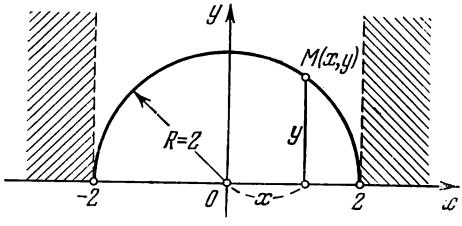


FIG. 49

If to every value of the variable x there corresponds one value of the variable y, then y is termed a *single-valued function* of x; if at least to some values of the variable x there correspond several (two, three, etc.) or an infinite number of values of the variable y, then the latter is called *multiple-valued* (double-valued, triple-valued, etc.) *function* of x.

For instance, $y = x^2$ is a single-valued function of x; $y = \sin x$ is also a single-valued function of x. The function $y = \pm \sqrt{x}$ is a double-valued function of x; $y = \operatorname{Arcsin} x$ is a multiple-valued (infinite-valued) function of x.

Henceforth the word 'function' will be understood as a single-valued function, unless otherwise specified.

Sec. 36. Simplest Functional Relations

1. Direct Proportional Relation

Definition. *Two variables are said to be directly proportional if they have a fixed (constant) ratio.*

Here are some examples of directly proportional quantities: the length of the circumference of a circle and its radius; the path

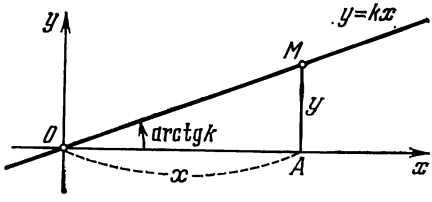


FIG. 50

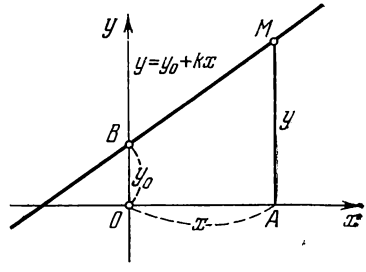


FIG. 51

covered during uniform motion and the time spent; linear stretching of an elastic rod and the load applied, and many others.

Let x and y be directly proportional quantities and let for $x = 1$ the quantity y attains the value equal to k. By definition, we have $\frac{x}{1} = \frac{y}{k}$; hence

$$y = kx, \tag{36-1}$$

where the constant k is called the *proportionality factor*. Function (36-1) is termed a *homogeneous linear function*; its graph is a straight line passing through the origin with the slope k (Fig. 50).

2. Linear Relation

Definition. *Two variable quantities x and y are linearly related if*

$$y = y_0 + kx, \tag{36-2}$$

where k and y_0 are constants.

(36-2) is called a *linear* function; its graph is a straight line (Fig. 51) with the initial segment y_0 and slope k.

The distance of a point performing uniform rectilinear motion from the start and the time, the length of a bar and its temperature are examples of linearly related quantities.

3. Inverse Proportional Relation

Definition. *Two variable quantities are inversely proportional if one is proportional to the reciprocal of the other.*

Examples of inverse proportional quantities are: the velocity of uniform motion and time required to cover a given distance; the

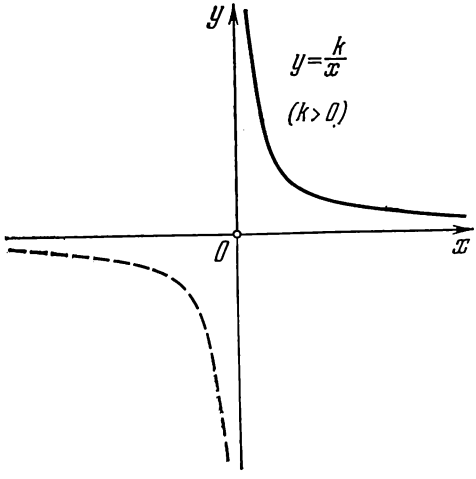


FIG. 52

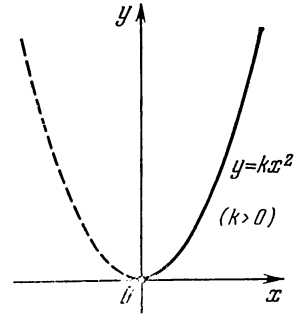


FIG. 53

volume occupied by gas (at a constant temperature) and pressure; current intensity (at a constant electromotive force) and resistance of a circuit, and so on.

Let x and y be inversely proportional quantities and suppose that if $x = 1$, then $y = k$. By definition, we have

$$\frac{x}{1} = \frac{k}{y},$$

hence

$$= \frac{k}{x}.$$

For $k > 0$ the graph of this function represents an entire equilateral hyperbola or its part (Fig. 52). For $k < 0$ we get a hyperbola (or its part) situated in the second and fourth quadrants.

4. Quadratic Relation

In the simplest case a *quadratic relation* has the form

$$y = kx^2, \tag{36-3}$$

where k is a constant. The graph of this function is a parabola (entire curve or its part) (Fig. 53); for $k > 0$ it is situated above the x-axis and for $k < 0$ below this axis.

Examples of quadratic relations are: the area of a circle and radius of this circle; the path covered by a body at free fall and time elapsed, and so forth.

5. Sinusoidal Relation

When studying periodic processes we often come across the *sinusoidal relation*

$$y = A \sin (\omega x + \varphi), \tag{36-4}$$

where A is the *amplitude*, ω the *frequency* and φ the *initial phase*. Function (36-4) is called the *harmonic*; y is a *periodic function* with the period

$$T = \frac{2\pi}{\omega},$$

i.e. the values of the function $y = y(x)$ at the points x and $x + T$ are the same. Indeed, we have

$$y(x + T) = A \sin [\omega (x + T) + \varphi] = A \sin (\omega x + 2\pi + \varphi) = A \sin (\omega x + \varphi) = y(x).$$

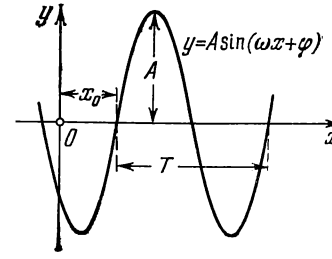


FIG. 54

Harmonic (36-4) can be reduced to the form

$$y = A \sin \omega (x - x_0), \tag{36-5}$$

where $x_0 = -\frac{\varphi}{\omega}$. Hence it is clear that the graph of the harmonic is a *deformed sinusoid* with an amplitude A and period T shifted along the x-axis by the quantity x_0 (Fig. 54)*.

Examples of sinusoidal relations are: deviation of air particles from equilibrium caused by a sound wave of constant frequency and time; intensity of single-phase sinusoidal current and time, and so on.

Sec. 37. Methods of Representing Functions

To represent or, which is the same, to specify a function means to indicate its domain of definition and the rule according to which for any given value of the independent variable the corresponding

* For more detail see Sec. 42.

value of the function is found. There are three basic methods of representing functions: (1) analytical, (2) tabular, and (3) graphical.

1. The Analytical Method

If a function is represented by means of a formula, then it is said to be specified *analytically*. For instance, in the formula for the volume of a sphere

$$V=\frac{4}{3}\pi R^3$$

the volume V is a function of radius R represented analytically.

If a function

$$y=f(x)$$

is specified by a formula, then its *function symbol* f denotes the totality of operations that should be performed in a definite sequence on the value of the argument x in order to get the corresponding value of the function y (or, which is the same, the value of the function $f(x)$).

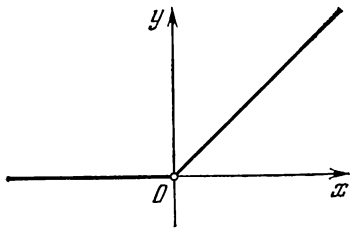


FIG. 55

Let, for instance,

$$f(x)=\sqrt[3]{x^2-1}. \qquad (37\text{-}1)$$

Here the function symbol f designates the following series of operations:

(1) squaring the argument x;

(2) subtracting the number 1 from the result obtained;

(3) taking the cube root of this difference.

Knowing f and assigning various values to the argument x, we get the corresponding values of the function $f(x)$. For instance, for our function (37-1) we have

$$f(-1)=\sqrt[3]{(-1)^2-1}=0,$$

$$f(0)=\sqrt[3]{0^2-1}=-1,$$

$$f(1)=\sqrt[3]{1^2-1}=0$$

and so on.

A similar meaning is obtained by the expressions

$$f(x+h)=\sqrt[3]{(x+h)^2-1}$$

and so on.

In some cases a function may be specified by *several formulas.*

Let, for instance,

$$f(x)=\begin{cases} 0, & \text{if } x\leqslant 0, \\ x, & \text{if } x>0. \end{cases}$$

This function is quite defined since for each value of the argument x we can indicate the corresponding value of the function $f(x)$. Namely, if x is negative or equal to zero, then $f(x)$ is equal to zero, for example,

$$f(0)=0, \quad f\left(-\frac{1}{2}\right)=0, \quad f(-1)=0$$

and so on.

If x is positive, then $f(x)$ is equal to the value of the argument, for instance,

$$f\left(\frac{3}{4}\right)=\frac{3}{4}, \qquad f(5)=5$$

and so on.

Thus, two formulas

$$f(x)=0, \quad \text{if } x\leqslant 0,$$

and

$$f(x)=x, \quad \text{if } x>0,$$

define *one* function (Fig. 55).

2. The Tabular Method

Suppose we want to establish a relationship between the yearly average temperature $t°$ and the height h of a certain terrain above sea level expressed in kilometres. Let us tabulate the results of our observations:

h	0	1	2	3	4	5	6	7	8
$t°$	+7.9	+4.6	+0.1	−5.0	−10.7	−16.9	−23.7	−30.8	−38.0

From the above table we see that the yearly average temperature varies with the height of a terrain above sea level, and to each value of the height h there corresponds a definite value of temperature $t°$. Consequently, the yearly average temperature t^0 is a function of the height of a terrain h above sea level, the relation between the variables $t°$ and h being established by a table. Such mode of representing a function is called *tabular*.

Knowing the analytic expression of a function, we can represent this function by means of a table for certain values of the argument

we are interested in. Suppose we have a function

$$y = x^3.$$

Assigning to x a number of numerical values and computing the corresponding values of y, we get the following table:

x	...	−3	−2	−1	0	1	2	3	...
y	...	−27	−8	−1	0	1	8	27	...

We see that if a function is specified analytically (i.e. by means of a formula), then it is possible to compile a table for it, or, in other words, to *tabulate this function.*

We usually tabulate functions frequently used in practice but having a complicated formula when represented analytically. For instance, the tables of the functions $\sin x$, $\cos x$, etc. (so-called tables of natural trigonometric quantities), $\log x$ (tables of logarithms) and so on are widely known and frequently used. There are formulas for these functions but they are too complicated for practical use since they are expressed by means of infinite series (see Sec. 192, Chapter 21).

There arises a question: Is it always possible to pass over from a tabular representation of a function to its analytical expression, i.e. to write such a function by a formula?

It should be noted in this connection that a table gives not all the values of a function. The intermediate values of the function can be found only approximately (so-called *interpolation of a function*). Therefore, in the general case it is impossible to find an exact analytical expression of a function from its tabular data.

But it is always possible to construct one or several formulas which for the tabular values of the argument will yield the corresponding tabular values of the function. Such formulas are termed *interpolation* ones.

3. The Graphical Method

The two above considered methods of representation of a function (analytical and tabular) lack visuality. The *graphical method* is devoid of this disadvantage. In this case the relation between the argument x and the function y is established with the aid of a graph (Fig. 56). Here, to find for a certain value of the argument, say x, the corresponding value y of the function, we have to lay off a segment $OA = x$ in the corresponding direction on the x-axis and then to erect a perpendicular AM to intersect the graph at point M. Taking the length of this perpendicular with an appropriate sign,

we get the number

$$y = f(x).$$

Assigning to x various values and using this technique, we shall have the corresponding values of the function y which, if necessary, can be tabulated.

Graphical representation of a function may be exemplified by the so-called *barogram* (the record of a barograph) showing graphically how atmospheric pressure changes with time.

To construct the graph of a function $y = f(x)$ specified analytically, we have to compile a table of values of the given function and then, regarding x as the abscissa of a point, and y as its ordinate,

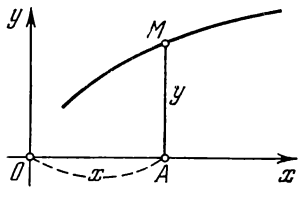


FIG. 56

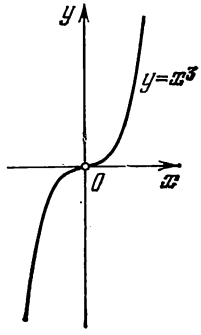


FIG. 57

to plot a system of points in the plane. Joining these points with a line which also takes into account the intermediate values of the function, we obtain an approximate graphical representation of the given function.

For instance, using the table given under Item 2 of the present section, we plot the graph of the function

$$y = x^3$$

which is a *cubic parabola* (Fig. 57).

Sec. 38. The Concept of Function of Several Variables

The concept of function of one independent variable is naturally spread to the case of several variables.

Definition. *A variable u is called a function (single-valued) of several variables, for instance, of two: x and y if to each pair of values of x and y (permissible values) there corresponds a definite value of the quantity u.*

Here, the variables x and y are called *independent variables* or *arguments*; the set of their values under consideration is called the *domain of definition* or the *domain of existence* of the function u. The domain of existence of a function of two variables x and y is, generally speaking, the set of points of the plane Oxy.

The fact that u is a function of x and y is expressed by the following brief notation

$$u = f(x, y),$$

where f is called the *function symbol*; it symbolizes the rule of correspondence between the independent variable and function. Of course, instead of f any letter may be used for this purpose.

Example 1. The area U of a rectangle whose sides are equal to x and y is expressed by the formula

$$U = xy.$$

Obviously, U is a function of two arguments x and y defined in the domain $x > 0$, $y > 0$.

Example 2. The combined equation of state for gases has the form

$$vp = RT,$$

where v is the volume occupied by a given mass of gas, p the pressure of the gas, T the absolute temperature, and R a constant quantity. Solving this equation with respect to v, we get

$$v = \frac{RT}{p}.$$

We see that the volume v is a function of two variables: the pressure p and absolute temperature T, and this function is defined in the domain $p > 0$, $T > 0$.

A function u of three variables x, y, and z in the general form may be denoted in the following way

$$u = f(x, y, z).$$

Example 3. The volume $V = xyz$ and the total surface $S = 2xy + 2yz + 2zx$ of a rectangular parallelepiped with linear dimensions x, y, and z are functions of three arguments x, y, and z defined in the domain $x > 0$, $y > 0$, and $z > 0$.

Sec. 39. Implicit Function

A function is called *explicit* if it is specified by a formula whose right-hand member contains no dependent variable. For instance, the function $y = x^2$ is an explicit one.

A function y of the argument x is called *implicit* if it is specified by the equation

$$F(x, y) = 0 \tag{39-1}$$

not resolved in the dependent variable. For instance, the function y $(y > 0)$ defined by the equation $x^2 + y^2 = 1$ is an implicit one.

To obtain the explicit expression of the function y defined by (39-1), it is sufficient to resolve this equation in y. Since for a given value of the argument x equation (39-1) may have several (and even an infinite number of) roots y, in the general case an implicit function is multiple-valued.

The set of values of the argument x for each of which equation (39-1) has at least one real root y represents the *domain of existence* of the corresponding implicit function. It should be noted that not every equation (39-1) defines an implicit function. For instance, the equation

$$x^2 + y^2 + 1 = 0,$$

obviously, defines no function whatever (in the real domain).

Example. Let x and y be related by the equation

$$x^2 + y^3 = 1.$$

Here y is an *implicit function* of the argument x. Resolving this equation with respect to y, we get

$$y = \sqrt[3]{1 - x^2}.$$

This last formula gives us y as an explicit function of x.

Sometimes it is rather difficult to resolve (39-1) with respect to y. For instance, the Kepler equation

$$y - \varepsilon \sin y = x \qquad (0 < \varepsilon < 1)$$

cannot be resolved by elementary means with respect to y. In such a case we have to study the function y directly using the equation defining this function.

Sec. 40. Inverse Function

Let y be a function of the argument x:

$$y = f(x). \tag{40-1}$$

Assigning values to x, we shall obtain corresponding values of y. However, regarding y as the argument and x as a function, we may assign values to y and compute the corresponding values of x. In this case equation (40-1) will define x as an implicit function of y.

The latter function is called *inverse* with respect to the given function y.

Assuming that (40-1) is resolved in x, we get an explicit expression of the inverse function

$$x = \varphi(y), \tag{40-2}$$

where the function $\varphi(y)$, for all permissible values of y, satisfies the condition

$$f[\varphi(y)] = y. \tag{40-3}$$

Example 1. In the formula for the volume of a ball

$$V = \frac{4}{3}\pi R^3 \tag{40-4}$$

the radius R is the argument and the volume V is the function. Resolving equation (40-4) in R, we get the function inverse to the given one:

$$R = \sqrt[3]{\frac{3V}{4\pi}}.$$

Sometimes they stick to the standard notation in which x designates the independent variable and y the function, i.e. the dependent variable. Then an inverse function should be written in the form

$$y = \varphi(x).$$

For instance, we may assert that $y = 2^x$ and $y = \log_2 x$ are mutually inverse functions.

The inverse function of a single-valued function may turn out to be *multiple-valued* (as in Fig. 58), i.e. to a given value of y there may correspond several values $x_1, x_2, x_3, \ldots$ of the inverse function $x = \varphi(y)$ (Fig. 58). In some cases we succeed in making the inverse function single-valued by introducing additional limitations for its possible values.

Example 2. The double-valued function

$$x = \pm\sqrt{y}$$

is an inverse function with respect to the function $y = x^2$. If we agree to take only the arithmetic value of the root, then the inverse function will be single-valued.

Obviously, the function inverse to function (40-2) is function (40-1). Therefore, the functions with the function symbols f and φ related by (40-3) are *mutually inverse* functions. One of them is called the *direct* function, the other *inverse*.

It is worth noting that one and the same curve

$$y = f(x)$$

represents the graph of a given function as well as the graph of the function inverse to it depending on which of the axes (Ox or Oy) the values of the argument are laid off.

If we agree to denote the independent variable by x and the dependent variable by y, then to get the graph of the inverse func-

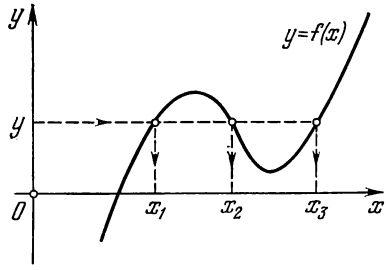


FIG. 58

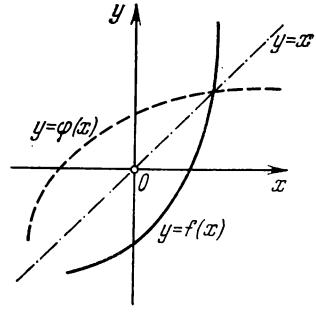


FIG. 59

tion $y = \varphi(x)$ from the graph of the given function $y = f(x)$, it is, obviously, sufficient to mirror reflect the latter graph about the bisector of the first and third quadrants (Fig. 59).

Sec. 41. Classification of Functions of One Argument

Depending on the character of the operations to be performed on the value of the argument in order to get the corresponding value of the function, the following classification of functions is established.

(1) If on the value of the argument x and on some of the constants the following operations: addition, subtraction, multiplication, raising to an integral positive power (and a finite number of times) are performed, then we have an *integral rational function* or a *polynominal*. The general form of such a function is

$$P(x) = a_0x^m + a_1x^{m-1} + \ldots + a_{m-1}x + a_m,$$

where m is an integral positive number or zero, and the coefficients $a_0, a_1, \ldots, a_{m-1}, a_m$ are constant numbers.

(2) A function representable in the form of a quotient, both the numerator and denominator being integral rational functions,

$$R(x) = \frac{a_0x^m + a_1x^{m-1} + \ldots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n},$$

is called a *fractional rational function.*

Integral rational functions and fractional rational functions constitute the class of *rational functions.*

(3) If, in addition to the above mentioned five algebraic operations performed on the argument x, roots are taken a finite number of times and the result is not a rational function, then we have an *irrational function*. For instance,

$$f(x) = \sqrt{\frac{5x^2+4x-7}{3x^2-8x+4}} + (\sqrt[5]{x}+1)^3.$$

Here a root is usually understood as its arithmetic value.

Rational and irrational functions constitute the class of *explicit algebraic functions*.

(4) In the more general case the *algebraic function* is defined as a multiple-valued implicit function y defined by the equation

$$p_0(x)\, y^n + p_1(x)\, y^{n-1} + \ldots + p_{n-1}(x)\, y + p_n(x) = 0,$$

where n is a positive integer, and the coefficients $p_0(x)$, $p_1(x)$, ... , $p_{n-1}(x)$, $p_n(x)$ are integral rational functions of x and, moreover, the coefficient $p_0(x)$ is not identically equal to zero*. For instance, the root of the equation

$$y^5 + y - x^2 = 0$$

is an algebraic function. It should be noted here that this function is not an explicit algebraic function, since an algebraic equation of the fifth and higher degrees is, generally speaking, unsolvable in radicals.

(5) Every function not belonging to the class of algebraic functions is called a *transcendental function*.

The simplest transcendental functions (so-called *elementary transcendental* functions) are:

(a) the exponential function a^x, where a is a positive number not equal to unity;

(b) the logarithmic function $\log_a x$, where $a > 0$ and $a \neq 1$;

(c) the trigonometric functions: $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$;

(d) the inverse trigonometric functions: $\operatorname{Arcsin} x$, $\operatorname{Arccos} x$, $\operatorname{Arctan} x$, $\operatorname{Arccot} x$, $\operatorname{Arcsec} x$, $\operatorname{Arccsc} x$.

Algebraic, elementary transcendental functions and their finite combinations are called *elementary functions*. This is the main stock of functions we are going to deal with within the framework of our course.

As a rule, we shall use only single-valued elementary functions putting, if necessary, additional limitations on the multiple-valued functions under consideration.

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* In the general case algebraic functions are considered in the complex domain.

Sec. 42. The Graphs of the Basic Elementary Functions

Below we shall consider the graphs of some basic elementary functions.

I. *The power function*

$$y = x^n, \tag{42-1}$$

where n is an integer.

The function is defined for $-\infty < x < +\infty$, if $n \geqslant 0$, for $0 < |x| < +\infty$, if $n < 0$.

If $n \geqslant 0$ ($n = 0, 1, 2, \ldots$), then the graphs of (42-1) represent

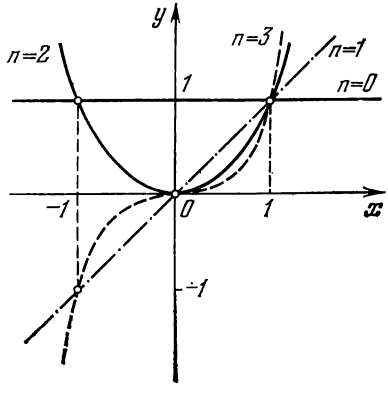

FIG. 60

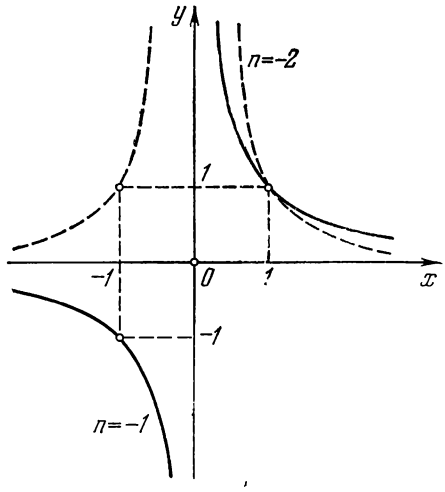

FIG. 61

parabolas of orders zero, one, two, etc., respectively (see Fig. 60).

If $n < 0$ ($n = -1, -2, \ldots$), then the graphs of functions (42-1) represent *hyperbolas* of various orders (Fig. 61).

II. *The radical*

$$y = \sqrt[n]{x}, \tag{42-2}$$

where n is a natural number.

The domain of definition of the function: $0 \leqslant x < +\infty$ for even n and $-\infty < x < +\infty$ for odd n.

Since $x = y^n$, (42-2) is an inverse function with respect to power function (42-1). Therefore, the graphs of the radical for various n are parabolas or parts of them (Fig. 62).

III. *The exponential function*

$$y = a^x,$$

where a is a constant number, and $a > 0$, $a \neq 1$.

This function is defined for all values of x. It has positive values and increases monotonically from 0 to $+\infty$ for $a > 1$, and decreases monotonically from $+\infty$ to 0 for $0 < a < 1$ (Fig. 63).

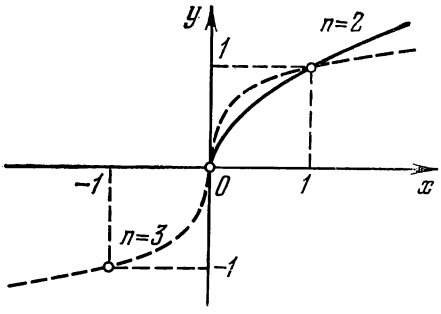

FIG. 62

IV. *The logarithmic function*

$$y = \log_a x \quad (a > 0,\ a \neq 1). \tag{42-3}$$

The domain of definition: $0 < x < +\infty$.

Since from (42-3) we have

$$x = a^y,$$

the logarithmic function is an inverse function with respect to the exponential function. Therefore the graph of the logarithmic function

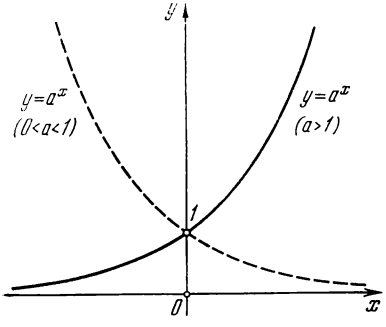

FIG. 63

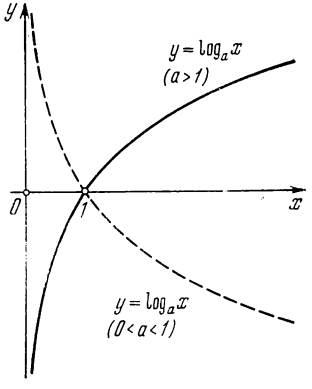

FIG. 64

is obtained from the graph of the exponential function as a mirror image of the latter about the bisector of the first and third quadrants (Fig. 64).

V. *Trigonometric functions.*

In higher mathematics the argument of a trigonometric function is a *number* that may be considered as the measure of the corresponding angle expressed in radians.

Here we shall confine ourselves to a review of the most important trigonometric functions:

(a) $y = \sin x$;

the function is defined for all values of x. The function $\sin x$ is

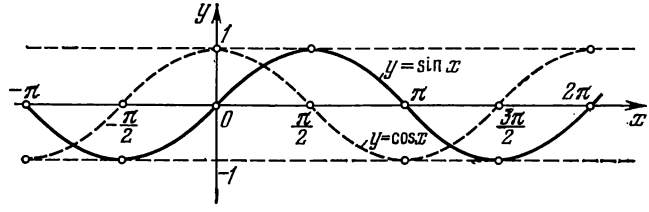

FIG. 65

limited ($|\sin x| \leqslant 1$) and periodic with the period 2π (i.e. the values of the function are repeated as the argument changes by 2π); its graph is a *sinusoid* (Fig. 65).

(b) $y = \cos x$;

this function possesses the properties similar to those of the function $\sin x$. Its graph is a *cosinusoid* presenting a sinusoid but shifted to the left by $\frac{\pi}{2}$ (Fig. 65). Indeed, $\cos x = \sin\left(x + \frac{\pi}{2}\right)$.

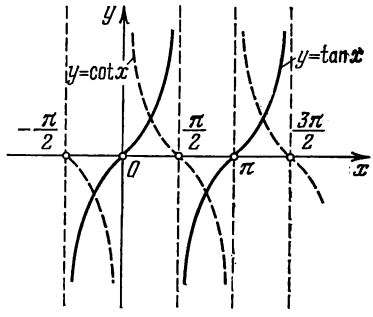

FIG. 66

(c) $y = \tan x$;

the function is defined for $x \neq \frac{2k+1}{2}\pi$ ($k = 0, \pm 1, \pm 2, \ldots$); has a period π. Its graph is a *tangensoid* (Fig. 66).

(d) $y = \cot x$;

it is defined for $x \neq k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$); has a period π. Its graph is a *cotangensoid* which is geometrically identical with the tangensoid (Fig. 66).

VI. *Inverse trigonometric functions*:

(a) $y = \arcsin x$, (42-4)

i.e. *y is an arc taken within the limits*

$$-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}, \quad (42\text{-}5)$$

whose sine is equal to x:

$$\sin y = x \tag{42-6}$$

(the *principal value*). Function (42-4) is uniquely defined on the interval $[-1, 1]$; its graph is a part of sinusoid (the arc AB in Fig. 67).

If we convert equality (42-6) without putting the condition

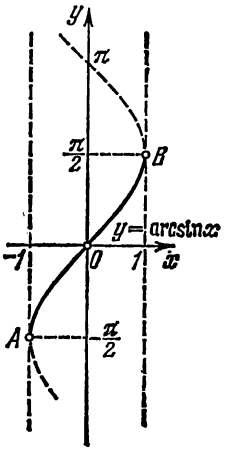

FIG. 67

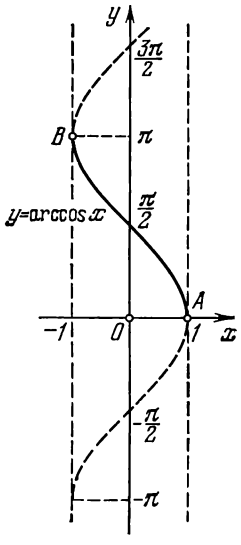

FIG. 68

(42-5), i.e. if we find all the values of y whose sines are equal to x, then we get a *multiple-valued function*

$$y = \operatorname{Arcsin} x,$$

whose graph is a sinusoid running along the y-axis. By virtue of the properties of arcs having the same sine, we have the following formula

$$\operatorname{Arcsin} x = (-1)^k \arcsin x + k\pi \quad (k = 0, \pm 1, \pm 2, \ldots).$$

(b) $$y = \arccos x, \tag{42-7}$$

i.e. y *is an arc taken within the limits*

$$0 \leqslant y \leqslant \pi \tag{42-8}$$

whose cosine is equal to x:

$$\cos y = x \tag{42-9}$$

(the *principal value*). Function (42-7) is uniquely defined on the interval $[-1, 1]$; its graph is a part of cosinusoid (the arc AB in Fig. 68).

Solving (42-9) with respect to y, we get in the general case a multiple-valued function

$$y = \operatorname{Arccos} x,$$

whose graph is a cosinusoid running along the y-axis. The following formula is valid here

$$\operatorname{Arccos} x = \pm \arccos x + 2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

(c) $y = \arctan x$, (42-10)

i.e. *y is an arc taken within the limits*

$$-\frac{\pi}{2} < y < \frac{\pi}{2}, \tag{42-11}$$

whose tangent is equal to x:

$$\tan y = x \tag{42-12}$$

(the *principal value*).

Function (42-10) is defined on the interval $-\infty < x < +\infty$

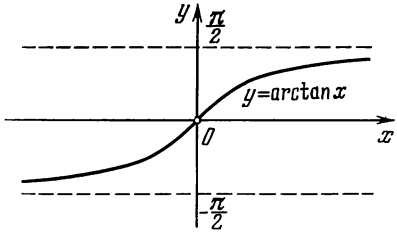

FIG. 69

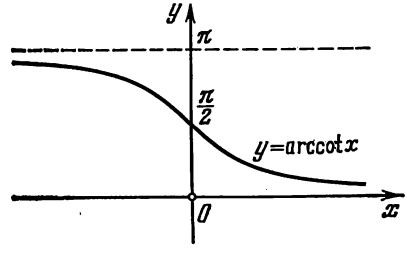

FIG. 70

uniquely; its graph is an arc of tangensoid (Fig. 69).

Solving (42-12) with respect to y, in the general case we shall have a *multiple-valued* function

$$y = \operatorname{Arctan} x,$$

whose graph consists of an infinite number of shifted tangensoids (42-10). The following formula is true

$$\operatorname{Arctan} x = \arctan x + k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

(d) $y = \operatorname{arccot} x$, (42-13)

i.e. *y is an arc taken within the limits*

$$0 < y < \pi, \tag{42-14}$$

whose cotangent is equal to x:

$$\cot y = x. \tag{42-15}$$

Function (42-13) is uniquely defined on the interval $-\infty < x < +\infty$; its graph is an arc of cotangensoid (Fig. 70).

If in (42-15) for each x we define all values of y whose cotangent is equal to x, then we shall get a multiple-valued function

$$y = \operatorname{Arccot} x \tag{42-16}$$

whose graph consists of an infinite number of shifted cotangensoids (42-13). We have

$$\operatorname{Arccot} x = \operatorname{arccot} x + k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

The above considered graphs should be well remembered. Making

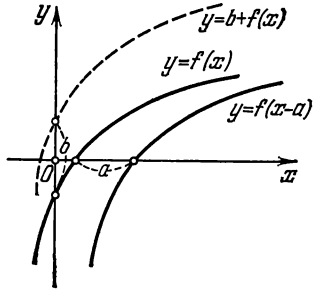

FIG. 71

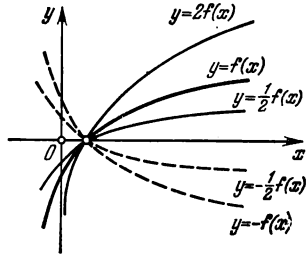

FIG. 72

use of them, we can plot numerous graphs of elementary functions regarding the latter as "*transformed basic elementary functions*".

Let the graph of the function

$$y = f(x) \tag{42-17}$$

be known (Fig. 71).

Let us consider the most important transformations of this graph.

(1) The graph

$$y = f(x - a)$$

represents the initial graph (42-17) *shifted in the direction of the x-axis by a quantity equal to a* (Fig. 71).

(2) The graph

$$y = b + f(x)$$

is obtained from (42-17) as the result of *translating the latter along the y-axis by a quantity equal to b* (Fig. 71).

(3) The graph

$$y = cf(x) \qquad (c \neq 0) \tag{42-18}$$

is obtained from the graph of the function $f(x)$ for $0 < c < 1$ by *compressing* the ordinates of the latter $\frac{1}{c}$ times, and for $1 < c <$

$< +\infty$ by *stretching* its ordinates c times with the corresponding abscissas remaining unchanged (Fig. 72).

If $-\infty < c < 0$, then graph (42-18) is a *mirror image* of the graph $y = -cf(x)$ *about the x-axis* (Fig. 72).

(4) The graph

$$y = f(kx) \qquad (k \neq 0) \tag{42-19}$$

is obtained from the graph of the function $y = f(x)$ for $0 < k < 1$ *by extending* $\frac{1}{k}$ times the abscissas of its points, and for $1 < k < +\infty$ *by reducing* k times the abscissas of its points with their ordinates remaining unchanged (Fig. 73).

If $-\infty < k < 0$, then (42-19) is a *mirror image* of the graph

$$y = f(-kx)$$

about the y-axis (Fig. 73).

The above formulated rules are geometrically obvious, and we leave their proofs to the reader.

Combining transformations (1) to (4), we get the possibility to

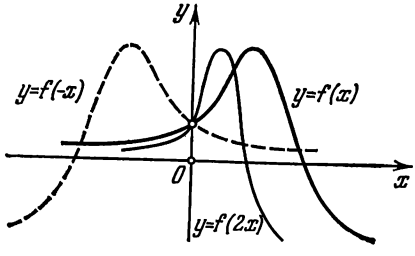

FIG. 73

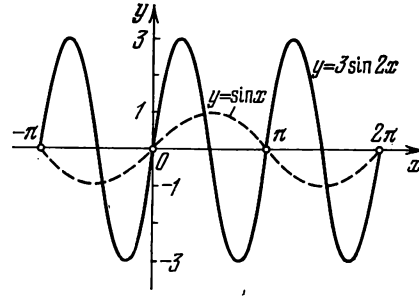

FIG. 74

construct the graphs of relatively composite functions proceeding from the graphs of simple functions.

Example. Construct the graph of the function

$$y = 3 \sin 2x.$$

By transformation rules (3) and (4), this graph represents a sinusoid $y = \sin x$ the abscissas of whose points are reduced by half and the ordinates are extended three-fold (by absolute value, preserving the sign; Fig. 74).

When constructing the graph of a function, it is important to take into account *its symmetry* and *periodicity*.

Definition 1. *A function $f(x)$ is said to be even if the change of the sign of any value of the argument does not affect the value of the func-*

tion, i.e. if

$$f(-x) = f(x).$$

Here are a few examples of even functions: $x^0 = 1$, x^2, $\cos x$, etc.

The graph of an even function $y = f(x)$ is, obviously, *symmetric with respect to the y-axis* (Fig. 75). Therefore, for an even function

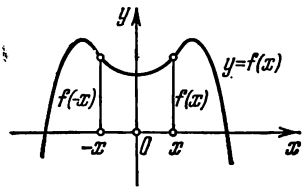

FIG. 75

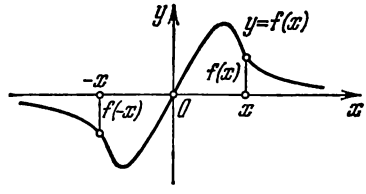

FIG. 76

it is sufficient to construct only the right-hand half of the graph ($x \geqslant 0$), since its left-hand half ($x \leqslant 0$) is a mirror image of the right-hand half with respect to the axis of ordinates.

Definition 2. *A function $f(x)$ is called odd if the change of the sign of any value of the argument results in the change of the sign of the function while its absolute value remains invariable*, i.e. if

$$f(-x) = -f(x).$$

These are odd functions: x, x^3, $\sin x$, $\tan x$, $\arcsin x$, $\arctan x$, etc.

The graph of an odd function $y = f(x)$ is, obviously, *symmetric about the origin* (Fig. 76). Therefore, to construct the graph of an

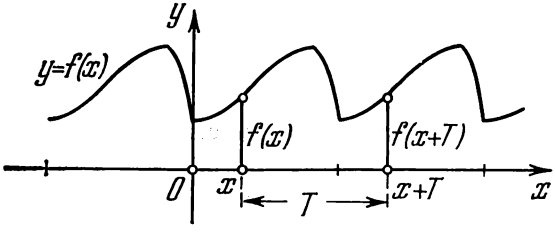

FIG. 77

odd function, it is sufficient to depict its right-hand half ($x \geqslant 0$), since its left-hand half ($x \leqslant 0$) is obtained by rotating the right-hand half through an angle of 180°.

Definition 3. *A function $f(x)$ is said to be periodic if there exists a positive number T (a period of the function)* such that

$$f(x + T) \equiv f(x)$$

(Fig. 77). Periodic functions were dealth with in Sec. 36 of the present chapter. For instance, $\sin x$ (period 2π), $\cos x$ (period 2π), $\tan x$ (period π), $\cot x$ (period π) are periodic functions.

To construct the graph of a periodic function it is sufficient to plot it on a segment whose length is equal to the period (so-called *principal domain*), and then to construct the periodic continuation of the graph by assigning equal values of ordinates to the points whose abscissas differ by a number multiple to the period.

Sec. 43. Interpolation of Functions

Consider the function $y = f(x)$ represented by the first two columns of the table

x	y	Δy	$\Delta^2 y$
.	.	.	.
x_{-3}	y_{-3}	Δy_{-3}	$\Delta^2 y_{-3}$
x_{-2}	y_{-2}	Δy_{-2}	$\Delta^2 y_{-2}$
x_{-1}	y_{-1}	Δy_{-1}	$\Delta^2 y_{-1}$
x_0	y_0	Δy_0	$\Delta^2 y_0$
x_1	y_1	Δy_1	$\Delta^2 y_1$
x_2	y_2	Δy_2	.
x_3	y_3	.	.
.	.	.	.

We shall assume that the tabular values . . ., x_{-2}, x_{-1}, x_0, x_1, x_2, . . . of the argument x are *equally spaced.* In other words, the difference

$$\Delta x_i = x_{i+1} - x_i = h \qquad (i = 0, \pm 1, \pm 2, \ldots) \tag{43-1}$$

is a constant (Δ is the symbol of difference). The constant value h is called the *pitch of the table.* For the purpose of investigation of the behaviour of the function y let us supplement our table with the *differences* Δy *of the first order* (the so-called *first differences*)

$$\Delta y_i = y_{i+1} - y_i \qquad (i = 0, \pm 1, \pm 2, \ldots). \tag{43-2}$$

If $y = y_0 + kx$ is a linear function, then its difference $\Delta y_i = kh$ is a constant value.

Analogously, we can write the differences of the second order (the second differences):

$$\Delta^2 y_i = \Delta y_{i+1} - \Delta y_i = (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i) =$$
$$= y_{i+2} - 2y_{i+1} + y_i \qquad (i = 0, \pm 1, \pm 2, \ldots) \tag{43-3}$$

and so on.

If y is a linear function, then its second differences $\Delta^2 y_i$ are equal to zero. For the quadratic function $y = a + bx + cx^2$ the second differences $\Delta^2 y_i$ are constant (check this!).

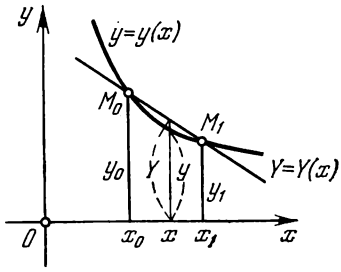

FIG. 78

Interpolation is understood as the process of finding a value of a function between two known values by a procedure other than the law which is given by the function itself.

Let the pitch h of a table be small and the differences Δy_i almost constant. Suppose that x_0 is the nearest *least* tabular value for a given non-tabular value x, i.e. $x_0 < x < x_1$. On the interval (x_0, x_1) y may be regarded approximately as a linear function Y such that $Y(x_0) = y_0$ and $Y(x_1) = y_1$. Geometrically this means that we substitute the arc $\overset{\frown}{M_0 M_1}$ of the curve by the corresponding chord $\overline{M_0 M_1}$ (Fig. 78). Since the slope of the chord $\overline{M_0 M_1}$ is equal to

$$k = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h},$$

we have

$$y \approx Y = y_0 + \frac{\Delta y_0}{h}(x - x_0) \tag{43-4}$$

(so-called *linear interpolation*).

Introducing the quantity

$$\frac{x - x_0}{h} = t \tag{43-5}$$

(expressing the distance between the points x and x_0 measured in pitches), we can rewrite approximate formula (43-4) in the following form

$$y = y_0 + t\,\Delta y_0 \tag{43-6}$$

Using (43-6), we can also carry out inverse interpolation, i.e. given a value of the function y $(y_0 \leqslant y \leqslant y_1)$, find the corresponding value of the argument x. Indeed, we have

$$t = \frac{y - y_0}{\Delta y_0}. \tag{43-7}$$

Hence, by (43-5), we get

$$x = x_0 + th. \tag{43-8}$$

Example 1. The function $y = y(x)$ is represented by the following table

x	1	1.02	1.04
y	1.21	1.44	1.69

Applying linear interpolation, find y (1.005). What is the numerical value of x if $y(x) = 1.5$?

Here the pitch $h = 0.02$. Putting $x_0 = 1$, we have

$$t = \frac{1.005 - 1}{0.020} = \frac{1}{4}.$$

Hence, by (43-6), we find

$$y = 1.21 + \frac{1}{4} \cdot 0.02 = 1.215.$$

For inverse interpolation we put $x_0 = 1.02$ and $y_0 = 1.44$. Hence, $\Delta y_0 = 1.69 - 1.44 = 0.25$. By (43-7) and (43-8), we find

$$t = \frac{1.5 - 1.44}{0.25} = \frac{0.06}{0.25} = 0.24$$

and

$$x = 1.02 + 0.24 \cdot 0.02 = 1.0248.$$

It should be noted that for $x \in (x_0, x_1)$ we obtain a similar formula for linear interpolation if instead of the nearest least tabular value x_0 its nearest *greatest* tabular value x_1 is used which is sometimes more advantageous. Namely, we have

$$y \approx y_1 + \frac{\Delta y_0}{h}(x - x_1). \qquad (43\text{-}4')$$

To get more accurate results, it is sometimes advisable to resort to *quadratic interpolation*, substituting on the interval (x_0, x_2) the function y with a quadratic trinomial $\tilde{Y}$ such that

$$\tilde{Y}(x_0) = y_0, \quad \tilde{Y}(x_1) = y_1, \quad \tilde{Y}(x_2) = y_2, \qquad (43\text{-}9)$$

where x_0 is, as before, the nearest least tabular value for a given non-tabular value x. Geometrically this means (Fig. 79) that we approximately replace the arc $\overset{\frown}{M_0 M_1 M_2}$ of the graph of the function $y = y(x)$ by a parabola with a vertical axis passing through the points M_0, M_1, M_2.

We write the function Y in the following artificial form:

$$\tilde{Y} = a + b(x - x_0) + c(x - x_0)(x - x_1). \qquad (43\text{-}10)$$

Putting $x = x_0$, by virtue of (43-9), we get

$$\tilde{Y}(x_0) = y_0 = a;$$

hence

$$a = y_0. \tag{43-11}$$

Analogously, for $x = x_1$ we shall have

$$\tilde{Y}(x_1) = y_1 = a + b(x_1 - x_0);$$

whence, using (43-11) and taking into account that $x_1 - x_0 = h$, we find

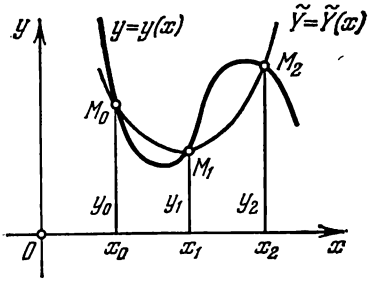

FIG. 79

$$b = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}. \tag{43-12}$$

Finally, for $x = x_2$ we have

$$\tilde{Y}(x_2) = y_2 = a + b(x_2 - x_0) + c(x_2 - x_0)(x_2 - x_1).$$

Hence, since $x_2 - x_0 = 2h$ and $x_2 - x_1 = h$, then, taking into account (43-11) and (43-12), we get

$$c = \frac{y_2 - y_0 - \frac{\Delta y_0}{h} \cdot 2h}{2h^2} = \frac{(y_2 - y_1) + (y_1 - y_0) - 2\Delta y_0}{2h^2} = \frac{\Delta y_1 - \Delta y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}. \tag{43-13}$$

Thus, we finally have

$$y \approx \tilde{Y} = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) \tag{43-14}$$

(the *formula for quadratic interpolation*).

Putting

$$\frac{x - x_0}{h} = t \tag{43-15}$$

and

$$\frac{x - x_1}{h} = \frac{(x - x_0) - (x_1 - x_0)}{h} = t - 1, \tag{43-16}$$

we get a more convenient formula for quadratic interpolation:

$$y \approx y_0 + t\,\Delta y_0 + \frac{t(t-1)}{2}\Delta^2 y_0. \tag{43-17}$$

Example 2. The function $y = y(x)$ is represented by the first two columns of the table

x	y	Δy	$\Delta^2 y$
0.20	1.2214	626	33
0.25	1.2840	659	33
0.30	1.3499	692	
0.35	1.4191		

Find y (0.27), using the formula for quadratic interpolation.

We choose $x_0 = 0.25$ as the initial value (the chosen row is underlined in the given table). The pitch of the table $h = 0.05$.

We then find the differences $\Delta y_0 = 0.0659$ and $\Delta^2 y_0 = 0.0033$ (for brevity, decimal digits are omitted in the table).

We have

$$t = \frac{0.27 - 0.25}{0.05} = 0.4.$$

Hence, by formula (43-17) we get

$$y(0.27) = 1.2840 + 0.4 \cdot 0.0659 + \frac{0.4 \cdot (-0.6)}{2} \cdot 0.0033 =$$

$$= 1.2840 + 0.0264 - 0.0004 = 1.3100.$$

Exercises

1. In a triangle ABC whose base $AC = b$ and altitude $BD = h$ a straight line EF is drawn parallel to the base AC at a distance x from it. Express the area of the trapezoid $AEFC$ as a function of x; determine the domain of this function and construct its graph.

2. Determine the domains of the following functions:

(a) $y = \sqrt{x-2}$; (c) $y = \frac{1}{\sqrt{x-x^2}}$;

(b) $y = \sqrt{x^2-1}$; (d) $y = \log(1+x)$.

3. Find $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(-x)$, $f\left(\frac{1}{x}\right)$, $f(x+1)$, if $f(x) = x^2 - 3x + 2$.

4. Knowing that $f(1) = -2.23$ and $f(2) = 1.05$, find approximately $f(1, 3)$ regarding the function as linear on the interval $[1, 2]$ (*linear interpolation*).

5. The results of measuring the quantities x and y are tabulated:

x	10	15	25
y	10	20	40

Find the relation between x and y, knowing that it is linear.

6. Find an integral rational function of the second degree

$$f(x) = ax^2 + bx + c,$$

such that

$$f(0) = -3, \quad f(1) = 0, \quad f(2) = 5.$$

7. Let $\varphi(x) = x^2$ and $\psi(x) = 2^x$. Find $\varphi[\psi(x)]$ and $\psi[\varphi(x)]$.

8. Find $[f(x)]$ and $f\{f[f(x)]\}$, if $f(x) = \frac{1}{1-x}$.

9. Find explicit inverse functions for:

(a) $y = 2x + 3$; (b) $y = \sqrt[3]{1-x^3}$; (c) $y = \sin\frac{x}{2}$.

10. Construct the graphs of the following elementary functions:

(a) $y = x^{\frac{2}{3}}$; (b) $y = \frac{1}{x-2}$; (c) $y = 1 - \left(\frac{1}{2}\right)^x$;

(d) $y = \log(x+2)$; (e) $y = 2\cos\left(x - \frac{\pi}{4}\right)$;

(f) $y = \sin^2 x$. (*Hint.* $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$);

(g) $y = 4\sin\left(3x - \frac{\pi}{3}\right)$;

(h) $y = -\cot\frac{x}{2}$;

(i) $y = \arcsin\frac{x+1}{2}$; (j) $y = \frac{1}{2} + \frac{1}{\pi}\arctan x$.

11. Approximate the real roots of the equation

$$x^3 - 3x + 1 = 0$$

by constructing the graph of the function $y = x^3 - 3x + 1$.

12. Find approximately the least positive root of the equation $\tan x = x$ by constructing the graphs of the functions $y = \tan x$ and $y = x$.

13. The function $y = y(x)$ is represented by the table:

x	0.6	0.7
y	1.8221	2.0138

Using linear interpolation, find $y(0.63)$. Evaluate x if $y(x) = 2$.

Chapter 7

The Theory of Limits

Sec. 44. Real Numbers

The *quantity* is understood in mathematics as everything that can be measured, its physical essence being all the same to us. Therefore, mathematical conclusions possess *generality*, i.e. they are applicable to all quantities in general. The process of measuring a quantity consists in comparing it with another homogeneous quantity (i.e. with a quantity of the same nature) taken for the unit of measurement. The result of measurement of a quantity is a *number* or a numerical value of the quantity measured. If a quantity being measured and the unit of measurement are *commensurable* with each other (i.e. have a common measure), then the result of measurement is a *rational number*

$$x = \frac{m}{n},$$

where m and n are integers. If a quantity under measurement and the unit of measurement are *incommensurable* with each other, then the result of measurement is an irrational number (for instance, $\sqrt{2}$, π, etc.) which can be represented in the form of an infinite aperiodic decimal fraction

$$x = p_0, p_1 p_2 \cdots p_n \cdots$$

If we take a finite number of digits after the decimal point in this fraction, then we shall get some rational numbers which will evaluate the measured quantity with any degree of accuracy. Therefore, in practical measurements we can manage with rational numbers. But when formulating general laws, it is impossible to avoid irrational numbers (for instance, the area of a circle $S = \pi R^2$, where π is an irrational number). Rational and irrational numbers are called *real* numbers*. Geometrically real numbers are represented by

* Henceforward, the word "number" should be understood as "a real number" unless otherwise is stated.

means of a number axis* Ox (Fig. 80). Numbers are placed on this axis according to a chosen scale: rational (integers 0, ± 1, ± 2, . . . and fractions $\pm \frac{1}{2}$, $\pm \frac{2}{3}$, . . . and so on) and irrational. As a result, all real numbers find their places on the number axis, occupying the latter without gaps, i.e. *to each real number there*

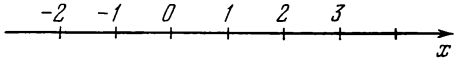

FIG. 80

corresponds a definite point on the number axis and, conversely, every point on the number axis represents a real number. Therefore, instead of "a real number" we often say "a point".

The set of all real numbers x is usually completed with two symbols $-\infty$ and $+\infty$ with the following properties

$$-\infty < x < +\infty.$$

Such system of real numbers is called *extended.* It is assumed that the following arithmetic is true:

(a) $x \pm \infty = \pm \infty$;

(b) $\frac{x}{\pm \infty} = 0$;

(c) $x \cdot (\pm \infty) = \pm \infty$, if $x > 0$

and

$$x \cdot (\pm \infty) = \mp \infty, \quad \text{if} \quad x < 0.$$

Real numbers may be positive and negative. In some cases we have to consider the *absolute value* of a real number ignoring its sign.

Definition. *The absolute value (or the modulus) of a real number is the arithmetic value of this number.*

The absolute value of the number a is denoted as $|a|$. For instance,

$$|-5| = 5, \quad |+3| = 3.$$

In general, if x is a real number, then

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Obviously, for any number x there exists the equality

$$|-x| = |x|.$$

If all real numbers are arranged on the number axis, then the absolute value $|x|$ of any number x represents the distance of the

* Synonyms: "number line", "number scale".

corresponding point A with the abscissa x from the origin O: $|x| = OA$ (Fig. 81).

Hence, it follows that if the absolute value of the number x satisfies the inequality

$$|x| < a \quad (\text{or } |x| \leqslant a), \tag{44-1}$$

then the number x obeys the following limitation:

$$-a < x < a \quad (\text{or, respectively, } -a \leqslant x \leqslant a), \tag{44-2}$$

i.e. x belongs to the interval $(-a, a)$ (or to the segment $[-a, a]$).

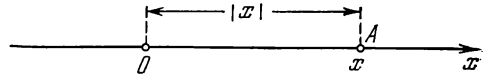

FIG. 81

In particular, for any number x the following inequality holds

$$-|x| \leqslant x \leqslant |x|.$$

Conversely, if there exists one of the dual inequalities (44-2), then the corresponding inequality (44-1) is fulfilled.

A more general statement: if

$$|x - x_0| < a \quad (\text{or } |x - x_0| \leqslant a),$$

then, taking into account that $|x - x_0|$ is equal to the distance between the points x and x_0, we have

$$x_0 - a < x < x_0 + a \quad (\text{or } x_0 - a \leqslant x \leqslant x_0 + a)$$

and conversely (Fig. 82).

The absolute value of a real number possesses the following properties:

(1) *The absolute value of a sum of two or several numbers is less than, or equal to, the sum of absolute values of these numbers.*

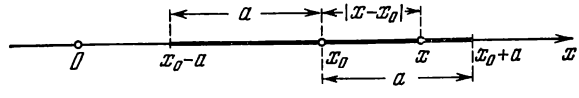

FIG.82

Indeed, first suppose that x and y are real numbers having the same sign, i.e. $xy \geqslant 0$. Obviously, we have

$$|x + y| = |\pm|x| \pm |y|| = |\pm(|x| + |y|)| = |x| + |y|$$

(for instance, $|-3 - 5| = |-(3 + 5)| = 3 + 5$).

Let now x and y be real numbers of opposite signs, i.e. $xy < 0$ and let us assume for definiteness that $|x| \geqslant |y|$. Then we have

$$|x + y| = |\pm|x| \mp |y|| = |\pm(|x| - |y|)| = |x| - |y| < |x| + |y|$$

(for instance, $|-5 + 2| = |-(5 - 2)| = 5 - 2 < 5 + 2$).

Thus, for any real numbers x and y the following inequality holds

$$| x + y | \leqslant | x | + | y |, \tag{44-3}$$

the equality sign taking place if and only if the numbers x and y have the same sign.

Note. Inequality (44-3) is readily spread over any finite number of terms, for instance

$$| x + y + z | = | (x + y) + z | \leqslant | x + y | + | z | \leqslant$$
$$\leqslant | x | + | y | + | z |.$$

(2) *The absolute value of a difference of two numbers is greater than, or equal to, the difference of the absolute values of the minuend and the subtrahend.*

Indeed, by virtue of property (1), we have

$$| x | = | y + (x - y) | \leqslant | y | + | x - y |.$$

Hence

$$| x - y | \geqslant | x | - | y |.$$

(3) *The absolute value of a product of two or several numbers is equal to the product of the absolute values of the factors.* For instance

$$| xy | = | x | | y |.$$

(4) *The absolute value of a quotient is equal to the quotient of the absolute values of the dividend and the divisor (if the denominator is non-zero)*, i.e. if $y \neq 0$, then

$$\left| \frac{x}{y} \right| = \frac{|x|}{|y|}.$$

(5) *The absolute value of an integral positive or an integral negative power is equal to the same power of the absolute value of the base*, i.e.

$$| x^n | = | x |^n.$$

We leave the proof of almost obvious assertions (3) to (5) to the reader.

Sec. 45. Errors of Approximate Numbers

When measuring a quantity with a precise value a, we usually obtain only its *approximate value* x. The difference $a - x$ is called the *error* of the approximate number x. We shall call the number a *precise* and the number x *approximate*. If $x \leqslant a$, then x is a *minor approximation*, and if $x \geqslant a$, then x is a *major approximation*.

Definition 1. *The absolute error* Δ_0 *of an approximate number* x *is the absolute value of the difference between the appropriate precise*

number a and the given approximate number x, that is

$$\Delta_0 = |a - x|. \tag{45-1}$$

If the precise number a is unknown, then formula (45-1) does not allow us to determine the absolute error Δ_0 of the approximate number x. In this case we confine ourselves to the *upper bound* of the absolute error Δ_0, i.e. we find a positive number Δ as close to Δ_0 as possible, such that

$$\Delta_0 \leqslant \Delta. \tag{45-2}$$

The number Δ satisfying inequality (45-2) is called the *limiting absolute error* of the approximate number x. Obviously, we have

$$x - \Delta \leqslant a \leqslant x + \Delta; \tag{45-3}$$

instead of inequality (45-3), use is also made of the following brief notation

$$a = x \pm \Delta. \tag{45-3'}$$

We often know two approximate numbers x_1 and x_2 between which a precise number a is contained:

$$x_1 \leqslant a \leqslant x_2.$$

Then we may put

$$a = x \pm \Delta,$$

where

$$x = \frac{1}{2}(x_2 + x_1) \quad \text{and} \quad \Delta = \frac{1}{2}(x_2 - x_1).$$

The absolute error itself, without taking into consideration the nature of the measured quantity, does not characterize the accuracy of measurement. For instance, if the length of a table $a_1 = 2$ m and the length of a railway $a_2 = 200$ km are measured with one and the same absolute error $\Delta_1 = \Delta_2 = 0.1$ m, then it does not mean that the measurements are of equal accuracy. Obviously, the second measurement is more accurate than the first one. For estimating the accuracy of measurements we introduce the concept of *relative error*.

Definition 2. *The relative error δ_0 of an approximate number x is the ratio of the absolute error Δ_0 of this number to the absolute value of the corresponding precise number a,* i.e.

$$\delta_0 = \frac{\Delta_0}{|a|}. \tag{45-4}$$

Hence

$$\Delta_0 = |a|\,\delta_0, \tag{45-5}$$

i.e. *the absolute error of an approximate number is equal to its relative error multiplied by the absolute value of the corresponding precise number.*

If the precise number a is unknown or too complicated, then we give the *upper estimate* of the number δ_0. The number δ satisfying the inequality

$$\delta_0 \leqslant \delta,$$

is called the *limiting relative error* of the approximate number x. Obviously, if $x > 0$, then we may put

$$\delta = \frac{\Delta}{x - \Delta},$$

where Δ is the limiting absolute error of the number x such that $x - \Delta > 0$.

Example 1. What is the limiting relative error δ of the number $x = 3.14$ substituting the number π?

Since $3.14 < \pi < 3.142$, the absolute error Δ_0 of the number x satisfies the inequality $\Delta_0 < 0.002$. Hence

$$\delta_0 = \frac{\Delta_0}{\pi} < \frac{0.002}{3.14} < 6.4 \cdot 10^{-4}.$$

Consequently, we may take

$$\delta = 0.064\%.$$

Since in many cases it is difficult to find the precise number a, in practice we put $a \approx x$, where x is an approximate number sufficiently close to a, and use the following approximation formulas

$$\delta_0 \approx \frac{\Delta_0}{|x|} \qquad (45\text{-}4')$$

and

$$\Delta_0 \approx |x| \delta_0 \qquad (45\text{-}5')$$

($\approx$ is the sign of approximate equality). The corresponding formulas are also valid for the limiting errors.

Example 2. The result of a measurement to within 0.5% is $x = 25.7$ m.

Determine the limiting absolute error Δ of this measurement.

From formula (45-5′) we have

$$\Delta \approx 25.7 \cdot \frac{1}{2} \cdot 0.01 \approx 0.13 \text{ m}.$$

Hence, we may put the measured value a to be equal to

$$a = 25.7 \text{ m} \pm 0.13 \text{ m}.$$

Now we are going to introduce certain notions connected with representation of numbers in the decimal number system, confining ourselves to consideration of positive numbers*. In the decimal representation of a number any non-zero digit and zero if it does not serve to designate the decimal digit or does not substitute the unknown or rejected digit is called a *significant digit* of this number. For instance, the number 0.0507 has three significant digits: 5, 0, and 7. The number written as 27,600 does not allow us to judge on the number of its significant digits. If, for instance, this number has four significant digits, then it should be written in the form $2.760 \cdot 10^4$. Significant digits of an approximate number are subdivided into *correct* and *incorrect*.

Definition 3. *An approximate number is said to have n correct significant digits (reckoning from left to right) if the absolute error of this number does not exceed half the unity of its nth digit.*

For instance, if the number $x = 2.356$ has three correct digits 2, 3, 5, then the absolute error of this number is

$$\Delta_0 \leqslant \frac{1}{2} \cdot 0.01 = 0.005.$$

Mathematical tables are compiled in such a manner that all the digits placed in them are correct. For instance, for a four-digit table of logarithms it is guaranteed that the absolute error of the mantissa of each number $\Delta_0 \leqslant \frac{1}{2} \cdot 10^{-4}$.

In some cases the absolute error of an approximate number may reach the *unity* of its *n*th digit. Then the given number is said to have *n correct digits in a broad sense*. If the absolute error of an approximate number may reach two unities of its *n*th digit, then we say that the first $n - 1$ significant digits of the number are *correct*, its *n*th digit being *doubtful*.

The concept of correct digits not always can be understood literally, i.e. in the sense that if an approximate number has *n* correct digits, then the first *n* digits of the approximate number coincide with the first *n* digits of the precise number. For instance, if $a = 1$ is a precise number and $x = 0.999$ an approximate number, then all the digits of the latter are, obviously, correct in a broad sense though none of the digits of the precise number coincide with the corresponding digit of the given approximate number. But in the most cases the literal understanding will be correct.

The number of correct digits of an approximate number characterizes the accuracy of measurement and makes it possible to find the limiting relative error of this number.

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* The effect of sign can be given special consideration.

Example 3. The approximate number

$$x = 8.3047$$

has two correct digits. What is the limiting relative error δ of this number?

Here the absolute error

$$\Delta_0 \leqslant \frac{1}{2} \cdot 0.1 = 0.05.$$

By formula (45-4′), we estimate the relative error

$$\delta_0 \approx \frac{0.05}{8.3047} = 0.006.$$

Consequently, we may approximately put

$$\delta = 0.6\%.$$

Conversely, knowing the limiting relative error of an approximate number, we can determine the number of its correct digits.

Example 4. The limiting relative error of the approximate number

$$x = 623.809$$

is $\delta = 0.2\%$. How many correct digits has this number?

Applying formula (45-5′), we find the estimate of the absolute error of our approximate number

$$\Delta_0 \approx 0.2 \cdot 0.01 \cdot 623.809 \approx 1.2.$$

Not quite rigorously we may consider that the number x has three correct digits in a broad sense.

Generally speaking, there is no sense to preserve incorrect digits in the final notation of an approximate number. In the last resort we may retain one spare digit. Therefore the incorrect digits of an approximate number are usually rejected or, as we say, this approximate number is *rounded off*. Cumbersome exact numbers are also frequently rounded off.

Rule for rounding off numbers: (1) *If the first of the discarded digits of a number is less than* 5, *then its remaining digits are kept unchanged*; (2) *if the first of the digits dropped exceeds, or is equal to*, 5, *then the first digit kept is increased by unity.*

For instance, rounding off the number $\pi = 3.141592...$ to five, four, three significant digits, we get the following approximate numbers 3.1416, 3.142, and 3.14, respectively.

We should like to mention here the following particular case: If the digit 5 is dropped, and no significant digits come after it, the rounding is done to the closest even number; that is, *the last*

retained digit is left unchanged if it is even and is increased if it is odd (the even-digit rule).

When rounding off an approximate number, we, generally speaking, increase its error by adding the *rounding-off error* to its absolute error.

When the rules for rounding off are applied, the rounding-off error, obviously, does not exceed half the unity of the last decimal digit retained.

Hence it follows that:

(1) *if an exact number is rounded off to n significant digits, then the approximate number thus obtained will have n correct decimal digits;*

(2) *if an approximate number with n correct decimal digits is rounded off to n significant digits, then the new approximate number thus obtained will have n correct decimal digits in a broad sense.*

Sec. 46. Limit of a Function

In mathematical analysis we deal, as a rule, with *dimensionless* quantities, i.e. with quantities devoid of physical contents. Collections of values of such quantities represent certain number sets. Proceeding from this fact and using the logical symbols $\forall$ ("for any") and $\exists$ ("there exists"), we can formalize the definition of function set forth in Sec. 35, Chapter 6.

Definition 1. *Let X and Y be given sets of numbers. If by virtue of some correspondence f associating the elements of the set Y with those of the set X, $\forall x \in X$ $\exists y \in Y$ (unique), then y is called a single-valued function of x defined on the set X.*

This fact is briefly denoted in the following manner:

$$y = f(x) \qquad (x \in X)^*. \tag{46-1}$$

By definition, the *set of values of function* (46-1) is contained in Y, i.e. $\{f(x)\} \subset Y$.

We may say that the function f *maps* the set X into the set Y (Fig. 83).

If $\{f(x)\} = Y$, i.e. any element $y \in Y$ is a value of the function f, then the function f is said to *map* the set X onto the set Y.

Example. The function $f(x) = \sin x$ $(0 < x < 2\pi)$ maps the interval $X = (0, 2\pi)$ onto the segment $Y = [-1, 1]$.

Let the function $y = f(x)$ establish *one-to-one correspondence* between the elements of the sets X and Y, i.e. $\forall x \in X$ there exists

* Strictly speaking, function (46-1) should be understood as the correspondence f itself by virtue of which for any $x \in X$ we try to find its partner $y \in Y$. In addition, $f(x)$ represents the *value of the function* f at point x. But in practice, the symbol $f(x)$, where x attains all possible values is also called the *function*.

one and only one its *image* $y = f(x) \in Y$ and, conversely, $\forall y \in Y$ there can be found a unique *preimage* $x \in X$ such that $f(x) = y$. Then the function $x = f^{-1}(y)$ $(y \in Y)$ establishing the correspondence between the elements of the sets Y and X is called *inverse* to the function $y = f(x)$. In other words, the inverse function f^{-1} is a mapping of the set Y onto the set X. Obviously, $y = f(x)$ and $x = f^{-1}(y)$ are *mutually inverse* functions.

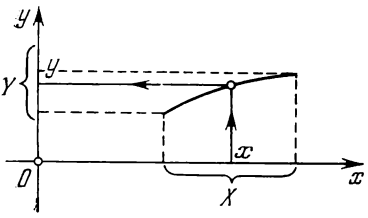


FIG. 83

Definition 2. *The neighbourhood U_a of a point a (a is a real number) will be understood as any interval $\alpha < x < \beta$ surrounding this point ($\alpha < a < \beta$) with the point a removed* (Fig. 84).

The *neighbourhood* U_∞ of the symbol $\infty \equiv \pm\infty$ will be understood as the exterior of any segment $[\alpha, \beta]$ (Fig. 85), i.e. $U_\infty = (-\infty, \alpha) \cup (\beta, +\infty)$. Naturally, the symbol ∞ is not contained in its neighbourhood.

Remark. It is generally accepted to understand the neighbourhood of a point a as any interval $I_a = (\alpha, \beta)$ containing the point a, i.e. if I_a is the neighbourhood of a point a, then $I_a \ni a$. In our

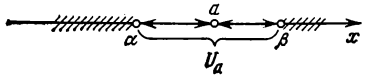


FIG. 84

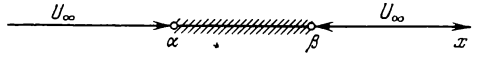


FIG. 85

definition of the neighbourhood U_a of a point a, for the sake of convenience of our further reasoning, we exclude the point a itself from it, i.e. we assume $a \notin U_a$. Such a set of points is usually called a *deleted* (or *punctured*) neighbourhood of the point a.

Taking some liberties in formulating our definitions, we call the set of points $U_a = (\alpha, a) \cup (\alpha, \beta)$ simply the *neighbourhood of point a* (in our sense). Such definition of the neighbourhood conforms to its ordinary understanding. For instance, it is only natural to assume that the neighbourhood of the city of Moscow does not contain the city itself! Moreover, in defining *one-sided neighbourhoods* of a point a: $U_a^- = (\alpha, a)$ (*left-hand neighbourhood*) and $U_a^+ = (a, \beta)$ (*right-hand neighbourhood*) the point a is always excluded! (see Sec. 47 of the present chapter).

Thus, henceforth the neighbourhood of a point a will be understood as any interval surrounding this point with the point a removed, unless otherwise stated.

In those cases when it will be convenient to consider that the neighbourhood of a point a contains the point a itself we shall call it a "complete neighbourhood of a point a".

As it is easy to be convinced: (1) the sum (or union) of any number of neighbourhoods of a point a and (2) the product (or intersection) of a finite number of neighbourhoods of a point a are also neighbourhoods of this point.

For a positive number δ a neighbourhood U_a of a finite point a will be called a δ-*neighbourhood* if $U_a = (a - \delta, a) \cup (a, a + \delta)$, i.e. if $\forall x$

$$0 < |x - a| < \delta$$

(see Fig. 86).

Let a function $f(x)$ be defined on a set X. A finite point a is called the *limiting point* (or the *point of accumulation*) of this set if *any* of its δ-neighbourhood U_a contains an infinite number of elements

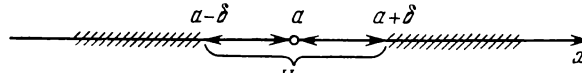


FIG. 86

$x \in X$, i.e. $U_a \cap X \neq \phi \forall U_a$. In the simplest case we may assume that the function $f(x)$ is defined in some neighbourhood of the point a, at the point a itself the function $f(x)$ not necessarily having a sense.

And so, let a be the limiting point of the set X which is the domain of definition of the function $f(x)$.

Definition 3. *The number A is called the limit of the function $f(x)$ as $x \to a$ (a is a number), i.e.*

$$\lim_{x \to a} f(x) = A,$$

if for any $\varepsilon > 0$ there exists a δ-neighbourhood $U_a = \{x \mid 0 < |x - a| < \delta\}$, $\delta = \delta(\varepsilon)$ depends on ε, such that

$$|f(x) - A| < \varepsilon \quad \text{for} \quad x \in U_a^*. \tag{46-2}$$

Of course, inequality (46-2) must be fulfilled for all those x for which the function $f(x)$ is defined, i.e. for $x \in X \cap U_a$. According to the definition of the limiting point in each neighbourhood U_a the set of such values is not empty.

Note 1. According to the sense of the definition of the limit of a function, the numbers ε and $\delta = \delta(\varepsilon)$ may be assumed sufficiently

* For simplicity, we use here the δ-neighbourhood of the point a, i.e. its *symmetrical* neighbourhood. But the definition remains valid for any neighbourhood $\tilde{U}_a = (\alpha, a) \cup (a, \beta)$, since it, obviously, contains the δ-neighbourhood of the point a, where $\delta = \min(a - \alpha, \beta - a) > 0$.

small.

Definition 4. *The assertion*

$$\lim_{x\to\infty} f(x) = A$$

is equivalent to the following:

$$|f(x) - A| < \varepsilon \quad \text{for} \quad |x| > \Delta, \tag{46-3}$$

where $\Delta = \Delta(\varepsilon)$ *depends on* ε.

The set of all points x for which $|x| > \Delta$ is, obviously, a *symmetrical* neighbourhood U_∞ of the symbol ∞, assuming here that for any such neighbourhood $U_\infty \cap X \neq \phi$. We may say conventionally that ∞ is the limiting point of the set X which is the domain of definition of the function $f(x)$.

Combining definitions 2 and 3, we get a general definition of the limit of a function, as $x \to a$, which is suitable both for a finite a and for $a = \infty$.

General definition of the limit of a function. *Let* $f(x)$ *be a function defined on a set* X, *and* a *the limiting point of this set. The number* A *is the limit of the function* $f(x)$, *as* $x \to a$, *if and only if for any* $\varepsilon > 0$ *there exists a neighbourhood* U_a *of the point* a* *such that*

$$|f(x) - A| < \varepsilon \qquad \forall x \in U_a \cap X \tag{46-4}$$

(where $U_a \cap X \neq \varnothing$).

This fact has the following brief notation:

$$\lim_{x\to a} f(x) = A, \tag{46-5}$$

or

$$f(x) \to A \quad \text{for} \quad x \to a. \tag{46-5'}$$

Example 1. Show that

$$\lim_{x\to 2} x^2 = 4. \tag{46-6}$$

For convenience of our reasoning we shall assume that $1 < x < 3$, i.e. $|x - 2| < 1$.

Let $\varepsilon > 0$ be an arbitrary number. We have

$$|x^2 - 4| = |x - 2|\,|x + 2| =$$

$$= |x - 2|\,(x + 2) < 5\,|x - 2| < \varepsilon,$$

if

$$|x-2| < \frac{\varepsilon}{5} \quad \text{and} \quad |x-2| < 1.$$

---

* Not necessarily a symmetrical one.

Hence, we may put

$$\delta = \min\left(\frac{\varepsilon}{5}, 1\right) < 0.$$

Thus, equality (46-6) has been proved. Note that δ-neighbourhood of the point $x = 2$ is complete, i.e. contains the point 2.

Example 2. Show that

$$\frac{x}{x+1} \to 1 \quad \text{for} \quad x \to \infty. \tag{46-7}$$

We have

$$\left|\frac{x}{x+1} - 1\right| = \frac{1}{|x+1|} = \frac{1}{|x-(-1)|} \leqslant \frac{1}{|x|-|-1|} = \frac{1}{|x|-1} < \varepsilon,$$

if only

$$|x| > 1 + \frac{1}{\varepsilon} = \Delta,$$

which is equivalent to the assertion (46-7).

Note 2. It should not be understood that the function $f(x)$ constantly remains less than its limit.

Three cases are possible here:

(1) the function does not exceed its limit, for instance, $\frac{x^2}{x^2+1} \to 1$, as $x \to \infty$, and $\frac{x^2}{x^2+1} < 1$;

(2) the function is not less than its limit, for instance, $x^2 \to 0$ as $x \to 0$, and $x^2 > 0$ for $x \neq 0$;

(3) the function fluctuates about its limit attaining values either less or greater than the limit; for instance,

$$2 + \frac{\sin x}{x} \to 2 \quad \text{for} \quad x \to \infty.$$

Note 3. When considering the limit of the function $f(x)$ for $x \to a$, we could assume, for simplicity, that the function $f(x)$ is *defined in some neighbourhood of the point a.*

But the simplest examples show that this is inconvenient for applications.

Example 3. Let

$$f(x) = \frac{\sqrt{\sin x}}{x^2} \qquad (x > 0).$$

This function is defined on the set

$$X = (0, \pi] \cup [2\pi, 3\pi] \cup \ldots,$$

which is not a neighbourhood of the infinite point ∞. Nevertheless, from our point of view, we have

$$\lim_{x \to \infty} f(x) = 0.$$

Let us note one simple proposition.

Theorem 1. *If the function $f(x) = c$ is constant within some neighbourhood of a point a, then*

$$\lim_{x \to a} f(x) = c,$$

and c is the only limit of this function as $x \to a$.

We leave the proof of this theorem to the reader.

A function having a limit should not be mixed up with a *bounded function.*

Definition 5. *The function $f(x)$ is called bounded on a given set X if there exists a positive number M such that*

$$|f(x)| \leqslant M \quad \text{for} \quad x \in X.$$

If there is no such number M, the function $f(x)$ is called *unbounded.*

Lemma. *The function $f(x)$ having a limit A as $x \to a$ is bounded in some neighbourhood of the point a.*

Indeed, choosing $\varepsilon = 1$, we have $|f(x) - A| < 1$ for $x \in U_a$, where U_a is the appropriate neighbourhood of the point a. Hence, for all permissible values of the argument x* we get

$$|f(x)| = |[f(x) - A] + A| \leqslant |f(x) - A| + |A| < 1 + |A| = M,$$

if only $x \in U_a$.

Note 4. The converse statement is not true: a bounded function may have no limit.

For instance, the function $f(x) = \sin \frac{1}{x}$ is bounded for $0 < |x| < +\infty$ and has no limit as $x \to 0$.

Let us set forth one more theorem which establishes the relation between the boundaries of a function and its limit.

Theorem 2. *Let there exist*

$$\lim_{x \to a} f(x) = A$$

and

$$M < f(x) < N \tag{46-8}$$

in some neighbourhood U_a of point a. Then

$$M \leqslant A \leqslant N. \tag{46-9}$$

* That is such $x \in U_a$ for which the function $f(x)$ has sense.

Proof. Indeed, let $A < M$. Putting $\varepsilon = M - A > 0$, in some neighbourhood V_a of the point a we shall have

$$|f(x) - A| < M - A, \text{ i.e. } -(M - A) < f(x) - A < M - A.$$

Hence, choosing $x \in V_a \cap U_a$, we get $f(x) < M$ which contradicts the left-hand inequality of (46-8).

The supposition $A > N$ is disproved in a similar manner. Thus, inequalities (46-9) have been proved.

Note 5. Theorem 2 remains true if one or both inequalities of (46-8) are not strict.

Corollary. *A positive function may have no negative limit.*

Note 6. The notion of the limit of a function of one variable is naturally spread over to the function of several variables.

Consider, for instance, the function of two variables $f(x, y)$ specified on some set X of the plane Oxy.

The *neighbourhood* $U_{a, b}$ of a point $M_0(a, b)$ (a and b are finite) will be understood as the interior of any rectangle $\{\alpha_1 < x < \beta_1, \alpha_2 < y < \beta_2\}$ constructed round the point M_0 (i.e. $\alpha_1 < a < \beta_1$, $\alpha_2 < b < \beta_2$) with the point M_0 removed.

In this case the statement

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = A$$

means that $\forall \varepsilon > 0\ \exists U_{a, b}$ such that $\forall M(x, y) \in U_{a, b}$ the following inequality is valid

$$|f(x, y) - A| < \varepsilon. \tag{46-10}$$

Of course, we assume here that in any neighbourhood $U_{a, b}$ there will be found points $M(x, y)$ for which the function $f(x, y)$ has sense (the *limiting point*).

This definition is readily generalized for the case when a or b, or both of them are ∞.

Sec. 47. One-Sided Limits of a Function

Now we are going to consider the so-called *one-sided* (or unilateral) *limits of a function.*

Let us introduce the concept of a *left-hand* and a *right-hand* neighbourhoods of a point a (a is a number).

Definition 1. (1) *Any interval* $U_a^- = (\alpha, a)$ *whose right-hand end is a point a is termed its left-hand neighbourhood.*

(2) *Analogously, any interval* $U_a^+ = (a, \beta)$ *whose left-hand end is a point* a *is termed its right-hand neighbourhood.*

The notation $x \to a - 0$ means that x takes on only the values belonging to a left-hand neighbourhood of a point a, i.e. $x \to a$, $x < a$.

Analogously, the notation $x \to a + 0$ means that $x \to a$, $x > a$.

Definition 2. (1) *The formula*

$$\lim_{x \to a-0} f(x) = A,$$

where the function $f(x)$ *is defined on a set* X *and* a *is the limiting point of this set* (a *is finite*), *and* A *is a number, denotes that* $\forall \varepsilon > 0\ \exists U_a^-$ *such that*

$$|f(x) - A| < \varepsilon \quad \text{for} \quad x \in X \cap U_a^{-*} \tag{47-1}$$

(*a left-hand limit of the function, or the limit on the left*).

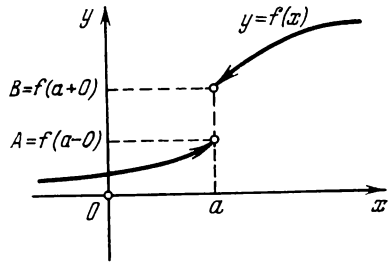


FIG. 87

(2) *Analogously, the formula*

$$\lim_{x \to a+0} f(x) = B$$

(*B is a number*) *has the following sense*:

$$|f(x) - B| < \varepsilon \quad \text{for} \quad x \in X \cap U_a^{+**} \tag{47-2}$$

where $\varepsilon > 0$ *is arbitrary and* U_a^+ *depends on* ε (*the right-hand limit of the function, or the limit on the right*).

For the numbers A and B the following notation is used (Fig. 87):

$$A = f(a - 0)$$

and

$$B = f(a + 0).$$

If the function $f(x)$ is defined at the point a, then its value at this point is designated by $f(a)$. Of course, it may not coincide with the numbers $f(a - 0)$ and $f(a + 0)$.

Definition 3. *A neighbourhood of the symbol* $-\infty$ *is understood as any interval* $(-\infty, \alpha)$, *and a neighbourhood of the symbol* $+\infty$ *as any interval* $(\beta, +\infty)$.

The formulas

$$\lim_{x \to -\infty} f(x) = A' \quad \text{and} \quad \lim_{x \to +\infty} f(x) = B' \tag{47-3}$$

---

* We may, of course, confine ourselves to considering left-hand δ-neighbourhoods of the point a: $U_a^- = (a - \delta, a)$, where $\delta = \delta(\varepsilon) > 0$.

** It is usually assumed that $U_a^+ = (a, a + \delta)$, where $\delta = \delta(\varepsilon) > 0$.

are deciphered in the following way:

$$|f(x) - A'| < \varepsilon \quad \text{for} \quad x \in (-\infty, \Delta_1),$$

and

$$|f(x) - B'| < \varepsilon \quad \text{for} \quad x \in (\Delta_2, +\infty),$$

where ε is arbitrary, $x \in X$, and

$$\Delta_i = \Delta_i(\varepsilon) \qquad (i = 1, 2).$$

Example. Let $f(x) = \operatorname{sgn} x = \frac{x}{|x|}$ $(x \neq 0)$ (Fig.88). We have

$$\lim_{x \to -0} f(x) = -1, \quad \lim_{x \to +0} f(1) = 1^*.$$

Note. For a limit of the function $f(x)$ to exist, as $x \to a$ (a is a

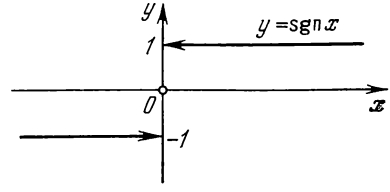


FIG. 88

number), it is necessary and sufficient that the equality

$$f(a - 0) = f(a + 0)$$

is fulfilled.

Sec. 48. Limit of a Sequence

The *sequence*

$$x_1, x_2, \ldots, x_n, \ldots \tag{48-1}$$

is understood as a function $x_n = f(n)$ defined on the set of natural numbers $X = \{1, 2, \ldots\}$.

By analogy with the limit of a function at a point at infinity, we introduce here the concept of the limit of a sequence. Namely, a number a is the *limit of the sequence* x_n $(n = 1, 2, \ldots)$

$$\lim_{n \to \infty} x_n = a^{**}$$

---

* For brevity, we use here the following notations: $-0 \equiv 0 - 0$ and $+0 \equiv 0 + 0$.

** Strictly speaking, we must write $n \to +\infty$. But since n is natural, $n \to \infty$ and $n \to +\infty$ have one and the same meaning.

if for any $\varepsilon > 0$ there exists a number N, depending on ε, such that for all natural $n > N$ the following inequality is fulfilled

$$| x_n - a | < \varepsilon.$$

Example. Let

$$x_1 = 0.9, \quad x_2 = 0.99, \quad x_3 = 0.999, \ldots$$

We have

$$x_n = 1 - \frac{1}{10^n} \qquad (n = 1, 2, \ldots).$$

Let $\varepsilon > 0$ be an arbitrary number. Then

$$|x_n - 1| = \frac{1}{10^n} < \varepsilon,$$

if $n > \log \frac{1}{\varepsilon} = N$. Hence,

$$\lim_{n \to \infty} x_n = 1.$$

Sec. 49. Infinitesimals

Definition. *A function $\alpha(x)$ is called an infinitesimal as $x \to a$ (a is a real number or the symbol ∞) if for any $\varepsilon > 0$ there exists a neighbourhood U_a of the point a such that*

$$| \alpha(x) | < \varepsilon \quad \text{for} \quad x \in U_a^*. \tag{49-1}$$

Condition (49-1) is equivalent to the following:

$$\lim_{x \to a} \alpha(x) = 0, \tag{49-2}$$

i.e. the *limit of the infinitesimal $\alpha(x)$ is equal to zero and conversely*. In other words:

$$\alpha(x) \to 0 \quad \text{as} \quad x \to a. \tag{49-3}$$

We analogously define the infinitesimal function as $x \to a - 0$ and $x \to a + 0$, and also as $x \to -\infty$ or $x \to +\infty$.

Note. If

$$\lim_{x \to a} f(x) = A, \tag{49-4}$$

then, by virtue of the definition of the limit of a function, we get that the difference $f(x) - A$ is an infinitesimal. Thus, from formula (49-4) we obtain a representation of the function $f(x)$ having the

* As usual, inequality (49-1) must be fulfilled for those x for which the function $\alpha(x)$ is defined. It is assumed here that the set of such values is non-empty in any neighbourhood U_a of the point a.

limit A as $x \to a$ in the form

$$f(x) = A + \alpha(x), \tag{49-5}$$

where $\alpha(x) \to 0$ as $x \to a$.

Conversely, if (49-5) is valid for the function $f(x)$, then the number A is the limit of the function as $x \to a$. From formula (49-5) there follows an important lemma on preserving the sign of a function.

Lemma. *If* $\lim\limits_{x \to a} f(x) = A \neq 0$, *then in some neighbourhood* U_a *of a point* a *the sign of the function* $f(x)$ $(x \in X)$ *coincides with the sign of the number* A.

Indeed, let $\varepsilon = |A| > 0$. Choosing a neighbourhood U_a so that $|\alpha(x)| < |A|$ for $x \in U_a$, by (49-5), we shall have

$$\operatorname{sgn} f(x) = \operatorname{sgn} A^*,$$

where $x \in U_a \cap X$.

Example. A point M moves along the x-axis, the law of its motion being

$$x = 2^{-t}\sin t \qquad (t, \text{ time}).$$

Obviously,

$$|x| \leqslant 2^{-t} < \varepsilon,$$

if $t > \log_2 \frac{1}{\varepsilon} = T$. Therefore,

$$\lim_{t \to +\infty} x = 0.$$

Hence, the point M performs damped oscillations about the origin.

Note. In some neighbourhood U_a the function $f(x) \equiv 0$ is, by definition (49-1), an *infinitesimal* as $x \to a$.

It should be noted that no constant function $f(x) = c \neq 0$, where the number c is arbitrarily small by its absolute value, may be called an infinitesimal. Therefore, so-called infinitely small physical quantities (for instance, the mass of a molecule, size of an atom, charge of an electron, and so on) are not infinitesimals from mathematical point of view.

Sec. 50. Infinitely Large Quantities

Definition. *The function* $f(x)$ *is called an infinitely large quantity as* $x \to a$ *(where* a *is a number or the symbol* ∞*)*

$$f(x) \to \infty \quad \text{as} \quad x \to a, \tag{50-1}$$

* The notation sgn x is read "sign x". The function sgn x is defined in the following way: sgn $x = +1$ if $x > 0$; sgn $0 = 0$; sgn $x = -1$ if $x < 0$ (cf. Fig. 88).

if for any $E > 0$ *there exists a neighbourhood* U_a *of the point* a *such that*

$$|f(x)| > E \quad \text{for} \quad x \in U_a \tag{50-2}$$

for all permissible values of the argument x.

If the function $f(x)$ is infinitely large as $x \to a$, then we conventionally write

$$\lim_{x \to a} f(x) = \infty. \tag{50-3}$$

Example. $\tan x \to \infty$ as $x \to \frac{\pi}{2}$.

The notations

$$\lim_{x \to a} f(x) = -\infty, \quad \lim_{x \to a} f(x) = +\infty$$

respectively mean: $f(x) < -E$ for $x \in U_a$ and $f(x) > E$ for $x \in U_a$ ($E > 0$ is arbitrary and the neighbourhood U_a depends on E).

The following statement is readily proved.

Lemma. (1) *If* $f(x) \to \infty$ *as* $x \to a$, *then* $1/f(x) \to 0$ *as* $x \to a$; (2) *if* $\alpha(x) \to 0$ *as* $x \to a$ ($\alpha(x) \neq 0$ *for* $x \neq a$), *then* $1/\alpha(x) \to \infty$ *as* $x \to a$.

Note. An unbounded function is not necessarily infinitely large. For instance, the function

$$f(x) = \frac{1}{x} \sin \frac{1}{x}$$

is not bounded in any neighbourhood of the point $x = 0$, but it is not infinitely large as $x \to 0$.

Sec. 51. Basic Properties of Infinitesimals

Theorem 1. *The algebraic sum of a finite number of infinitesimal functions as* $x \to a$* *is an infinitesimal function as* $x \to a$.

Proof. For the sake of simplicity, we confine ourselves to three functions:

$$\alpha(x) \to 0, \quad \beta(x) \to 0 \quad \text{and} \quad \gamma(x) \to 0 \quad \text{as} \quad x \to a.$$

Consider their algebraic sum

$$\alpha(x) + \beta(x) - \gamma(x).$$

---

* Here and throughout this section we assume that all the functions under consideration are automatically defined on some *common set* X for which a is a limiting point. The values of x in question are such that $x \in X$.

Let $\varepsilon > 0$ be an arbitrary positive number. Then $\frac{\varepsilon}{3}$ will also be a positive number.

By virtue of definition of the infinitesimal, there exist three neighbourhoods U'_a, U''_a, U'''_a characterized by the number $\frac{\varepsilon}{3}$ such that

$$|\alpha(x)| < \frac{\varepsilon}{3} \quad \text{for} \quad x \in U'_a, \tag{51-1}$$

$$|\beta(x)| < \frac{\varepsilon}{3} \quad \text{for} \quad x \in U''_a, \tag{51-2}$$

$$|\gamma(x)| < \frac{\varepsilon}{3} \quad \text{for} \quad x \in U'''_a. \tag{51-3}$$

The intersection $U_a = U'_a \cap U''_a \cap U'''_a$ represents the neighbourhood of the point a in which the above inequalities will be fulfilled simultaneously. Thus,

$$|\alpha(x) + \beta(x) - \gamma(x)| \leqslant |\alpha(x)| + |\beta(x)| + |-\gamma(x)| =$$

$$= |\alpha(x)| + |\beta(x)| + |\gamma(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

if $x \in U_a$ and $x \in X$. And this just means that

$$\alpha(x) + \beta(x) - \gamma(x) \to 0 \quad \text{as} \quad x \to a.$$

The theorem has been proved.

In particular, *the difference between two infinitesimal functions as* $x \to a$ *is an infinitesimal function as* $x \to a$.

Definition. *The function* $f(x)$ *is said to be bounded as* $x \to a$ *if it is bounded in some neighbourhood* U_a *of the point* a.

Theorem 2. *The product of a bounded function as* $x \to a$ *by an infinitesimal function as* $x \to a$ *is an infinitesimal function as* $x \to a$.

Proof. Let

$$|f(x)| \leqslant M \qquad (M > 0) \quad \text{for} \quad x \in V_a,$$

where V_a is some neighbourhood of the point a and

$$\alpha(x) \to 0 \quad \text{as} \quad x \to a.$$

Then for an arbitrary $\varepsilon > 0$ there exists a neighbourhood $U_a \subset V_a$, such that

$$|\alpha(x)| < \frac{\varepsilon}{M} \quad \text{for} \quad x \in U_a. \tag{51-4}$$

Hence we have

$$|f(x)\,\alpha(x)| = |f(x)|\,|\alpha(x)| < M \cdot \frac{\varepsilon}{M} = \varepsilon,$$

if $x \in U_a$. Thus,

$$f(x)\,\alpha(x) \to 0 \quad \text{as} \quad x \to a.$$

Theorem 3. *The product of a finite number of infinitesimal functions as* $x \to a$ *is an infinitesimal function as* $x \to a$.

Proof. (1) Let us first consider two functions $\alpha(x) \to 0$ and $\beta(x) \to 0$ as $x \to a$.

Putting $0 < \varepsilon < 1$ and reasoning as in Theorem 1, we get convinced that there exists a neighbourhood U_a such that

$$|\alpha(x)| < \varepsilon, \quad |\beta(x)| < \varepsilon \quad \text{for} \quad x \in U_a.$$

Hence

$$|\alpha(x)\,\beta(x)| = |\alpha(x)| \cdot |\beta(x)| < \varepsilon \cdot \varepsilon < \varepsilon,$$

if $x \in U_a$.

Consequently, $\alpha(x)\,\beta(x) \to 0$ as $x \to a$.

(2) If we have, for instance, three functions $\alpha(x) \to 0$, $\beta(x) \to 0$, $\gamma(x) \to 0$ as $x \to a$, then, using the first part of the proof, we get

$$\alpha(x)\,\beta(x)\,\gamma(x) = [\alpha(x)\,\beta(x)]\,\gamma(x) \to 0$$

as $x \to a$.

Corollary. *The nth power* $[\alpha(x)]^n$ *of an infinitesimal function* $\alpha(x) \to 0$ *as* $x \to a$ *is an infinitesimal function as* $x \to a$ (*n is a positive integer*).

Note. As to the ratio of two infinitesimals $\alpha(x) \to 0$ and $\beta(x) \to 0$ as $x \to a$, it may be a function of arbitrary behaviour as $x \to a$.

Example. Let $\alpha(x) = x$, $\beta(x) = 2x + x^2$, $\gamma(x) = x^2$. Here, as $x \to 0$, we have

$$\alpha(x) \to 0, \quad \beta(x) \to 0, \quad \gamma(x) \to 0;$$

$$\frac{\beta(x)}{\alpha(x)} = 2 + x \to 2; \quad \frac{\gamma(x)}{\alpha(x)} = x \to 0, \quad \frac{\alpha(x)}{\gamma(x)} = \frac{1}{x} \to \infty$$

We can compare infinitesimals by means of the operation of division.

Definition 1. *Two infinitesimals* $\alpha(x)$ *and* $\beta(x)$ *as* $x \to a$ *have the same order as* $x \to a$ *provided their ratio has a finite non-zero limit*, i.e.

$$\lim_{x \to a} \frac{\alpha(x)}{\beta(x)} = k \neq 0.$$

Definition 2. *As* $x \to a$, *the order of the infinitesimal* $\beta(x)$ *is said to be higher than the order of the infinitesimal* $\alpha(x)$ (or, which is the same, *the order of the infinitesimal* $\alpha(x)$ *is lower than the order of the infinitesimal* $\beta(x)$) *if the ratio* $\beta(x)/\alpha(x)$ *is an infinitesimal function as* $x \to a$, i.e.

$$\lim_{x \to a} \frac{\beta(x)}{\alpha(x)} = 0.$$

In this case we write

$$\beta(x) = o[\alpha(x)] \quad \text{as} \quad x \to a.$$

Definition 3. *The infinitesimal* $\beta(x)$ *is said to have order* n (*n natural number*) *relative to the infinitesimal* $\alpha(x)$ *as* $x \to a$ *if*

$$\lim_{x \to a} \frac{\beta(x)}{\alpha^n(x)} = k \neq 0.$$

But if $\beta(x) = o[\alpha^n(x)]$ as $x \to a$ (i.e. $k = 0$), then the order of $\beta(x)$ is higher than n as compared with $\alpha(x)$.

Sec. 52. Basic Limit Theorems

It is also assumed here that the functions considered in each of the following theorems are defined on a common set X for which point a is a limiting point (point of accumulation).

Theorem 1. *If each of the terms of an algebraic sum of a finite number of functions has a limit as* $x \to a$, *then there exists a limit of this algebraic sum as* $x \to a$ *and it is equal to the same algebraic sum of the limits of the terms.*

Proof. Let us, for instance, have an algebraic sum of three functions

$$f(x) + g(x) - h(x),$$

where

$$\lim_{x \to a} f(x) = A, \quad \lim_{x \to a} g(x) = B, \quad \lim_{x \to a} h(x) = C.$$

Since the functions differ from their limits by infinitesimals, we get

$$f(x) = A + \alpha(x), \quad g(x) = B + \beta(x), \quad h(x) = C + \gamma(x), \tag{52-1}$$

where $\alpha(x) \to 0$, $\beta(x) \to 0$, $\gamma(x) \to 0$ as $x \to a$. From the above equalities, using the theorem on an algebraic sum of infinitesimals (Sec. 51, Theorem 1), we have

$$f(x) + g(x) - h(x) = (A + B - C) + [\alpha(x) + \beta(x) - \gamma(x)], \tag{52-2}$$

where $\alpha(x) + \beta(x) - \gamma(x) \to 0$ as $x \to a$. From (52-2) it follows that the sum $f(x) + g(x) - h(x)$ differs from the number $A + B - C$ by an infinitesimal and, hence, *this number* is the limit of the given sum. Thus, we have

$$\lim_{x \to a} [f(x) + g(x) - h(x)] = A + B - C = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) - \lim_{x \to a} h(x), \tag{52-3}$$

which was to be proved.

Corollary. *A function can have only one limit as* $x \to a$.
Indeed, if

$$f(x) \to A \quad \text{and} \quad f(x) \to A' \quad \text{as} \quad x \to a,$$

then, by virtue of Theorem 1, we get

$$f(x) - f(x) = 0 \to A - A' \quad \text{as} \quad x \to a.$$

Since the limit of a constant function is equal to the function itself and is unique (see Sec. 46, Theorem 1), we have $A - A' = 0$, i.e. $A' = A$.

Note. It was assumed in the initial condition of the theorem that each of the functions has a limit and it was to be proved that their sum also has a limit. The converse is, generally speaking, not true: from the fact of existence of the limit of a sum it does not follow that each term of this sum has a limit. For instance, we have

$$\lim_{x \to \infty} (\sin^2 x + \cos^2 x) = 1,$$

while $\lim\limits_{x \to \infty} \sin^2 x$ and $\lim\limits_{x \to \infty} \cos^2 x$ do not exist, and therefore, here

$$\lim_{x \to \infty} (\sin^2 x + \cos^2 x) \neq \lim_{x \to \infty} \sin^2 x + \lim_{x \to \infty} \cos^2 x.$$

Thus, the statement: the limit of a sum is equal to the sum of the limits of the terms is not rigorous.

A similar note should be borne in mind when considering the limit of a product (Theorem 2) and the limit of a quotient (Theorem 4).

Theorem 2. *If each of the factors of a product of a finite number of functions has a limit as* $x \to a$, *then the limit of the product as* $x \to a$ *is equal to the product of the limits of the factors.*

Proof. (1) Let us first consider the product of two functions $f(x)\, g(x)$, and let

$$\lim_{x \to a} f(x) = A \quad \text{and} \quad \lim_{x \to a} g(x) = B.$$

We have

$$f(x) = A + \alpha(x), \quad g(x) = B + \beta(x), \tag{52-4}$$

where $\alpha(x) \to 0$ and $\beta(x) \to 0$ as $x \to a$. Hence, we get

$$f(x)\, g(x) = AB + \gamma(x), \tag{52-5}$$

where

$$\gamma(x) = A\beta(x) + B\alpha(x) + \alpha(x)\, \beta(x). \tag{52-6}$$

From the basic properties of infinitesimals (Sec. 51, Theorems 1, 2, and 3) it follows that $\gamma(x) \to 0$ as $x \to a$. Therefore, by (52-5), we

have

$$\lim_{x\to a}[f(x)\,g(x)]=AB=\lim_{x\to a}f(x)\lim_{x\to a}g(x). \tag{52-7}$$

(2) Let us now consider, for instance, a product of three functions $f(x)\,g(x)\,h(x)$, having finite limits as $x\to a$. Using the first part of the proof, we find

$$\lim_{x\to a}[f(x)\,g(x)\,h(x)]=\lim_{x\to a}\{f(x)\,[g(x)\,h(x)]\}=$$

$$=\lim_{x\to a}f(x)\lim_{x\to a}[g(x)\,h(x)]=\lim_{x\to a}f(x)\lim_{x\to a}g(x)\lim_{x\to a}h(x).$$

Corollary 1. *A constant factor may be taken outside the limit sign.* Indeed, if c is a constant function, then

$$\lim_{x\to a}[cf(x)]=\lim_{x\to a}c\lim_{x\to a}f(x)=c\lim_{x\to a}f(x).$$

Corollary 2. *If a function $f(x)$ has a limit as $x\to a$, then the limit of the nth power of the function as $x\to a$ (where n is a positive integer) is equal to the nth power of the limit of that function*, i.e.

$$\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a}f(x)\right]^n.$$

Example 1.

$$\lim_{x\to\infty}\frac{(x+10)\,(x+20)^2\,(x+30)^3}{x^6}=$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{10}{x}\right)\left(1+\frac{20}{x}\right)^2\left(1+\frac{30}{x}\right)^3\right]=$$

$$=\lim_{x\to\infty}\left(1+\frac{10}{x}\right)\lim_{x\to\infty}\left(1+\frac{20}{x}\right)^2\lim_{x\to\infty}\left(1+\frac{30}{x}\right)^3=1\cdot1\cdot1=1.$$

Lemma. *Let $f(x)\to A\neq0$ as $x\to a$. Then the inverse function $1/f(x)$ is bounded in some neighbourhood U_a of the point a.*

Indeed, let us put $\varepsilon=|A|>0$. By the definition of the limit of a function, we have

$$|f(x)-A|<\frac{\varepsilon}{2}=\frac{|A|}{2}\quad\text{for}\quad x\in U_a$$

for all permissible values of x. Hence we get

$$|f(x)|=|A-[A-f(x)]|\geqslant$$

$$\geqslant|A|-|A-f(x)|>|A|-\frac{|A|}{2}=\frac{|A|}{2}>0\quad\text{for}\quad x\in U_a.$$

Thus

$$\left|\frac{1}{f(x)}\right|=\frac{1}{|f(x)|}<\frac{2}{|A|},$$

if $x\in U_a$, which was to be proved.

Theorem 3. *If a function $f(x)$ has a non-zero limit as $x \to a$, then the limit of the inverse function $1/f(x)$ as $x \to a$ is equal to the inverse of the limit of the given function*, i.e.

$$\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{\lim\limits_{x \to a} f(x)}. \tag{52-8}$$

Proof. Indeed, let

$$\lim_{x \to a} f(x) = A \neq 0.$$

Then, by the lemma, bearing in mind that the product of a bounded function by an infinitesimal is an infinitesimal (Sec. 51, Theorem 2), we shall have

$$\frac{1}{f(x)} - \frac{1}{A} = \frac{1}{A} \cdot \frac{1}{f(x)} \cdot [A - f(x)] \to 0 \quad \text{as} \quad x \to a.$$

Hence, we get

$$\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{A} = \frac{1}{\lim\limits_{x \to a} f(x)}.$$

Theorem 4. *If both the dividend $f(x)$ and the divisor $g(x)$ have limits as $x \to a$ and the limit of the divisor is non-zero, then the limit of their quotient (fraction) as $x \to a$ is equal to the quotient of the limits of the dividend (the numerator of the fraction) and the divisor (the denominator of the fraction)*, i.e.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}. \tag{52-9}$$

Proof. Let $\lim\limits_{x \to a} g(x) \neq 0$. Then, applying both the theorem on the limit of a product (Theorem 2) and the theorem on the limit of the inverse of a function (Theorem 3), we get

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \lim_{x \to a} \left[f(x) \cdot \frac{1}{g(x)} \right] = \lim_{x \to a} f(x) \lim_{x \to a} \frac{1}{g(x)} =$$

$$= \lim_{x \to a} f(x) \cdot \frac{1}{\lim\limits_{x \to a} g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}.$$

Example 2.

$$\lim_{x \to 1} \frac{x^2 - 1}{x^3 - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)} = \lim_{x \to 1} \frac{x+1}{x^2+x+1} =$$

$$= \frac{\lim\limits_{x \to 1} (x+1)}{\lim\limits_{x \to 1} (x^2+x+1)} = \frac{2}{3}.$$

The following theorem is given without proof.

Theorem 5. *If a function $f(x)$ has a limit as $x \to a$ and there exists $\sqrt[n]{f(x)}$ (where n is a natural number) both at the point a and in some of its neighbourhoods U_a, then*

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}. \tag{52-10}$$

Sec. 53. Some Tests for the Existence of the Limit of a Function

Not every function has a limit, even if the former is bounded. For instance, $\sin x$ has no limit as $x \to \infty$, though $|\sin x| \leqslant 1$.

There are various tests for finding a limit of a function when its direct determination is impractical. Here we confine ourselves to considering two tests.

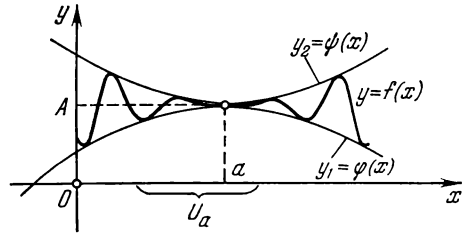


FIG. 89

Theorem on an intermediate function. *Let in some neighbourhood U_a of a point a a function $f(x)$ be contained between two functions $\varphi(x)$ and $\psi(x)$ tending to one and the same limit A as $x \to a$* (Fig. 89), *i.e.*

$$\varphi(x) \leqslant f(x) \leqslant \psi(x) \tag{53-1}$$

and

$$\lim_{x \to a} \varphi(x) = \lim_{x \to a} \psi(x) = A. \tag{53-2}$$

Then the function $f(x)$ has the same limit:

$$\lim_{x \to a} f(x) = A. \tag{53-3}$$

Proof. From (53-1) we have

$$\varphi(x) - A \leqslant f(x) - A \leqslant \psi(x) - A.$$

Hence

$$|f(x) - A| \leqslant \max(|\varphi(x) - A|, |\psi(x) - A|). \tag{53-4}$$

By (53-2), for any $\varepsilon > 0$ there exists a neighbourhood U_a such that

$$|\varphi(x) - A| < \varepsilon \quad \text{and} \quad |\psi(x) - A| < \varepsilon \quad \text{for} \quad x \in U_a. \tag{53-5}$$

Therefore, from (53-4) we get

$$|f(x) - A| < \varepsilon \quad \text{for} \quad x \in U_a, \tag{53-6}$$

i.e. equality (53-3) holds.

Definition. (1) *A function $f(x)$ is called increasing (non-decreasing) on a given set X if from the inequality $x_1 < x_2$ ($x_1, x_2 \in X$) there follows the inequality $f(x_1) < f(x_2)$ ($f(x_1) \leqslant f(x_2)$, respectively).*

(2) *A function $f(x)$ is called decreasing (non-increasing) on a given set X if from the inequality $x_1 < x_2$ ($x_1, x_2 \in X$) there follows the inequality $f(x_1) > f(x_2)$ ($f(x_1) \geqslant f(x_2)$, respectively).*

An increasing (non-decreasing) or decreasing (non-increasing) function is said to be *monotone* on the given set X.

Theorem. *Let the function $f(x)$ be monotone and bounded for $x < a$, or for $x > a$. Then there exists respectively its left-hand limit*

$$\lim_{x \to a-0} f(x) = f(a-0)$$

or its right-hand limit

$$\lim_{x \to a+0} f(x) = f(a+0).$$

In spite of the obviousness of this theorem, its proof cannot be set forth here.

Note. A similar statement is true for $a = -\infty$ or for $a = +\infty$.

Corollary. *A monotonically increasing or monotonically decreasing bounded sequence x_n ($n = 1, 2, \ldots$) has a limit.*

Example. Consider the sequence of perimeters P_3, P_6, P_{12}, ... of regular n-gons ($n = 3, 6, 12, \ldots$) inscribed in a circle of radius R and obtained as a result of doubling the number of their sides.

It is easy to get convinced that

$$P_3 < P_6 < P_{12} < \ldots,$$

i.e. the perimeter P_n monotonically increases with n. At the same time the quantity P_n is bounded, since the perimeter of each inscribed regular n-gon never exceeds the perimeter of any circumscribed polygon, in particular, the perimeter of the circumscribed square, i.e. $P_n < 8R$.

Hence, there exists

$$\lim_{n \to \infty} P_n = C,$$

which is taken for the *length of the circumference* of a circle.

Sec. 54. The Limit of $\frac{\sin x}{x}$ as $x \to 0$

Theorem. *The limit of the ratio of the sine of an infinitely small arc to the arc itself expressed in radians is equal to unity*, i.e.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \qquad (54\text{-}1)$$

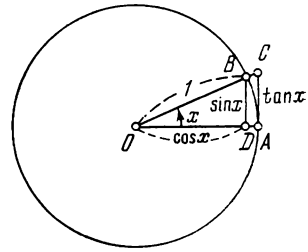


FIG. 90

Proof. (1) Let first $x > 0$; and since the arc x tends to zero, we may assume that $0 < x < \frac{\pi}{2}$.

In a trigonometric circle of radius $R = 1$ we construct the angle $x = \angle AOB$ (Fig. 90); let DB be the length of the perpendicular dropped from the point B onto the radius OA and AC the length of the tangent line to the circle at the point A to the point of its intersection with the extended radius OB. Obviously, we have

$$\text{area } \triangle OAB < \text{area of sector } OAB < \text{area } \triangle OAC.$$

Since $DB = \sin x$ and $AC = \tan x$, according to the formulas of elementary geometry, we get

$$\frac{1}{2} \sin x < \frac{1}{2} x < \frac{1}{2} \tan x,$$

i.e.

$$\sin x < x < \tan x. \qquad (54\text{-}2)$$

Dividing all terms of the last dual inequality by a positive quantity $\sin x$, we shall have

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \qquad (54\text{-}3)$$

or

$$\cos x < \frac{\sin x}{x} < 1. \qquad (54\text{-}4)$$

Let $x \to +0$; then, obviously, we get $\cos x \to 1$*. Thus, from inequality (54-4) it follows that the function $\sin x/x$ is contained between

*Indeed, since, by (54-2), $|\sin x| \leqslant |x|$ $\left(|x| < \frac{\pi}{2}\right)$, the sine of an infinitely small arc is an infinitesimal. Hence, $1 - \cos x = 2 \sin^2 \frac{x}{2} \to 0$ as $x \to 0$, i.e. $\lim_{x \to 0} \cos x = 1$.

two functions having a common limit equal to 1. On the basis of the theorem on an intermediate function (Sec. 53), we get

$$\lim_{x \to +0} \frac{\sin x}{x} = 1. \tag{54-5}$$

(2) Let now $x < 0$; we have

$$\frac{\sin x}{x} = \frac{\sin (-x)}{-x},$$

where $-x > 0$. Therefore,

$$\lim_{x \to -0} \frac{\sin x}{x} = 1. \tag{54-5'}$$

From formulas (54-5) and (54-5') it, obviously, follows the equality (54-1) (see Sec. 47, Note 5).

Note. From (54-2) it follows that if $0 < |x| < \frac{\pi}{2}$, then

$$|\sin x| = \sin |x| < |x|.$$

Hence, since $|\sin x|$ does not exceed unity, the below inequality holds for any x

$$|\sin x| \leqslant |x|, \tag{54-6}$$

an equality taking place only for $x = 0$. Inequality (54-6) is frequently used for estimating the sines of small arcs.

Sec. 55. The Number e

Consider the expression

$$\left(1 + \frac{1}{n}\right)^n,$$

where n is a natural number.

Let us assign infinitely increasing values to n and compute the corresponding values of $\left(1 + \frac{1}{n}\right)^n$. We shall get the following table

n	1	2	10	100	1000	10 000	...
$\left(1+\frac{1}{n}\right)^n$	2	2.25	2.594	2.705	2.717	2.718	...

We see that with an increase in n the expression $\left(1 + \frac{1}{n}\right)^n$ changes still slower and slower and, apparently, tends to some limit approximately equal to 2.718. Let us prove that it is really so.

Theorem. *The sequence*

$$\left(1+\frac{1}{n}\right)^n \qquad (n=1,\ 2,\ \ldots)$$

tends to a finite limit contained between 2 *and* 3.

Proof. Making use of the binomial formula (see Sec. 89, Chapter 11), we have

$$\left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{1\cdot 2}\cdot\left(\frac{1}{n}\right)^2+$$
$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\left(\frac{1}{n}\right)^3+\ldots+\frac{n(n-1)\ldots[n-(n-1)]}{1\cdot 2\cdot 3\ldots n}\cdot\left(\frac{1}{n}\right)^n,$$

or

$$\left(1+\frac{1}{n}\right)^n = 2+\frac{1}{2}\left(1-\frac{1}{n}\right)+\frac{1}{2\cdot 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots$$
$$\ldots+\frac{1}{2\cdot 3\ldots n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right). \qquad (55\text{-}1)$$

For $n>1$ all the terms in formula (55-1) are positive, and with an increase in the exponent n the number of terms increases and each corresponding term becomes greater.

Hence, the sequence

$$\left(1+\frac{1}{n}\right)^n,$$

grows together with the exponent n beginning with the least value equal to 2.

On the other hand, obviously, each term of the right-hand member of (55-1) will increase if all factors of the denominators are substituted by twos and each expression in parentheses is replaced by unity. Therefore

$$\left(1+\frac{1}{n}\right)^n < 2+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}.$$

By virtue of a known formula for a geometric progression, we have

$$\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}=\frac{\frac{1}{2}-\frac{1}{2^n}}{1-\frac{1}{2}}=1-\frac{1}{2^{n-1}}<1.$$

Hence

$$\left(1+\frac{1}{n}\right)^n<3.$$

Thus, as n increases unboundedly, the terms of the sequence $\left(1+\frac{1}{n}\right)^n$ steadily increase remaining greater than 2 but less than 3.

Consequently, by the corollary to the theorem from Sec. 53, there exists a finite limit of this sequence which, obviously, belongs to

the segment [2, 3] (see Sec. 46, Theorem 2). And this limit is called the number e. Thus,

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

The approximate value of this number is

$$e = 2.7182818284\ldots$$

We can prove that the function

$$\left(1+\frac{1}{x}\right)^x \qquad (x\in(-\infty,\ -1)\cup(0,\ +\infty))$$

as $x\to\infty$ tends to the number e:

$$e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

Let us give another expression for e. Putting $\frac{1}{x}=\alpha(\alpha>-1)$, we have

$$e=\lim_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}.$$

With the aid of the number e it is convenient to express many limits.

Example 1. Find

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x.$$

Putting

$$\frac{2}{x}=\alpha,$$

we have

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x=\lim_{\alpha\to 0}(1+\alpha)^{\frac{2}{\alpha}}=\lim_{\alpha\to 0}\left[(1+\alpha)^{\frac{1}{\alpha}}\right]^2=e^2.$$

The function of the form

$$y = e^x, \tag{55-2}$$

where $e = 2.71828\ldots$, belongs to *exponential* functions. We also use the notation

$$e^x = \exp x.$$

The graph of function (55-2) is shown in Fig. 91. This exponential function plays an important role in mathematical analysis and its applications.

Example 2. Let a certain chemical reaction proceed so that at each moment of time the rate of formation of a substance is proportional to the quantity of this substance available at this moment.

We denote by Q_0 the initial quantity of this substance (i.e. the quantity of the substance at the moment $t = 0$) and subdivide the interval of time $(0, t)$ into n small intervals:

$$\left(0, \frac{t}{n}\right), \left(\frac{t}{n}, \frac{2t}{n}\right), \ldots, \left(\frac{(n-1)t}{n}, \frac{nt}{n}\right).$$

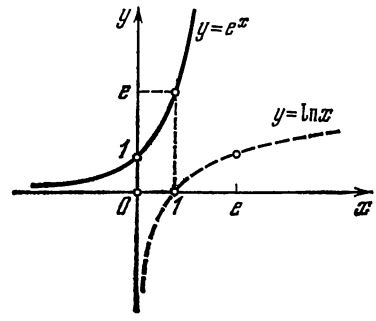


FIG. 91

If during each of these quite short intervals of time the rate of reaction is regarded as constant, then the quantities of the substance available at the moments

$$\frac{t}{n}, \frac{2t}{n}, \ldots, \frac{nt}{n} = t$$

will be respectively equal to

$$Q_1 = Q_0 + kQ_0 \cdot \frac{t}{n} = Q_0\left(1 + \frac{kt}{n}\right),$$

$$Q_2 = Q_1 + kQ_1 \cdot \frac{t}{n} = Q_1\left(1 + \frac{kt}{n}\right) = Q_0\left(1 + \frac{kt}{n}\right)^2,$$

. . . . . .

$$Q_n = Q_{n-1} + kQ_{n-1}\frac{t}{n} = Q_{n-1}\left(1 + \frac{kt}{n}\right) = Q_0\left(1 + \frac{kt}{n}\right)^n,$$

where k is a given proportionality factor (the *compound interest law*). But, according to the initial condition, the quantity of the substance continuously grows. Therefore, to get an exact formula, we have to assume that the number of our intervals increases without bound, and each of them tends to zero.

Hence, considering that $\frac{t}{n} \to 0$, for the quantity of the substance Q available at the moment t we shall have the following formula:

$$Q = \lim_{n\to\infty}\left[Q_0\left(1 + \frac{kt}{n}\right)^n\right].$$

This limit can be readily expressed in terms of the number e. Indeed, introducing the notation

$$\frac{kt}{n} = \alpha, \quad \text{where} \quad \alpha \to 0,$$

we get

$$Q = Q_0 \lim_{\alpha \to 0} \left[(1+\alpha)^{\frac{1}{\alpha}} \right]^{kt},$$

i.e.

$$Q = Q_0 e^{kt}. \tag{55-3}$$

This is just the law governing the growth of the substance under the above mentioned conditions.

Formula (55-3) is applied in studying a number of processes, such as decay of radium (where $k < 0$), reproduction of bacteria, and so on. Hence, it is clear what an important role is played by the number e in mathematical analysis and its applications.

Sec. 56. Natural Logarithms

If the base of logarithms is equal to the number e, then they are called *natural* or *Napierian** and are designated in the following manner

$$\log_e x = \ln x.$$

Higher mathematics uses almost exclusively natural logarithms, since many formulas for them, as we shall see later on, turn out to be simpler as compared with those for logarithms of other systems**.

Let us derive the relations between the natural logarithm of a number and the logarithm of this number to the base a ($a > 0$, $a \neq 1$). Let us have

$$y = \log_a x;$$

hence

$$a^y = x.$$

Taking the logarithm of this equality to the base e, we find

$$y \ln a = \ln x.$$

Whence

$$y = \frac{1}{\ln a} \ln x$$

or

$$\log_a x = \frac{1}{\ln a} \ln x. \tag{56-1}$$

* After John Napier (1550-1617), a Scottish mathematician, inventor of logarithms.

** Besides, in applications we often come across exponential relationships of the form (55-3); therefore it is more convenient to use logarithms to the base e.

This formula expresses the logarithm of the number x to the base a in terms of the natural logarithm of this number.

Note that, putting $x = e$ in formula (56-1), we have

$$\log_a e = \frac{1}{\ln a} \ln e = \frac{1}{\ln a}.$$

Putting in formula (56-1) $a = 10$, we get

$$\log x = \log_{10} x = M \ln x, \tag{56-2}$$

where

$$M = \frac{1}{\ln 10} = \log e = 0.43429$$

is the *modulus* of natural logarithms.

Conversely, from formula (56-2) we find

$$\ln x = \frac{1}{M} \log x, \tag{56-3}$$

where

$$\frac{1}{M} = \ln 10 = 2.30258.$$

Sec. 57. Asymptotic Formulas

Let $\varphi(x)$ and $\psi(x)$ be functions defined in a neighbourhood of a point a.

Generalizing the definition from Sec. 51, we shall say that

$$\psi(x) = o(\varphi(x)) \quad \text{as} \quad x \to a, \tag{57-1}$$

if

$$\psi(x) = \alpha(x)\, \varphi(x), \tag{57-2}$$

where $\alpha(x) \to 0$ as $x \to a$.

If $\varphi(x) \neq 0$ in some neighbourhood of the point a, then from (57-2) we have

$$\lim_{x \to a} \frac{\psi(x)}{\varphi(x)} = 0 \tag{57-3}$$

(cf. Sec. 51).

Definition. *If, as $x \to a$, the below equality is valid*

$$f(x) = \varphi(x) + o(\varphi(x)), \tag{57-4}$$

then $\varphi(x)$ is called the asymptotic term (or asymptotic expression) for the function $f(x)$ as $x \to a$.

We use the following notation: $f(x) \sim \varphi(x)$ as $x \to a$. If $\varphi(x) \neq 0$ for $x \in U_a$, then from formula (57-4) we get

$$\lim_{x \to a} \frac{f(x)}{\varphi(x)} = 1. \tag{57-5}$$

Now we find out the existence conditions for the function $f(x)$ of the *non-zero* asymptotic term

$$\varphi(x) = kx + b \quad \text{as} \quad x \to \infty. \tag{57-6}$$

Let

$$f(x) = kx + b + \alpha(x), \tag{57-7}$$

where $\alpha(x)$ is an infinitesimal as $x \to \infty$, i.e. $\alpha(x) = o(1)$ as $x \to \infty$, and, obviously, also $\alpha(x) = o(kx + b)$ as $x \to \infty$.

From (57-7) we have

$$\frac{f(x)}{x} = k + \frac{b}{x} + \frac{\alpha(x)}{x}. \tag{57-8}$$

Proceeding to the limit as $x \to \infty$ in equality (57-8) and taking into account that $\alpha(x)/x \to 0$ as $x \to \infty$, we get

$$k = \lim_{x \to \infty} \frac{f(x)}{x}. \tag{57-9}$$

From formula (57-7) we find

$$b = \lim_{x \to \infty} [f(x) - kx]. \tag{57-10}$$

Conversely, if there exist limits (57-9) and (57-10) of which at least one is non-zero, then asymptotic expansion (57-7) is true. Indeed, from formula (57-10) where k is defined by equality (57-9) we have

$$\lim_{x \to \infty} [f(x) - kx - b] = 0.$$

Hence there follows directly formula (57-7).

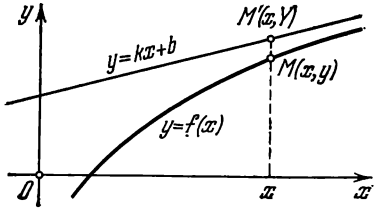


FIG. 92

The graph of the linear asymptotic expression $y = kx + b$ is called the *asymptote* to the curve $y = f(x)$ (Fig. 92); here the case $k = 0$, $b = 0$ is not excluded. In this case for the points $M(x, y)$ on the curve and $M'(x, Y)$ on the asymptote $Y - y = MM' \to 0$ as $x \to \infty$ *.

* If limits (57-9) and (57-10) exist as $x \to -\infty$ or as $x \to +\infty$, then asymptotic formula (57-7) is true under appropriate conditions. In this case the graph of the function $y = f(x)$ has a *left-hand asymptote* or a *right-hand asymptote*, respectively.

Example. Construct a linear asymptotic formula for the function $f(x) = \sqrt{x^2+x+1}$.

Using formulas (57-9) and (57-10), we have

$$k = \lim_{x\to+\infty} \frac{\sqrt{x^2+x+1}}{x} = \lim_{x\to+\infty} \sqrt{1+\frac{1}{x}+\frac{1}{x^2}} = 1$$

and

$$b = \lim_{x\to+\infty} (\sqrt{x^2+x+1} - x) = \lim_{x\to+\infty} \frac{x+1}{\sqrt{x^2+x+1}+x} =$$

$$= \lim_{x\to+\infty} \frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+1} = \frac{1}{2}.$$

Hence

$$\sqrt{x^2+x+1} \sim x+\frac{1}{2} \quad \text{as} \quad x \to +\infty.$$

Exercises

1. On the number axis construct the sets of points defined by the following inequalities: (a) $|x+2|<1$; (b) $|x-3| \geqslant 3$; (c) $0<|x-1|<\frac{1}{2}$; (d) $1 \leqslant |x| \leqslant 2$.

2. When determining the mass of a body there was obtained an approximate result $p = 2.57$ g with an absolute error $\Delta_0 \leqslant 0.01$ g. Determine the limiting relative error δ of the number p.

3. How many correct digits has the approximate number $x = 35.719$ if its relative error $\delta_0 \leqslant 1\%$?

In Exercises 4 to 10 find the limits:

4. (a) $\lim\limits_{x\to\infty} \frac{1000x}{1+x^2}$; (b) $\lim\limits_{x\to\infty} \frac{x^2-3x+2}{x^2-1}$; (c) $\lim\limits_{x\to1} \frac{x^2-3x+2}{x^2-1}$.

5. (a) $\lim\limits_{x\to9} \frac{\sqrt{x}-3}{x-9}$; (b) $\lim\limits_{x\to8} \frac{\sqrt[3]{x}-2}{x-8}$.

6. $\lim\limits_{x\to\infty} \frac{\sin x}{x}$. 7. $\lim\limits_{x\to0} \frac{\sin 5x}{x}$. 8. $\lim\limits_{x\to0} \frac{1-\cos x}{x^2}$.

9. $\lim\limits_{x\to a} \frac{\sin x - \sin a}{x-a}$. 10. $\lim\limits_{x\to0} \frac{\tan 2x}{x}$.

11. Find out the behaviour of the roots x_1 and x_2 of the quadratic equation

$$ax^2 + bx + c = 0,$$

if the coefficient $a \to 0$, and the coefficients $b \neq 0$ and c are constant.

12. Let

$$y_1 = x \quad \text{and} \quad y_2 = \sqrt{x^2+1}.$$

Show that

$$\lim_{x \to +\infty} (y_2 - y_1) = 0.$$

Find out the geometrical meaning of this equality.

Find the limits:

13. $\lim\limits_{n \to \infty} \left(1 + \frac{1}{2^n}\right)$.

14. $\lim\limits_{n \to \infty} \frac{1}{1+2^n}$.

15. $\lim\limits_{n \to \infty} \left(1 - \frac{1}{n}\right)^n$.

16. $\lim\limits_{n \to \infty} \left(1 + \frac{3}{n}\right)^n$.

17. $\lim\limits_{x \to 0} (1-x)^{\frac{2}{x}}$.

18. $\lim\limits_{x \to \infty} \left(\frac{x}{x+1}\right)^x$.

Chapter 8

Continuity of Functions

Sec. 58. Increments of an Argument and a Function. Continuity of a Function

Let x be a value of a given variable. Along with x, let us consider another value x_1 of this variable. Let us introduce the following definition.

Definition. *The difference between the new value of a variable and its initial value is called the increment of this variable*, i.e. in our case the increment of the given variable is equal to $x_1 - x$.

The increment is denoted by the Greek letter Δ (delta). For instance, $\Delta x = x_1 - x$ denotes the increment of the variable quantity x.

Adding the increment to the initial value of a variable, we get the *terminal value* of this variable. For instance, $x + \Delta x$ is the terminal value of the variable quantity x.

Suppose y is a function of the argument x, i.e.

$$y = f(x). \tag{58-1}$$

Let us assign the increment Δx to the argument x, then y will receive the corresponding increment Δy. This, obviously, can be written in the following way:

$$y + \Delta y = f(x + \Delta x). \tag{58-2}$$

From (58-1) and (58-2) it follows that

$$\Delta y = f(x + \Delta x) - f(x). \tag{58-3}$$

Example 1. Determine the increment of the argument x and the increment of the function

$$y = x^2,$$

if the argument x varies from -1 to 2.

Here, obviously,

$$\Delta x = 2 - (-1) = 3$$

and

$$\Delta y = 2^3 - (-1)^3 = 9.$$

Let us explain the concept of the increment of a function geometrically.

Let the curve AB be the graph of the function $y = f(x)$ (Fig. 93).

Consider on this curve a point M with the current coordinates x and y. Let us assign to the abscissa x of the point M (x, y) an increment Δx, then its ordinate y will receive an increment Δy and, as a result, the point $M(x, y)$ will occupy the position M' $(x+\Delta x, y + \Delta y)$. Let C be the point of intersection of the straight line passing through the point M parallel to the x-axis and the perpendicular $M'N'$ dropped from the point M' onto the x-axis. It is obvious that

$$MC = \Delta x, \quad CM' = \Delta y.$$

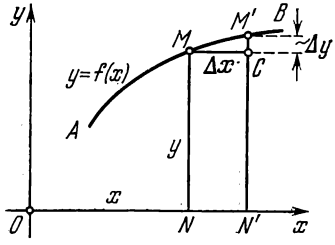


FIG. 93

It may occur that for some x, as Δx tends to zero, the point M' indefinitely approaches the point M and, hence, Δy also tends to zero. In this case the function $y = f(x)$ is called *continuous* for the given value of x. Or more precisely:

Definition 1. *A function $f(x)$ defined on a set X is called continuous for $x = x_1$ (or continuous at a point x_1) if:*

(1) *the function is defined for $x = x_1$* (i.e. $x_1 \in X$);

(2) *the increment of the function at the point x_1 tends to zero when the increment of the argument $\Delta x_1 = x - x_1$ tends to zero*, i.e.

$$\lim_{\Delta x_1 \to 0} [f(x_1 + \Delta x_1) - f(x_1)] = 0, \tag{58-4}$$

where the infinitesimal increment Δx_1 runs only through the values for which $f(x_1 + \Delta x_1)$ makes sense. Here, as always, we assume (see Sec. 46, Chapter 7) that x_1 is the *limiting point* of the set X and, thus, in any neighbourhood U_{x_1} there may be found points $x_1 + \Delta x_1 \in X$ different from x_1 $(\Delta x_1 \neq 0)$ for which the function $f(x)$ is defined.

In short, *a function is called continuous at a given point if at this point to an infinitesimal increment of the argument there corresponds an infinitesimal increment of the function.*

Using the concept of the limit of a function (Sec. 46, Chapter 7), we get a complete definition of the continuity of a function at a point: *the function $f(x)$ is continuous at a point x_1 if and only if* $\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon, x_1) > 0$ *such that*

$$|f(x) - f(x_1)| = |f(x_1 + \Delta x_1) - f(x_1)| < \varepsilon, \tag{58-5}$$

if $x = x_1 + \Delta x_1$ *and* $0 < |\Delta x_1| < \delta$ (Δx_1 — any permissible increment). It should be noted that inequality (58-5) is, obviously, fulfilled for $\Delta x_1 = 0$ as well, i.e. here the δ-neighbourhood of the point x_1 may be treated as complete: $|\Delta x_1| < \delta$.

Definition 2. *A function* $f(x)$ *is called continuous on a given set* X *if*: (1) *it is defined on this set* (*i.e.* $\forall x \in X\ \exists f(x)$); (2) *continuous at each point of this set, i.e.* $\forall x \in X$ *the below equality is valid*

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} [f(x + \Delta x) - f(x)] = 0, \tag{58-6}$$

where $x + \Delta x \in X$.

It should be noted that the set X is treated here as the domain of definition of the function, i.e. the points $x \notin X$ and $x + \Delta x \notin X$ are not considered.

For instance, the function $f(x)$ is continuous on the segment $[a, b]$ if: (1) this function is defined at each point of this segment, (2) $\forall x \in [a, b]$ equality (58-6) holds, where $x + \Delta x \in [a, b]$.

Example 2. The function

$$f(x) = \begin{cases} 1 & \text{for } x \in [0, 1], \\ 0 & \text{for } x \notin [0, 1] \end{cases}$$

is continuous on the segment $X = [0, 1]$, though it is not continuous on the axis $-\infty < x < +\infty$.

Example 3. Investigate for continuity the function

$$y = x^2.$$

Assigning the increment Δx to the argument x, we get

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2x \cdot \Delta x + (\Delta x)^2,$$

where Δy is the increment of the function y. Hence,

$$\Delta y = \Delta x \cdot (2x + \Delta x).$$

Obviously, whatever the fixed value of x, if Δx is an infinitesimal, then Δy will also be an infinitesimal. Consequently, the function x^2 is continuous for any value of the argument x. In other words, x^2 is a continuous function within the infinite interval $(-\infty, +\infty)$.

It is also easy to prove the continuity of the power function x^n, where n is a constant natural number.

Definition 3. *A point at which the continuity of a function is violated is called a point of discontinuity of this function.*

If $x = x_0$ is a point of discontinuity of the function $y = f(x)$, then the following two cases are possible:

(1) the function $f(x)$ is defined at $x = x_0$, and

$$\Delta y = f(x_0 + \Delta x_0) - f(x_0) \not\to 0$$

as $\Delta x_0 = x - x_0 \to 0$;

(2) the function $f(x)$ is not defined at $x = x_0$ and it is senseless to speak of the increment of the function at the point x_0. In this case let us agree to call $x = x_0$ a point of discontinuity of the function $f(x)$ only if the latter is defined in immediate proximity to the value x_0*.

If it is possible to change the function $f(x)$ or extend its definition at the point x_0 (i.e. to choose the number $f(x_0)$) so that the changed or supplemented function $f(x)$ becomes continuous at $x = x_0$, then

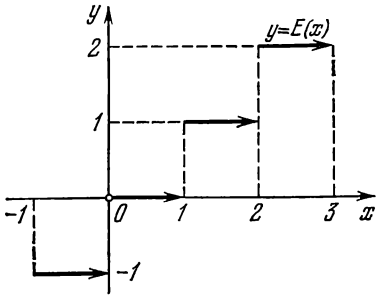


FIG. 94

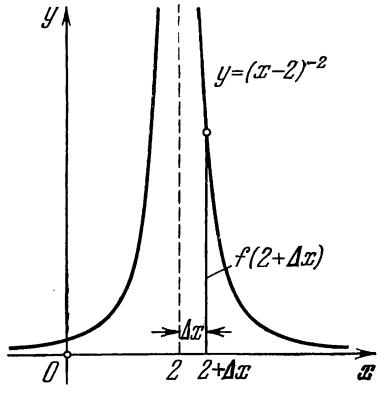


FIG. 95

this point is called a *point of removable discontinuity* of function $f(x)$. Otherwise, i.e. when the function $f(x)$ remains discontinuous for $x = x_0$ at any choice of the number $f(x_0)$, the value x_0 is called a *point of nonremovable discontinuity* of the function $f(x)$.

Example 4. Consider the function $E(x)$ equal to the integral part of the number x, i.e. if

$$x = n + q,$$

where n is an integer and $0 \leqslant q < 1$, then

$$E(x) = n$$

(Fig. 94). For instance, $E(\sqrt{2}) = 1$, $E(\pi) = 3$, $E(-1.5) = -2$ and so on.

The function $E(x)$ is discontinuous for each integral value of the argument x. Indeed, for instance, for $x = 1$ and sufficiently small Δx we have

$$E(1 + \Delta x) = 1, \quad \text{if} \quad \Delta x > 0,$$

* That is for any $\varepsilon > 0$ in the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ there will be found points at which the function $f(x)$ is defined.

and

$$E(1+\Delta x)=0, \quad \text{if} \quad \Delta x<0.$$

Hence, taking into account that

$$E(1)=1,$$

we get

$$E(1+\Delta x)-E(1)=\begin{cases} 0, & \text{if} \quad \Delta x>0, \\ -1, & \text{if} \quad \Delta x<0. \end{cases}$$

Consequently, the increment of the function

$$\Delta y = E(1+\Delta x) - E(1)$$

does not tend to zero as $\Delta x \to 0$, and therefore the function is discontinuous for $x = 1$.

An analogous reasoning can be conducted for any of the values $x = k$, where k is an integer.

Example 5. Let

$$f(x)=\frac{1}{(x-2)^2}.$$

This function is not defined for $x = 2$ but makes sense for all values of $x \neq 2$ (Fig. 95). Whatever value we assign to the number $f(2)$, we shall always have

$$f(2+\Delta x) - f(2)=\frac{1}{(\Delta x)^2}-f(2) \to \infty$$

as $\Delta x \to 0$. Thus, here for $x = 2$ at any choice of the value of $f(2)$ to an infinitesimal increment Δx of the argument there corresponds an infinitely large increment Δy of the function. Hence, this function has a point of nonremovable discontinuity for $x = 2$.

Sec. 59. Another Definition of the Continuity of a Function

In view of the importance of the notion of the continuity of a function, we are going to introduce another definition of the continuity of a function at a point which is equivalent to the above given.

Definition. *A function $f(x)$ is called continuous at $x = x_1$ if*: (1) *this function is defined for $x = x_1$*; (2) *there exists the equality*

$$\lim_{x \to x_1} f(x) = f(x_1) * \tag{59-1}$$

* Here, as usual, it is assumed that x_1 is the limiting point of the domain of definition of the function $f(x)$.

i.e. a function is continuous at a given point x_1 if and only if the limit of the function as $x \to x_1$ is equal to the value of the function at the limiting point (Fig. 96). Of course, it is assumed here that the variable x takes on only those values for which $f(x)$ makes sense.

In other words, for the function $f(x)$ which is continuous for the value x_1, from the fact that $x \to x_1$ it follows:

$$f(x) \to f(x_1).$$

It is easy to see that: (1) if the function $f(x)$ is continuous at $x = x_1$ in the above indicated sense (see Sec. 58), i.e. if

$$\lim_{\Delta x_1 \to 0} [f(x_1 + \Delta x_1) - f(x_1)] = 0, \tag{59-2}$$

then putting

$$x_1 + \Delta x_1 = x,$$

where, obviously, $x \to x_1$ as $\Delta x_1 \to 0$, and, using the theorem on the limit of an algebraic sum, we get

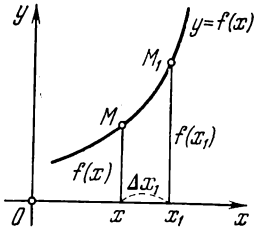


FIG. 96

$$\lim_{x \to x_1} f(x) = f(x_1). \tag{59-3}$$

Consequently, the function $f(x)$ is continuous at $x = x_1$ in our new sense as well.

(2) Obviously, that, conversely, from equality (59-3) there follows equality (59-2).

Hence, the equivalence of the two definitions has been completely proved.

For a function continuous on a set X, by virtue of formula (59-1), *for each value* $x_1 \in X$ the below equality is fulfilled

$$\lim_{x \to x_1} f(x) = f(x_1).$$

Since $x_1 = \lim_{x \to x_1} x$, hence we get

$$\lim_{x \to x_1} f(x) = f\left(\lim_{x \to x_1} x\right), \tag{59-4}$$

i.e. *if a function is continuous, then the symbols of the limit and the function are permutable.*

In detailed courses of analysis it is proved that formula (59-4) remains true for any continuous function $x = \varphi(t)$ such that $\varphi(t) \to x_1$ as $t \to t_1$. Thus, we have a *strong property of permutability of the function* $f(x)$ *and the limit*:

$$\lim_{t \to t_1} f(\varphi(t)) = f\left(\lim_{t \to t_1} \varphi(t)\right). \tag{59-5}$$

From definition 3 of the preceding section it follows that a *function is discontinuous at a given point if and only if either* (1) *there exists no limit of the function at this point, or* (2) *the limit of the function exists at the given point but it does not coincide with the value of the function at this point.*

Sec. 60. Continuity of Basic Elementary Functions

(1) Power function

$$y = x^n$$

(n is natural (see Fig. 60)) is continuous for any value of x (see Sec. 58, Example 2).

(2) Exponential function

$$y = a^x \qquad (a > 0)$$

(see Fig. 63) is continuous for any value of x*.

(3) Trigonometric function

$$y = \sin x$$

is continuous for each value of x (see Fig. 65).

Indeed, assigning an increment Δx to the argument x and denoting by Δy the corresponding increment of the function y, we have

$$y + \Delta y = \sin (x + \Delta x);$$

hence

$$\Delta y = \sin (x + \Delta x) - \sin x = 2 \sin \frac{\Delta x}{2} \cdot \cos \left(x + \frac{\Delta x}{2} \right).$$

By the note to the theorem from Sec. 54 for $\Delta x \neq 0$ we have

$$\left| \sin \frac{\Delta x}{2} \right| < \frac{|\Delta x|}{2};$$

besides,

$$\left| \cos \left(x + \frac{\Delta x}{2} \right) \right| \leqslant 1.$$

Therefore

$$|\Delta y| < 2 \cdot \frac{|\Delta x|}{2} \cdot 1 = |\Delta x|,$$

i.e.

$$\lim_{\Delta x \to 0} \Delta y = 0.$$

* The proof of this fact is given, for instance, in *A Course of Mathematical Analysis* by S. M. Nikolsky (Vol. 1), Mir Publishers, Moscow, 1977.

Hence, the function $\sin x$ is continuous within the interval $(-\infty, +\infty)$. It is proved in the same way that

$$y = \cos x$$

is a continuous function within the interval $(-\infty, +\infty)$ (see Fig. 65).

Sec. 61. Basic Theorems on Continuous Functions

Theorem 1. *The sum of a finite number of continuous functions is a continuous function**.

Proof. Indeed, if $f_1(x)$ and $f_2(x)$ are functions continuous on a set X and x_1 is any value from this set, then

$$\lim_{x \to x_1} [f_1(x) + f_2(x)] = \lim_{x \to x_1} f_1(x) + \lim_{x \to x_1} f_2(x) = f_1(x_1) + f_2(x_1),$$

i.e. the limit of the sum as $x \to x_1$ is equal to the value of this sum for $x = x_1$.

Consequently, the function $f_1(x) + f_2(x)$ is also continuous on the set X.

Theorem 2. *The product of a finite number of continuous functions is a continuous function.*

This theorem is proved in an analogous way.

Corollary. *The integral polynomial*

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n$$

is a continuous function.

Theorem 3. *The quotient obtained by the division of one continuous function by another continuous function is a function continuous at all points where the divisor is non-zero.*

The theorem is proved in a similar way.

Corollary. *The fractional rational function*

$$R(x) = \frac{a_0 + a_1 x + \ldots + a_n x^n}{b_0 + b_1 x + \ldots + b_m x^m}$$

is continuous everywhere except for those values of x where the denominator reduces to zero.

Theorem 4. *A continuous function of a continuous function is a continuous function.* In other words, a composite function consisting of continuous functions is continuous.

Proof. Let x_1 be an arbitrary point belonging to the domain of definition of a composite function $f(\varphi(x))$, where the function $u = \varphi(x)$ is continuous at the point x_1, and the function $f(u)$ is continuous at the point $u_1 = \varphi(x_1)$. On the basis of the strong property of permutability of a continuous function and its limit (see Sec. 58),

* It is assumed that all the functions under consideration are defined and continuous on a certain common set X containing no isolated points (for instance, on an open interval (a, b), or on a closed interval $[a, b]$, and so on).

we have

$$\lim_{x \to x_1} f(\varphi(x)) = f\left(\lim_{x \to x_1} \varphi(x)\right) = f(\varphi(x_1)),$$

i.e. the composite function $f(\varphi(x))$ is continuous at the point x_1.

By virtue of Theorem 4, for instance, the functions

$$(\sin x)^2 = \sin^2 x \quad \text{and} \quad \sin(x^2)$$

are continuous, since the functions x^2 and $\sin x$ are continuous.

The functions we are going to consider further on are continuous everywhere, except, possibly, for some individual values of the argument.

For instance, the function

$$\tan x = \frac{\sin x}{\cos x}$$

(see Fig. 66), by virtue of Theorem 3 of the present section, is continuous for all values of the argument x, except those for which

$$\cos x = 0,$$

i.e. except for the values $x = (2k - 1)\frac{\pi}{2}$, where k is any integer.

Analogously, the function

$$\cot x = \frac{\cos x}{\sin x}$$

(see Fig. 66) is continuous for $\sin x \neq 0$, i.e. for $x \neq k\pi$ (k is an integer).

The theorem on the continuity of an inverse function is also true. We introduce it without proof.

Theorem 5. *If a function $y = f(x)$ is continuous and strictly monotone* on the interval $\langle a, b \rangle$, then there exists a single-valued inverse function $x = \varphi(y)$ defined on the interval $\langle f(a), f(b) \rangle$, the latter being continuous and monotone in the same sense.*

By this theorem: the radical $\sqrt[n]{x}$ (n—natural) (see Fig. 62), logarithmic function $\log_a x$ ($a > 0$, $a \neq 1$) (see Fig. 64), principal values of the inverse trigonometric functions $\arcsin x$, $\arccos x$, $\arctan x$, $\operatorname{arccot} x$ (see Figs. 67-70) are continuous for any value of the argument x for which these functions are defined.

Sec. 62. Evaluation of Indeterminacies

It may happen that a function $f(x)$ is defined and continuous everywhere, except for some value $x = x_1$ for which the function $f(x)$ loses its sense (becomes indeterminate). There arises a question: Is

* That is $f(x)$ either strictly increases or strictly decreases on $\langle a, b \rangle$.

it possible to choose the number $f(x_1)$ so as to make the supplemented function $f(x)$ continuous for $x = x_1$?

By virtue of the above considered theorems, to do so it is necessary and sufficient that the equality

$$f(x_1) = \lim_{x \to x_1} f(x)$$

be fulfilled.

In this case the operation of finding the limit of the function $f(x)$ as $x \to x_1$ is called the *evaluation of indeterminacy*, and the limit itself

$$\lim_{x \to x_1} f(x),$$

if any, is termed the *true value* of the function $f(x)$ for $x = x_1$.

Example 1. Let

$$f(x) = \frac{x^2 - 4}{x - 2}.$$

The function loses sense for $x = 2$. Putting additionally

$$f(2) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} (x + 2) = 4,$$

we get a function which is continuous everywhere, for $x = 2$ as well.

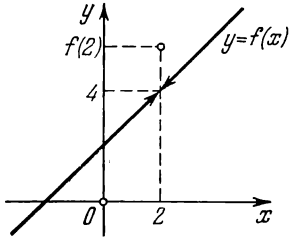


FIG. 97

If we put $f(2) \neq 4$, then the corresponding function will be discontinuous at $x = 2$ (Fig. 97).

Example 2. The function

$$f(x) = \frac{\sin x}{x}$$

is not defined for $x = 0$. Putting additionally

$$f(0) = \lim_{x \to 0} \frac{\sin x}{x} = 1,$$

we get a function defined and continuous for all values of the argument x.

Sec. 63. Classification of the Points of Discontinuity of a Function

A point x_0 of discontinuity of a function $f(x)$ is called the *point of discontinuity of the first kind* if there exist finite one-sided limits of the function (see Sec. 61) (Fig. 87):

$$\lim_{x \to x_0 - 0} f(x) = f(x_0 - 0), \qquad \lim_{x \to x_0 + 0} f(x) = f(x_0 + 0)$$

(in this case the function $f(x)$ should not necessarily be defined at the point x_0, i.e. $f(x_0)$ may not exist).

The magnitude

$$\delta = f(x_0 + 0) - f(x_0 - 0)$$

is called the *jump of the function* $f(x)$ at the point x_0.

All other points of discontinuity x_1 of the function $f(x)$ are said to be its *points of discontinuity of the second kind.* Among them, an important role is played by points of *infinite discontinuity* (so-called infinite discontinuities) x_1 for which there exist (finite or infinite) one-sided limits

$$\lim_{x \to x_1 - 0} f(x) \quad \text{and} \quad \lim_{x \to x_1 + 0} f(x)$$

at least one of them being infinite (see, for instance, Fig. 98).

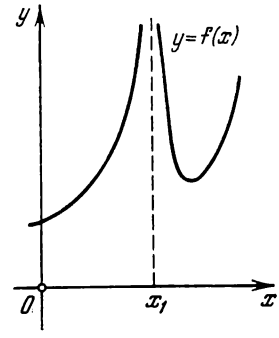


FIG. 98

In this case the straight line $x = x_1$ is called the *vertical asymptote* of the graph of the function $y = f(x)$.

A function having on a given interval a finite number of points of discontinuity of the first kind only is said to be *piecewise continuous.* Note that at the points of discontinuity a piecewise continuous function may not be defined.

It should be borne in mind that for the function $f(x)$ to be continuous at the point x_0, it is necessary and sufficient that the numbers

$$f(x_0 - 0) = f(x_0 + 0) = f(x_0)$$

be equal (i.e. that the jump of the function at the point x_0 be equal to zero).

Example. Determine the character of the point of discontinuity $x_0 = 0$ of the function

$$f(x) = \operatorname{arccot} \frac{1}{x}.$$

Here we have

$$\lim_{x \to -0} \operatorname{arccot} \frac{1}{x} = \pi \quad \text{and} \quad \lim_{x \to +0} \operatorname{arccot} \frac{1}{x} = 0.$$

Hence, $x_0 = 0$ is a point of discontinuity of the first kind.

Exercises

1. Determine the increment of the argument x and the increment of the function

$$y = \log x$$

if the argument x varies from 10 to 100.

2. Show that for the linear function

$$y = ax + b$$

the increment Δy is independent of x.

3. Prove the continuity of the function

$$y = \sqrt{x}.$$

4. Prove that the function $y = |x|$ is continuous.

In Exercises 5 to 7 determine the points of discontinuity of the given functions:

5. $f(x) = \dfrac{x^3}{(x+1)(x+2)(x+3)}$.

6. $f(x) = \tan\left(2x + \dfrac{\pi}{4}\right)$.

7. $f(x) = \dfrac{1}{\sin \pi x}$.

In Exercises 8 to 10 find the "true values" of the given functions:

8. $f(x) = \dfrac{x^3 - 1}{x - 1}$ for $x = 1$.

9. $f(x) = \dfrac{2 - \sqrt{x}}{4 - x}$ for $x = 4$.

10. $f(x) = \dfrac{1 - \cos 2x}{x^2}$ for $x = 0$.

11. Determine the points of discontinuity of the given functions and find out the character of these points: (a) $\dfrac{\sqrt{\sin(x^2)}}{x}$; (b) $\arctan \dfrac{1}{1-x}$; (c) $\sin \dfrac{\pi}{x}$.

Chapter 9

The Derivative of a Function

Sec. 64. A Tangent to a Curve

Let M be a fixed point of a given continuous curved line K (Fig. 99). Consider the secant MM' passing through the point M. It may happen that as the point M' moving along the curve indefinitely approaches the point M, the secant MM' tends to occupy a certain limiting position MT, i.e. the angle $\gamma = \angle M'MT \to 0$ as $M' \to M$. Then MT is called the *tangent line* (or, briefly, the *tangent*).

Definition. *The tangent line MT to a given continuous curve K at its given point M (the point of tangency) is the straight line occupying the limiting position of the secant line MM' passing through the point M and another point M' of the curve as M' moves along the curve to coincide with the given point M.*

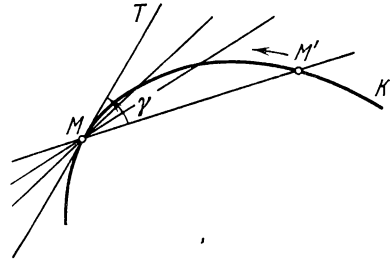


FIG. 99

If the secant line MM', as $M' \to M$, has no limiting position, we say that there exists no tangent line at the point M.

We are going now to show how the equation of the tangent line is found from the given equation of the curve.

Problem. *Knowing the equation of a continuous line*

$$y = f(x),$$

find the equation of the tangent line at its given point $M(x, y)$, assuming that such a tangent line exists.

Along with the point $M(x, y)$, let us take another point $M'(x + \Delta x, y + \Delta y)$ on our line (Fig. 100). On drawing the secant line MM' and straight lines MN and $M'N$ respectively parallel to the x- and y-axes, we get a right-angled triangle MNM' with the legs $MN = \Delta x$ and $NM' = \Delta y$.

Let the secant line MM' form an angle φ with the positive direction of the x-axis. Then, obviously, $\angle NMM' = \varphi$. From the right-angled triangle MNM' we determine the *slope of the secant lines*:

$$k' = \tan \varphi = \frac{\Delta y}{\Delta x}. \tag{64-1}$$

Let now $M' \to M$; then, obviously, $\Delta x \to 0$ and the secant line MM' tends to its limiting position to coincide with the *tangent line* MT at point M (we assume here that the tangent is existent). Let us denote by α the angle made by the tangent MT with the positive direction of the x-axis. As $\Delta x \to 0$, we shall have: $\varphi \to \alpha$ and if the tangent line MT is not perpendicular to the x-axis, then, by virtue of the continuity of tangent, we get

$$\tan \varphi \to \tan \alpha,$$

hence, proceeding to the limit as $\Delta x \to 0$ in equality (64-1), we find the *slope* $k = \tan \alpha$ of the *tangent line* MT:

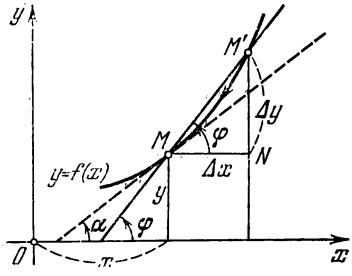


FIG. 100

$$k = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{64-2}$$

The limit standing in the right-hand side of equality (64-2) is called the *derivative* of the function $y = f(x)$ at point x and is denoted in the following way:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y' = f'(x) \tag{64-3}$$

(y' is read: "wai prime").

Thus, *the slope of the tangent to the graph of a function is equal to the value of its derivative at the point of tangency*, i.e.

$$k = f'(x). \tag{64-4}$$

Knowing the slope of a derivative, it is easy to write its equation. The tangent MT passes through the point of tangency $M(x, y)$; therefore its equation (see Sec. 16, Chapter 3) has the form

$$Y - y = k(X - x),$$

where X and Y are current coordinates. Substituting the value of the slope k and taking into consideration that the point M lies on the curve, we get the equation of the tangent to this curve:

$$Y - f(x) = f'(x)(X - x) \tag{64-5}$$

Note 1. If, for clearness, we denote the coordinates of the point of tangency by (x_1, y_1), and the current coordinates, as usually, by (x, y), then the *equation of the tangent* to the curve $y = f(x)$ at the point $M_1(x_1, y_1)$ has the form

$$y - y_1 = y_1'(x - x_1), \tag{64-5'}$$

where $y_1 = f(x_1)$ and $y_1' = f'(x_1)$.

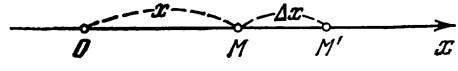
Note 2. When deriving the equation we assumed that there exists a tangent MT to the curve $y = f(x)$ at the point M. Conversely, it is easy to show that if for the function $y = f(x)$ there exists at point x a finite derivative, i.e. limit (64-3) (such function is called *differentiable at the point* x), then the graph of this function has a tangent (64-5) at the corresponding point which is not parallel to the y-axis.

Sec. 65. Velocity of a Moving Point

The problem on computing the velocity of nonuniform motion also leads to the concept of derivative.

Let us assume that a point M moves along a straight line which will be taken for the x-axis (Fig. 101). To each moment of time t

FIG. 101



there corresponds a definite distance $OM = x$. Hence, we may say that the abscissa x of a moving point is a function of time t:

$$x = f(t).$$

This equation is called the *equation of motion.* It expresses the *law of motion* of a point.

Problem. *Knowing the law of motion, find the velocity of a moving point for any moment of time.*

Let at some moment of time t the moving point occupy the position M, and $OM = x$. By the moment $t + \Delta t$ the point will occupy the position M', where $OM' = x + \Delta x$. Hence

$$x + \Delta x = f(t + \Delta t).$$

Consequently, the displacement of the point M for the time Δt will be

$$\Delta x = f(t + \Delta t) - f(t). \tag{65-1}$$

If during the interval of time $[t, t + \Delta t]$ the point M kept moving in one and the same direction, then Δx represents numerically the

path covered by the point for the time Δt*. The ratio

$$\frac{\Delta x}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} \tag{65-2}$$

expresses the average (mean) rate of change of the abscissa x during the time interval Δt which is usually termed the *average (mean) velocity of motion* of the point. The limit of the average velocity of motion, as Δt tends to zero, is called the *velocity of motion* at a given moment of time t. Denoting this velocity by v, we get

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

or

$$v = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \tag{65-3}$$

By analogy with the problem on the tangent line (see the preceding section), we may say that expression (65-3) represents the *derivative* of the function x with respect to the variable t, i.e.

$$v = f'(t).$$

Thus, *the velocity of rectilinear motion is equal to the derivative of path with respect to time***.*

Note. If $v = f'(t)$ preserves its constant sign over some interval $a < t < b$, then we can prove (see Sec. 86, Chapter 11) that for any moment $t \in (a, b)$ during a sufficiently small interval $[t, t+\Delta t]$ the point moves in one and the same direction. Thus, Δx represents the path covered by the point, and the above statement is locally strict (i.e. for a sufficiently small interval of time).

But if for some moment t_1 we have $f'(t_1)=0$, i.e. if for an infinitely small interval of time $\Delta t_1 = t - t_1$ the corresponding displacement Δx of the point is an infinitesimal of a higher order, then Δx is, generally speaking, not equal to the path covered by the point. Such a situation, for instance, takes place when a point performs rapidly damped oscillations about its position of equilibrium. In this case formula (65-3) is not adequate to our definition.

---

* In the general case, the displacement of a point *differs* from the distance covered by the point. For instance, if during the first second of movement the point displaces 10 m to the right and during the next second 10 m to the left, then the displacement of the point during the time interval $\Delta t = 2$ s will be equal to $\Delta x = 0$; whereas the distance covered by the point $s = 20$ m.

** More precisely: *the velocity is equal to the derivative of the abscissa of a moving point with respect to time.*

Sec. 66. The Derivative Defined Generally

In this section we develop the concept of the derivative which is one of the basic concepts of higher mathematics. This concept historically originated from two seemingly unrelated problems: how to draw a tangent line to a curve and how to compute the velocity of a moving body.

When solving these problems, we, generally speaking, had to perform one and the same operation: to find the limit of the ratio of the increment of the function to the increment of the argument.

We are going now to consider this problem in the general aspect.

For the sake of simplicity, we first assume that the function under consideration ($y = f(x)$) is defined either on a finite or infinite interval $X = (a, b)$ and is continuous on this interval. Let $x \in (a, b)$ be a fixed point within the interval (a, b). Now we assign to the argument x an increment $\Delta x \neq 0$ such that $x + \Delta x \in (a, b)$, then the function y will get a corresponding increment

$$\Delta y = f(x + \Delta x) - f(x). \tag{66-1}$$

Let us write the ratio

$$\frac{\Delta y}{\Delta x}. \tag{66-2}$$

This ratio shows how many times the increment of the function y is greater than the increment of the argument x on a given interval $[x, x + \Delta x]$; in other words, it yields *the average rate of change of the function y with respect to the argument x on the interval* $[x, x+\Delta x]$.

Let $\Delta x \to 0$; then also $\Delta y \to 0$ (by virtue of the discontinuity of the function y). We denote by $X_1 \subset (a, b)$ the set of points of the interval (a, b) for which the following passage to the limit makes sense

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{66-3}$$

Then the formula

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \qquad (x \in X_1) \tag{66-4}$$

defines a certain function $y' = f'(x)$ which is called the *derivative of the function* $f(x)$.

Definition. *The limit of the ratio of the increment of a given function* $y = f(x)$ *to the increment of the argument as the latter tends to zero is called the derivative of that function (provided that this limit exists).*

Thus, the derivative of the function $f(x)$ is a certain function $f'(x)$ *derived* (i.e. obtained according to definite rules) from the given function.

A function having a derivative on a set X_1 is said to be *differentiable* on this set (see Chapter 12).

If $x \in X_1$ is fixed, then, by (66-4), *the derivative y' represents the rate of change of the function y with respect to the argument x at point x.*

The derivative of a given function $y = f(x)$ is denoted in the following way

$y' = f'(x)$ (introduced by Lagrange*).

The following notations are also used:

$\frac{dy}{dx} = \frac{d}{dx} f(x)$ (suggested by Leibniz**)

(the meaning of this notation will be explained in Chapter 12), and

$\dot{y} = \dot{f}(x)$ (suggested by Newton***).

When it is not clear with respect to what argument (x, t, etc.) the function y is differentiated, for the corresponding derivatives the following symbols are used

y'_x, y'_t, and so on.

Note. Analogously the definition is given for the derivative of the function $y = f(x)$ defined on a set X containing no isolated points. In particular, the function $f(x)$ is differentiable on the segment $[a, b]$ if for each point $x \in [a, b]$ there exists the limit

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x},$$

and it is assumed that $x + \Delta x \in [a, b]$. Here

$$f'(a) = \lim_{\Delta x \to +0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

and

$$f'(b) = \lim_{\Delta x \to -0} \frac{f(b + \Delta x) - f(b)}{\Delta x}.$$

For the value of the derivative $y' = f'(x)$ of the function $y = f(x)$ at a fixed point $x = x_1$ the following notations are used:

$(y')_{x=x_1} = [f'(x)]_{x=x_1} = f'(x_1)$.

Here $f'(x_1)$ is a number.

* Joseph Louis Lagrange (1736-1813), a great French analyst, algebraist, number theorist, probabilist, physicist, and astronomer. Contributed especially to the calculus of variations, analytic mechanics, and astronomy.

** Gottfried Wilhelm von Leibniz (1646-1716), a great German philosopher, analyst, combinatorist, logician, inventor of calculus (independently of Newton) and of many of its symbols.

*** Sir Isaac Newton (1642-1727), a superb English mathematician, physicist, and astronomer. With Aristotle and Gauss, one of the three greatest mathematicians of all time. He and Leibniz invented calculus independently.

Using formula (66-1), we can rewrite the expression for the derivative as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \qquad (66\text{-}5)$$

With the aid of formula (66-5), relying on the theory of limits, we can find the derivatives of functions.

Example 1. Find the derivative of the function $y = x^2$.

Let x be an arbitrarily fixed value of the argument. Assigning to x an increment $\Delta x \neq 0$, we have

$$y + \Delta y = (x + \Delta x)^2.$$

Hence,

$$\Delta y = (x + \Delta x)^2 - x^2 = 2x \cdot \Delta x + (\Delta x)^2$$

and, consequently,

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x.$$

Thus

$$(x^2)' = 2x. \qquad (66\text{-}6)$$

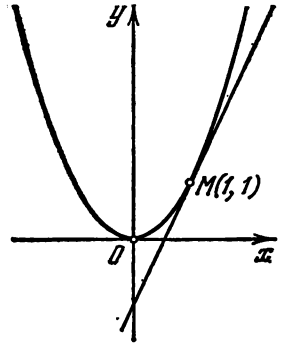


FIG. 102

The geometrical meaning of the derivative was explained in the course of solving the problem how to draw a tangent line to a curve (Sec. 64).

Geometrical meaning of the derivative. *For a given function $y = f(x)$ its derivative $y' = f'(x)$ for each value of x is equal to the slope of the tangent line to the graph of the function at the corresponding point.*

Example 2. Write the equation of the tangent line to the curve $y = x^2$ at the point M (1, 1) (Fig. 102).

We first find the derivative y' for $x = 1$. According to formula (66-6), we have

$$y' = 2x.$$

Hence

$$k = (y')_{x=1} = 2.$$

Consequently, the equation of the tangent line will be written in the following way:

$$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1.$$

Note that the tangent line to the graph of the function $y = f(x)$ forms at a given point an acute or an obtuse angle with the positive direction of the x-axis depending on whether the derivative of the

function at this point is positive or negative. If the derivative is equal to zero, then the tangent to the graph of the function at the corresponding point is, obviously, parallel to the x-axis. The converse assertions are also true.

Further, from the definition of the derivative it follows that the derivative y' gives the rate of change of the function $y = f(x)$ with respect to the argument x. For instance, if at some point x we have $y' = 2$, then this means that on a small interval $[x, x + \Delta x]$ the increment Δy of the function y is approximately two times greater than the increment of the argument x. The less $|\Delta x|$, the more accurate this relation is.

The sense of the derivative of the function $y = f(x)$ becomes especially obvious if the argument x is understood as time. Then the ratio

$$\frac{\Delta y}{\Delta x}$$

represents the *average rate of change of the function y for the interval of time* $[x, x + \Delta x]$, the limit of this ratio

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

being the *rate of change of the function y at the moment x.*

Thus, we have:

Physical meaning of the derivative. *For a function* $y = f(x)$ *varying with time x the derivative* y'_x *is the rate of change of the function y at a given moment x.*

The derivative makes it possible to study the character of change of a function. The greater the absolute value of the derivative, the more abruptly the function y changes as x varies, and, consequently, the steeper the graph of this function rises or falls. If the derivative of some function y is positive, then, obviously, this means that with an increase in the argument x the function y also increases; if the derivative of the function is negative, then this means that as the argument x grows the function y decreases. For more detail see Sec. 86, Chapter 11.

The derivative, this basic concept of calculus, is widely applied in geometry, physics, mechanics, chemistry, biology, and other sciences.

Sec. 67. Other Applications of the Derivative

The rate of physical, chemical, biological, and other processes, for instance, the rate of cooling of a body, reaction rate in chemistry, and so on, is also expresed by means of the derivative. We are going to explain this by giving a couple of examples.

Example 1. Let us assume that the temperature of a body U is a decreasing function of time

$$U = f(t).$$

Let t be a fixed moment of time. If t receives an increment Δt, then the temperature U changes (decreases) by ΔU. In this case the ratio

$$\frac{\Delta U}{\Delta t}$$

represents the *average rate* of cooling of the body, and the limit of this ratio as $\Delta t \to 0$, i.e.

$$\lim_{\Delta t \to 0} \frac{\Delta U}{\Delta t} = f'(t),$$

expresses the rate of cooling of the body at a given moment t.

Thus, *the rate of cooling of a body is equal to the derivative of the temperature of the body with respect to time.*

Example 2. Let us denote by x the amount of substance obtained in the course of a chemical reaction during an interval of time t. Obviously, x is a function of time t:

$$x = f(t).$$

If t receives an increment Δt, then x receives an increment Δx. Then the ratio

$$\frac{\Delta x}{\Delta t}$$

represents the *average rate* of the chemical reaction and the limit

$$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = f'(t)$$

expresses the *reaction rate at a given moment t.*

Thus, *the reaction rate is equal to the derivative of the reacting mass with respect to time.*

Sec. 68. Relation Between the Continuity and Differentiability of a Function

We saw (in Sec. 58) that the function

$$y = f(x) \tag{68-1}$$

is called *continuous at a point x* if at this point

$$\lim_{\Delta x \to 0} \Delta y = 0.$$

Function (68-1) is said to be *differentiable at a point* x if it has a derivative at this point, i.e. if there exists a finite limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y'. \tag{68-2}$$

There is a simple relation between these basic concepts of mathematical analysis.

Theorem. *If a function is differentiable at some point, then the function is continuous at this point. The converse is false: a continuous function can have no derivative.*

Proof. Let the function $y = f(x)$ be differentiable at point x, i.e. equality (68-2) is fulfilled for this function. We write the identity

$$\Delta y = \frac{\Delta y}{\Delta x} \cdot \Delta x \qquad (\Delta x \neq 0).$$

Hence

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta x = y' \cdot 0 = 0.$$

Consequently, the function $y = f(x)$ is continuous at the point x.

Corollary. *If a function is discontinuous at some point, then it has no derivative at this point.*

Continuous functions having no derivative at one point are exemplified by the function

$$y = |x|$$

(Fig. 103). This function is continuous for $x = 0$ but is not differentiable for this value, since there exists no tangent line at the point $x = 0$ of its graph.

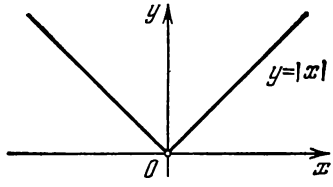


FIG. 103

Mathematicians succeeded in constructing the graphs of some continuous functions differentiable at none of the points (K. Weierstrass and others).

The superb Russian mathematician N. I. Lobachevsky was the first to introduce a distinct difference between the notions of continuity and differentiability of a function.

Note that the derivative $y' = f'(x)$ of a continuous function $y = f(x)$ *is not necessarily continuous.* If a function $f(x)$ has a continuous derivative $f'(x)$ on an interval $\langle a, b \rangle$, then the function is said to be *smooth* on this interval. A function $f(x)$ whose derivative allows only a finite number of points of discontinuity of the first kind on a given interval $\langle a, b \rangle$ is termed *piecewise smooth* on this interval.

Sec. 69. The Notion of an Infinite Derivative

If a function $y = f(x)$ is continuous at the point x_0 and

$$f'(x_0) = \lim_{\Delta x_0 \to 0} \frac{f(x_0 + \Delta x_0) - f(x_0)}{\Delta x_0} = \infty, \tag{69-1}$$

then the function $f(x)$ is said to have an *infinite derivative* at the point $x = x_0$. According to the geometrical meaning of the derivative (see Sec. 64), the derivative $y'_0 = f'(x_0)$ is equal to the slope of the tangent line at the point x_0. Therefore $\tan \alpha = \infty$ and, consequently, $\alpha = \frac{\pi}{2}$. Thus, condition (69-1) geometrically means that the graph of the function $y = f(x)$ has a vertical tangent at the point x_0.

Exercises

1. What is understood under: (a) the average path slope; (b) the path slope at a given point?

2. What is: (a) the average line density of a material bar; (b) the line density of a bar at a given point?

3. What should be understood under: (a) the average rate of change of a sea area; (b) the rate of change of a sea area at a given moment?

4. Give the definition for: (a) the average heat capacity of a body; (b) heat capacity of a body.

Chapter 10

Basic Derivative Theorems

Sec. 70. Introductory Notes

As we saw, the solutions of many problems were reduced to computing the derivatives of known functions. Therefore, it is important to know how to quickly find the derivatives of more or less composite functions.

The process of finding the derivative of a function is called *differentiation*, and a function that has a finite derivative on a given set is said to be *differentiable* on this set. The study of the derivative and its applications constitutes the subject of *differential calculus*.

The present chapter is dedicated to the basic rules for *differentiation of functions*.

Here we shall assume, unless otherwise is stated, that the functions under consideration are defined on a finite or an infinite interval.

Prior to indicating the basic rules for finding derivatives, we are going to calculate the derivatives of some simple functions.

Sec. 71. The Derivatives of Certain Simple Functions

The derivative of a function $y = f(x)$ can be found according to the following scheme:

(1) assign an increment $\Delta x \neq 0$ to the argument x, and find the corresponding terminal value $y + \Delta y = f(x + \Delta x)$ for the function y;

(2) obtain the increment Δy of the function by subtracting its initial value $y = f(x)$ from its new value $y + \Delta y$;

(3) form the ratio $\frac{\Delta y}{\Delta x}$;

(4) find the limit of this ratio provided $\Delta x \to 0$. The result of the passage to the limit $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y'$ is just the derivative y' of the function y with respect to the argument x if it, of course, exists.

Using this scheme, we shall find the derivatives of some simple elementary functions.

I. The derivative of a power function x^m, where m is a positive integer. Let

$$y = x^m.$$

We have

$$y + \Delta y = (x + \Delta x)^m$$

or, according to the binomial formula,

$$y + \Delta y = x^m + mx^{m-1}\Delta x + \frac{m(m-1)}{1 \cdot 2} x^{m-2}(\Delta x)^2 + \ldots + (\Delta x)^m;$$

hence

$$\Delta y = mx^{m-1}\Delta x + \frac{m(m-1)}{1 \cdot 2} x^{m-2}(\Delta x)^2 + \ldots + (\Delta x)^m$$

and

$$\frac{\Delta y}{\Delta x} = mx^{m-1} + \frac{m(m-1)}{1 \cdot 2} x^{m-2}\Delta x + \ldots + (\Delta x)^{m-1}.$$

Proceeding to the limit as $\Delta x \to 0$, we find

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = mx^{m-1}.$$

Consequently,

$$(x^m)' = mx^{m-1}. \tag{71-1}$$

Thus, we have the following theorem: *The derivative of a power function with a positive integral exponent is equal to the product of the exponent by the power function with exponent decreased by one.*

In particular, for $m = 1$ we get

$$(x)' = 1,$$

i.e. *the derivative of the independent variable is equal to unity.* We also have

$$(x^2)' = 2x, \quad (x^3)' = 3x^2$$

and so on.

Note. As it will be shown below (Sec. 79), formula (71-1) holds for any real constant exponent m (in particular, for a fractional one). Therefore, for instance, we have

$$(\sqrt{x})' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \qquad (x > 0);$$

i.e. *the derivative of a square root of an independent variable is equal to the inverse of the doubled root.*

For $x = 0$ the function $y = \sqrt{x}$ has the derivative $y' = \infty$. Here the derivative is *one-sided*, since $x \to +0$. Geometrically this means that the tangent line to the parabola $y = \sqrt{x}$ at the point $x = 0$ is perpendicular to the x-axis.

II. The derivative of sin x. Let

$$y = \sin x,$$

where the argument x is expressed in radians.

We have

$$y + \Delta y = \sin (x + \Delta x).$$

Hence

$$\Delta y = \sin (x + \Delta x) - \sin x$$

or

$$\Delta y = 2 \sin \frac{\Delta x}{2} \cdot \cos \left(x + \frac{\Delta x}{2} \right).$$

Dividing both sides of the last equality by Δx, we get

$$\frac{\Delta y}{\Delta x} = \frac{2 \sin \frac{\Delta x}{2}}{\Delta x} \cdot \cos \left(x + \frac{\Delta x}{2} \right),$$

or

$$\frac{\Delta y}{\Delta x} = \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \cos \left(x + \frac{\Delta x}{2} \right).$$

Proceeding to the limit as $\Delta x \to 0$ and using the theorem on the limit of a product, we have

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \lim_{\Delta x \to 0} \cos \left(x + \frac{\Delta x}{2} \right).$$

From the theorem on the limit of the ratio of the sine of an infinitely small arc to the arc itself (Sec. 54, Chapter 7) it follows that

$$\lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1.$$

Besides, by virtue of the continuity of the function $\cos x$, we have

$$\lim_{\Delta x \to 0} \cos\left(x + \frac{\Delta x}{2}\right) = \cos\left[\lim_{\Delta x \to 0}\left(x + \frac{\Delta x}{2}\right)\right] = \cos x.$$

Consequently,

$$y' = 1 \cdot \cos x = \cos x,$$

i.e.

$$(\sin x)' = \cos x. \tag{71-2}$$

Thus, we obtain the following theorem: *The derivative of* $\sin x$ *is equal to* $\cos x$.

III. The derivative of $\cos x$. Let

$$y = \cos x.$$

Then

$$y + \Delta y = \cos(x + \Delta x)$$

and, hence,

$$\Delta y = \cos(x + \Delta x) - \cos x$$

or

$$\Delta y = -2 \sin \frac{\Delta x}{2} \cdot \sin\left(x + \frac{\Delta x}{2}\right).$$

Whence

$$\frac{\Delta y}{\Delta x} = -\frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \sin\left(x + \frac{\Delta x}{2}\right).$$

Passing to the limit, as $\Delta x \to 0$, we get

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \cdot \lim_{\Delta x \to 0} \sin\left(x + \frac{\Delta x}{2}\right).$$

Since

$$\lim_{\Delta x \to 0} \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1$$

and

$$\lim_{\Delta x \to 0} \sin\left(x + \frac{\Delta x}{2}\right) = \sin x,$$

we finally find

$$y' = -\sin x,$$

i.e.

$$(\cos x)' = -\sin x. \tag{71-3}$$

Thus, we have the following theorem: *The derivative of* $\cos x$ *is equal to* $\sin x$ *taken with the opposite sign.*

Sec. 72. Basic Differentiation Rules

We now pass over to deriving the basic rules for differentiation of various functions.

Let us assume that all the functions under consideration are defined and differentiable on a common interval, and all used values of the argument x and new values $x + \Delta x$ belong to this interval.

I. The derivative of a constant. *The derivative of a constant quantity is equal to zero.*

A constant quantity c may be considered as a function

$$f(x) = c,$$

taking on one and the same value.

Let us assign an increment $\Delta x \neq 0$ to the argument x. Then, since the function $f(x)$ remains unchanged as the argument varies, we shall have

$$f(x + \Delta x) = c.$$

Subtracting the first equality from the second termwise, we get

$$f(x + \Delta x) - f(x) = 0;$$

whence

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = 0.$$

Passing now to the limit as $\Delta x \to 0$, we find

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 0,$$

i.e.

$$c' = 0. \tag{72-1}$$

Using the language of mechanics, we get an obvious interpretation of our theorem: *The velocity of a point at rest is equal to zero.*

II. The derivative of a sum. *The derivative of an algebraic sum of a finite number of differentiable functions is equal to the sum of the derivatives of these functions.*

Let, for instance,

$$y = u + v - w,$$

where u, v, and w are differentiable functions of x.

We assign an increment Δx to the argument x; then each function will receive its increment: Δu, Δv, and Δw, respectively. As a result, the function y will receive an increment Δy. Hence, we have

$$y + \Delta y = (u + \Delta u) + (v + \Delta v) - (w + \Delta w).$$

Subtracting the first equality from the second termwise, we find

$$\Delta y = \Delta u + \Delta v - \Delta w.$$

Dividing both sides of the last equality by Δx, we have

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} - \frac{\Delta w}{\Delta x}.$$

Proceeding now to the limit, as $\Delta x \to 0$, and taking into account that each term of the right-hand member has a limit, we find

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} - \lim_{\Delta x \to 0} \frac{\Delta w}{\Delta x},$$

or, bearing in mind the definition of the derivative, we finally get

$$y' = u' + v' - w'.$$

Thus, *if each of the functions u, v and w is differentiable, then an algebraic sum of these functions* (for instance, $u + v - w$) *is also differentiable, and*

$$(u + v - w)' = u' + v' - w'. \qquad (72\text{-}2)$$

Example 1. Find the derivative of the function

$$y = 2 - x + x^2.$$

Applying formula (71-1), we have

$$y' = (2)' - (x)' + (x^2)' = -1 + 2x.$$

Corollary. *If two differentiable functions differ from each other by a constant term, then their derivatives are equal to each other.*

Indeed, if $f(x)$ is a differentiable function and c is a constant term, then we have

$$[f(x) + c]' = f'(x) + (c)' = f'(x) + 0 = f'(x).$$

III. The derivative of a product. *The derivative of a product of two differentiable functions is equal to the product of the first factor by the derivative of the second plus the product of the second factor by the derivative of the first.*

Let

$$y = uv,$$

where u and v are some differentiable functions of x. We assign an increment Δx to x; then u will get an increment Δu, v an increment Δv, and y an increment Δy. We have

$$y + \Delta y = (u + \Delta u)(v + \Delta v),$$

or

$$y + \Delta y = uv + u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v.$$

Consequently,

$$\Delta y = u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v.$$

Hence

$$\frac{\Delta y}{\Delta x} = u \cdot \frac{\Delta v}{\Delta x} + v \cdot \frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x} \cdot \frac{\Delta v}{\Delta x} \cdot \Delta x.$$

Passing in the last equality to the limit as $\Delta x \to 0$ and bearing in mind that u and v are independent of Δx, we shall have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = u \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \cdot \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta x$$

or

$$y' = uv' + vu'.$$

Thus, *if either of the factors u and v has a derivative, then their product uv also has a derivative*; thus

$$(uv)' = uv' + vu'. \tag{72-3}$$

Example 2. Let

$$y = x^3 \sin x.$$

Using formula (72-3) and applying formulas (71-1) and (71-2), we have

$$y' = x^3 (\sin x)' + (x^3)' \sin x = x^3 \cos x + 3x^2 \sin x.$$

Corollary 1. *A constant may be factored out in front of the derivative sign.*

Indeed, if c is a constant factor, then we have

$$(cu)' = cu' + c'u,$$

hence, since $c' = 0$, we get

$$(cu)' = cu'.$$

Corollary 2. *If*

$$y = uvw,$$

where u, v and w are differentiable functions of x, then

$$y' = (uvw)' = [(uv)\,w]' = (uv)\,w' + (uv)'w = \\ = (uv)\,w' + (uv' + u'v)\,w = u'vw + uv'w + uvw'.$$

In general, *the derivative of a product of several differentiable functions is equal to the sum of the products of the derivative of each of these factors by all remaining ones.*

IV. The derivative of a quotient. *If both the numerator and the denominator of a fraction are differentiable functions and the denominator does not vanish, then the derivative of a fraction is also a fraction whose numerator is equal to the product of the denominator by the derivative of the numerator minus the product of the numerator by the derivative of the denominator, the whole expression divided by the denominator squared.*

Let

$$y = \frac{u}{v},$$

where u and v are differentiable functions of x and $v \neq 0$. Let us assign to the argument x an increment Δx; then u, v, y will receive, respectively, increments Δu, Δv, Δy, and we shall have

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}.$$

Subtracting the first equality from the second termwise, we get

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

or

$$\Delta y = \frac{v\,\Delta u - u\,\Delta v}{(v + \Delta v)\,v}.$$

Hence, we find

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{(v + \Delta v)\,v}. \tag{72-4}$$

Let $\Delta x \to 0$. Since the function v is differentiable at point x, it is continuous at this point (Sec. 68, Chapter 9), and, consequently,

$$\lim_{\Delta x \to 0} \Delta v = 0.$$

Therefore, passing to the limit in equality (72-4) and taking into account that the functions u and v have derivatives, we get

$$y' = \frac{vu' - uv'}{v^2},$$

or finally

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}. \tag{72-5}$$

Example 3. Let

$$y = \frac{x^2-1}{x^2+1};$$

then, applying (72-5), we shall have

$$y' = \frac{(x^2+1)(x^2-1)' - (x^2-1)(x^2+1)'}{(x^2+1)^2} = \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2} =$$

$$= \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

Corollary 1. *If the denominator of a fraction is a constant, then*

$$\left(\frac{u}{c}\right)' = \frac{cu' - uc'}{c^2} = \frac{u'}{c},$$

i.e.

$$\left(\frac{u}{c}\right)' = \frac{u'}{c}.$$

Note. The last result is obvious, since

$$\frac{u}{c} = \frac{1}{c} \cdot u$$

and, consequently,

$$\left(\frac{u}{c}\right)' = \frac{1}{c}u' = \frac{u'}{c}.$$

Corollary 2. *If the numerator of a common fraction is a constant, then*

$$\left(\frac{c}{v}\right)' = \frac{vc' - cv'}{v^2} = -\frac{cv'}{v^2}. \tag{72-6}$$

In particular, for $c = 1$ we find

$$\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}. \tag{72-7}$$

Example 4. If

$$y = \frac{1}{x^2-1},$$

then, by formula (72-7), we have

$$y' = -\frac{(x^2-1)'}{(x^2-1)^2} = -\frac{2x}{(x^2-1)^2}.$$

V. The derivative of a power with a negative integral exponent. Let m be a positive integer and

$$y = x^{-m},$$

or

$$y = \frac{1}{x^m}.$$

Applying formula (72-7) we get:

$$y' = -\frac{(x^m)'}{(x^m)^2} = -\frac{mx^{m-1}}{x^{2m}} = -mx^{-m-1}.$$

Hence,

$$(x^{-m})' = -mx^{-m-1}. \tag{72-8}$$

As we see, the rule is the same as for the differentiation of a power function with a positive integral exponent.

VI. The derivative of tan x. Let

$$y = \tan x = \frac{\sin x}{\cos x}.$$

Making use of formula (72-5), we find

$$y' = \frac{\cos x \cdot (\sin x)' - \sin x \cdot (\cos x)'}{\cos^2 x} = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

And so

$$(\tan x)' = \sec^2 x. \tag{72-9}$$

VII. The derivative of cot x. Let

$$y = \cot x = \frac{\cos x}{\sin x}.$$

Then we have

$$y' = \frac{\sin x \cdot (\cos x)' - \cos x \cdot (\sin x)'}{\sin^2 x} = \frac{\sin x \cdot (-\sin x) - \cos x \cdot \cos x}{\sin^2 x} =$$

$$= -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x.$$

Hence,

$$(\cot x)' = -\csc^2 x. \tag{72-10}$$

Sec. 73. The Derivative of a Composite Function

Consider a composite function

$$y = f[\varphi(x)]. \tag{73-1}$$

If in the chain of functional relations $y = f(z)$ and $z = \varphi(x)$ the argument x is the last one, then we shall call it the *independent*

variable (in order to stress the fact that the variation of this argument is independent of the behaviour of other variables).

Hence, we must distinguish between the notions of argument and independent variable. For instance, let

$$y = \sin z \quad \text{and} \quad z = x^2.$$

Here z is the argument of the function y, but, obviously, it is not an independent variable.

For the sake of simplicity, we assume that the function $y = f(z)$ is defined and differentiable in the interval (A, B), the function $z = \varphi(x)$ is defined and differentiable within the interval (a, b) and takes on values from the interval (A, B). Then function (73-1) is automatically defined and differentiable in the interval (a, b). Now there arises a question whether this function is differentiable.

Theorem. *If $y = f(z)$ and $z = \varphi(x)$ are differentiable functions with respect to their arguments, then the derivative of the composite function*

$$y = f[\varphi(x)]$$

exists and is equal to the product of the given function y with respect to the intermediate argument z by the derivative of the intermediate argument z with respect to the independent variable x, i.e.

$$y'_x = y'_z \cdot z'_x.$$

Proof. Let x be a permissible value of the independent variable. We assign a sufficiently small non-zero increment Δx to x; then the functions $z = \varphi(x)$ and $y = f(z)$ will receive the respective increments Δz and Δy. Since, by hypothesis, the derivative $y'_z = f'(z)$ exists, assuming that $\Delta z \neq 0$, we may write

$$\lim_{\Delta z \to 0} \frac{\Delta y}{\Delta z} = y'_z.$$

Hence

$$\frac{\Delta y}{\Delta z} = y'_z + \alpha$$

(see Sec. 49), where $\alpha \to 0$ as $\Delta z \to 0$ and, consequently,

$$\Delta y = (y'_z + \alpha)\,\Delta z.$$

Let us define the infinitesimal α for $\Delta z = 0$, putting $\alpha = 0$ for $\Delta z = 0$. Then the last equality will also hold for $\Delta z = 0$, since in this case, obviously, both sides of the equality are equal to zero. Dividing it by Δx, we get

$$\frac{\Delta y}{\Delta x} = (y'_z + \alpha) \cdot \frac{\Delta z}{\Delta x}.$$

Passing now to the limit as $\Delta x \to 0$ and taking into account that $\Delta z \to 0$ and, hence, $\alpha \to 0$, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (y'_z + \alpha) \cdot \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x},$$

or

$$y'_x = y'_z \cdot z'_y, \tag{73-2}$$

which was to be proved.

This theorem is briefly formulated in the following way:

A differentiable function of a differentiable function is a function also differentiable.

Note. In the Leibniz notation formula (73-2) takes on the form of an identity

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

Example 1. Find the derivative of the function

$y = \sin(x^2)$.

We put

$z = x^2$,

then

$y = \sin z$.

Hence

$z'_x = 2x$ and $y'_z = \cos z = \cos(x^2)$.

Consequently, in accordance with (73-1), we have

$y'_x = \cos(x^2) \cdot 2x = 2x \cos(x^2)$.

Example 2. Find the derivative of the function

$y = \sin^3 x = (\sin x)^3$.

We put $z = \sin x$; then $y = z^3$. Hence

$z'_x = \cos x$ and $y'_z = 3z^2 = 3 \sin^2 x$.

Consequently,

$y'_x = 3 \sin^2 x \cos x$.

If experienced, one may not write the intermediate variable z, introducing it mentally.

Example 3. Find the derivative of the function

$y = \sqrt{x^2 + 4x + 3}$.

Using the formula for the derivative of a square root (Sec. 71) and applying the rule for differentiating a composite function, we have

$$y' = \frac{1}{2\sqrt{x^2+4x+3}} \cdot (x^2+4x+3)' =$$

$$= \frac{1}{2\sqrt{x^2+4x+3}} \cdot (2x+4) = \frac{x+2}{\sqrt{x^2+4x+3}}.$$

Example 4. Find the derivative of the function

$$y = \tan^3 \frac{x}{2}.$$

We have

$$y' = 3\tan^2 \frac{x}{2} \cdot \left(\tan \frac{x}{2}\right)' = 3\tan^2 \frac{x}{2} \sec^2 \frac{x}{2} \cdot \left(\frac{x}{2}\right)' = \frac{3}{2} \tan^2 \frac{x}{2} \cdot \sec^2 \frac{x}{2}.$$

Sec. 74. The Derivative of an Inverse Function

Let

$$y = f(x) \tag{74-1}$$

be a differentiable function with respect to the argument x in some interval (a, b). If in equation (74-1) we regard y as the argument and x as the function, then this new function

$$x = \varphi(y),$$

where $f[\varphi(y)] \equiv y$, is called, as we know, *inverse* with respect to the given function. Our task is: knowing the derivative $y'_x = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ of the function $y = f(x)$, to find the derivative $x'_y = \lim\limits_{\Delta y \to 0} \frac{\Delta x}{\Delta y}$ of the inverse function $x = \varphi(y)$, assuming that the (latter exists and is continuous on the corresponding interval without resolving equation (74-1)).

Theorem. *For a differentiable function whose derivative is not equal to zero the derivative of the inverse function is equal to the inverse of the derivative of the given function.*

Proof. Let the function $y = f(x)$ be differentiable and $y'_x = f'(x) \neq 0$.

Let $\Delta y \neq 0$ be the increment of the independent variable y and Δx the corresponding increment of the inverse function $x = \varphi(y)$. We write the identity

$$\frac{\Delta x}{\Delta y} = 1 : \frac{\Delta y}{\Delta x}. \tag{74-2}$$

* We can prove that if under our conditions $\Delta y \neq 0$, then $\Delta x \neq 0$. Therefore equality (74-2) cannot lose its sense.

Passing to the limit in equality (74-2) as $\Delta y \to 0$ and taking into consideration that in this case also $\Delta x \to 0$ (by virtue of the continuity of the inverse function), we get

$$\lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = 1 : \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Hence

$$x'_y = \frac{1}{y'_x}, \tag{74-3}$$

where x'_y is the derivative of the inverse function.

Note. In the Leibniz notation formula (74-3) takes the form

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

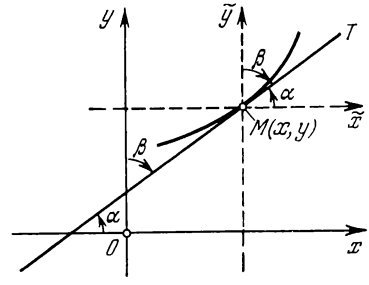


FIG. 104

Recalling the geometrical meaning of the derivative, we may give a simple illustration of formula (74-3). At the point $M(x, y)$ of the graph of the function $y = f(x)$ we construct a tangent line MT to this graph and draw straight lines $M\tilde{x}$ and $M\tilde{y}$ parallel to the x- and y-axes, respectively (Fig. 104). Denoting by α and β the angles formed by the tangent MT with the positive directions of the x- and y-axes, respectively, we have

$$\tan \alpha = \tan (\angle TM\tilde{x}) = y'_x$$

and

$$\tan \beta = \tan (\angle TM\tilde{y}) = x'_y.$$

Since

$$\alpha + \beta = \frac{\pi}{2},$$

hence it follows

$$\tan \alpha \cdot \tan \beta = y'_x \cdot x'_y = 1,$$

which is equivalent to formula (2).

Example. Let

$$y = x + x^3.$$

We have

$$y'_x = 1 + 3x^2,$$

and, consequently,

$$x'_y = \frac{1}{1+3x^2}.$$

Sec. 75. The Derivative of an Implicit Function

Let us consider several examples of differentiation of *implicit functions.*

Example 1. Find the derivative of the function y ($y > 0$) defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Resolving this equation with respect to y and taking the plus sign, by hypothesis, we shall obtain our function in the explicit form:

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Its differentiation is rather simple now.

But in some cases the given equation cannot be resolved with respect to y by elementary means and we have to regard y as an implicit function of x. Therefore we are going to indicate another method of finding the derivative of the implicit function y. Namely, assuming that instead of y its explicit expression is substituted in the given equation, we get an identity

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \equiv 1,$$

where y is a function of x. Obviously, if two functions are identically equal to each other, then their derivatives are also equal to each other. Therefore, taking the derivatives of both sides of the foregoing identity and applying the rule for differentiation of a composite function (see Sec. 73), we have

$$\frac{2x}{a^2} + \frac{2y}{b^2} y' = 0;$$

hence

$$y' = -\frac{b^2 x}{a^2 y}.$$

Example 2. Let y be a positive or a negative implicit function of x defined by the equation

$$y^2 = 2px.$$

Assuming that y is substituted by the corresponding explicit expression and differentiating with respect to x the left- and right-hand sides of the obtained identity, we have $2yy' = 2p$.

Hence

$$y' = \frac{p}{y}.$$

Note. If two functions $\varphi(x)$ and $\psi(x)$ are equal to each other not identically but for a certain value x_0 of the argument

$$\varphi(x_0) = \psi(x_0),$$

then from this, generally speaking, it does not follow that

$$\varphi'(x_0) = \psi'(x_0).$$

This is clearly seen from Fig. 105, where

$$\varphi'(x_0) = \tan \alpha_0 \quad \text{and}$$

$$\psi'(x_0) = \tan \beta_0.$$

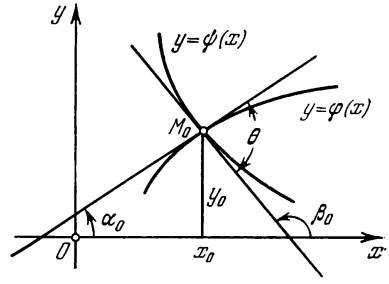


FIG. 105

Thus, *in the general case one cannot differentiate an equality termwise.*

Sec. 76. The Derivative of a Logarithmic Function

Let

$$y = \log_a x,$$

where $x > 0$ ($a > 0$, $a \neq 1$). It is required to find the derivative of this function making use of the scheme set forth in Sec. 71.

We assign to the argument x (x is fixed) an increment $\Delta x \neq 0$ such that $x + \Delta x > 0$. Then the function y will receive an increment Δy, and we shall have

$$y + \Delta y = \log_a (x + \Delta x);$$

hence

$$\Delta y = \log_a (x + \Delta x) - \log_a x$$

or, since the difference between two logarithms is equal to the logarithm of the quotient, we have

$$\Delta y = \log_a \left(1 + \frac{\Delta x}{x}\right).$$

Dividing both members of the last equality by Δx, we get

$$\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x} \log_a \left(1 + \frac{\Delta x}{x}\right).$$

Putting here

$$\frac{\Delta x}{x} = \alpha$$

($\alpha > -1$), we find

$$\frac{\Delta y}{\Delta x} = \frac{1}{x} \cdot \frac{1}{\alpha} \log_a (1 + \alpha),$$

or, by the known property of the logarithm:

$$\frac{\Delta y}{\Delta x} = \frac{1}{x} \cdot \log_a (1 + \alpha)^{\frac{1}{\alpha}}.$$

Let $\Delta x \to 0$, then, obviously, $\alpha \to 0$ (as the product of an infinitesimal Δx by a constant quantity $\frac{1}{x}$). Therefore

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{x} \cdot \lim_{\alpha \to 0} [\log_a (1 + \alpha)^{\frac{1}{\alpha}}].$$

The function $F(\alpha) = (1 + \alpha)^{\frac{1}{\alpha}}$ is, obviously, continuous for $\alpha \neq 0$. Since the logarithmic function is also continuous and

$$\lim_{\alpha \to 0} (1 + \alpha)^{\frac{1}{\alpha}} = e$$

(Sec. 55, Chapter 7), the symbols $\lim_{\alpha \to 0}$ and $\log_a$ may be permutated (see Sec. 59, Chapter 8)*

$$\lim_{\alpha \to 0} [\log_a (1 + \alpha)^{\frac{1}{\alpha}}] = \log_a [\lim_{\alpha \to 0} (1 + \alpha)^{\frac{1}{\alpha}}] = \log_a e.$$

Hence,

$$y' = \frac{1}{x} \log_a e.$$

Thus, we have

$$(\log_a x)' = \frac{1}{x} \log_a e. \tag{76-1}$$

Using the known relation (from Sec. 56)

$$\log_a e = \frac{1}{\ln a},$$

* Here we define by continuity the function $F(\alpha) = (1 + \alpha)^{1/\alpha}$ for $\alpha = 0$, setting $F(0) = e$.

formula (76-1) can be rewritten in the form

$$(\log_a x)' = \frac{1}{x \ln a}. \qquad (76\text{-}1')$$

Putting here, in particular, $a = e$ and bearing in mind that $\ln e = 1$, we get

$$(\ln x)' = \frac{1}{x}, \qquad (76\text{-}2)$$

i.e. *the derivative of the natural logarithm of an independent variable is equal to the inverse of this independent variable.*

Another particular case of importance is obtained for $a = 10$:

$$(\log x)' = \frac{M}{x},$$

where $M = \log e = 0.43429$ is the modulus of common logarithms.

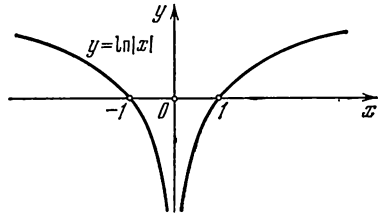


FIG. 106

The logarithmic function $y = \ln x$ is defined only for $x > 0$. For applications it is convenient to consider the function

$$y = \ln |x|,$$

which makes sense both for positive and for negative x, i.e. it is defined for $x \neq 0$ (Fig. 106). To find the derivative of this function let us represent it with the aid of two equalities:

$$y = \ln x \quad \text{for} \quad x > 0$$

and

$$y = \ln (-x) \quad \text{for} \quad x < 0.$$

Whence we get

$$y' = \frac{1}{x} \quad \text{for} \quad x > 0$$

and

$$y' = \frac{1}{-x} \cdot (-x)' = -\frac{1}{x} \cdot (-1) = \frac{1}{x} \quad \text{for} \quad x < 0.$$

Hence,

$$y' = \frac{1}{x} \quad \text{for} \quad x \neq 0,$$

i.e.

$$(\ln |x|)' = \frac{1}{x}.$$

Sec. 77. The Concept of a Logarithmic Derivative

Let

$$y = \ln z,$$

where

$$z = \varphi(x).$$

Then, applying the formula for differentiating a composite function, we get

$$y'_x = (\ln z)'_x = (\ln z)'_z z'_x,$$

or

$$y'_x = \frac{1}{z} z'_x.$$

Thus, we have

$$(\ln z)'_x = \frac{z'}{z}.$$

The derivative of the logarithm of a function is called the *logarithmic derivative* of the function.

Example. Find the derivative of the function

$$y = \ln (x^2 + 4x + 5).$$

Applying the last formula, we have

$$y' = \frac{(x^2+4x+5)'}{x^2+4x+5} = \frac{2x+4}{x^2+4x+5}.$$

Sec. 78. The Derivative of an Exponential Function

Let

$$y = a^x, \quad \text{where} \quad a > 0.$$

Then

$$\ln y = x \ln a.$$

Taking the derivatives of both sides with respect to x, we have*

$$\frac{1}{y} y' = \ln a;$$

* The existence of y' follows from the differentiability of a logarithmic function (see Secs. 76 and 74).

hence

$$y' = y \ln a,$$

or, finally,

$$(a^x)' = a^x \ln a. \tag{78-1}$$

Thus, *the derivative of an exponential function is equal to the function itself multiplied by the natural logarithm of the base.*

Example. Find the derivative of the function

$$y = 2^x.$$

By formula (78-1) we have

$$(2^x)' = 2^x \ln 2.$$

Putting in formula (78-1), in particular, $a = e$, we get

$$(e^x)' = e^x,$$

i.e. *the derivative of the exponential function e^x is equal to the function*

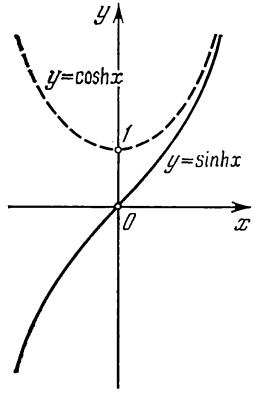


FIG. 107

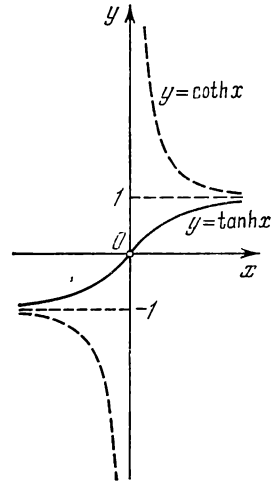


FIG. 108

itself. In this respect the function e^x is the simplest function in mathematical analysis.

In applications we often come across *hyperbolic functions* which are formally determined by the equalities

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2} \tag{78-2}$$

(Fig. 107) and

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \tag{78-3}$$

(Fig. 108). From formulas (78-2) we get the *principal relationship*

$$\cosh^2 x - \sinh^2 x = 1.$$

On the basis of formulas (78-2) and (78-3) we directly find the derivatives of the hyperbolic functions

$$(\cosh x)' = \sinh x, \qquad (\sinh x)' = \cosh x,$$
$$(\tanh x)' = \frac{1}{\cosh^2 x}, \qquad (\coth x)' = -\frac{1}{\sinh^2 x}.$$

Sec. 79. The Derivative of a Power Function

Consider the power function

$$y = x^\alpha \qquad (x > 0), \tag{79-1}$$

where α is any real number.

Taking the logarithm of both sides of (79-1), we get

$$\ln y = \alpha \ln x. \tag{79-2}$$

Hence

$$y = e^{\alpha \ln x}.$$

Therefore, by the theorem on the derivative of a composite function (Sec. 73), the power function y is differentiable.

Differentiating equality (79-2), we have

$$\frac{y'}{y} = \alpha \cdot \frac{1}{x}.$$

Hence

$$y' = \alpha \frac{y}{x} = \alpha x^{\alpha - 1}.$$

Thus, we obtain a *general rule* for differentiating a power function:

$$(x^\alpha)' = \alpha x^{\alpha - 1}, \tag{79-3}$$

i.e. *the derivative of* $y = x^n$ *is equal to* nx^{n-1}.

If power function (79-1) makes sense for $x \leqslant 0$, then formula (79-3) will also hold for $x \leqslant 0$.

Sec. 80. The Derivatives of Inverse Trigonometric Functions

The functions inverse to trigonometric functions are called the *inverse trigonometric* or the *inverse circular* functions (Arcsin x, Arccos x, Arctan x, Arccot x, etc.).

The principal values of the inverse trigonometric functions are obtained as a result of inversion of differentiable (with a non-zero derivative in the corresponding domain) trigonometric functions and, consequently, by virtue of the theorem about the derivative of an inverse function (Sec. 74) they are also *differentiable*. Let us find their derivatives.

I. The derivative of arcsin x. Let

$$y = \arcsin x, \tag{80-1}$$

where $-1 \leqslant x \leqslant 1$ and $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$ (see Sec. 42). The inverse function has the form

$$x = \sin y, \tag{80-2}$$

and

$$x'_y = \cos y \neq 0,$$

if $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Applying the rule for differentiation of an inverse function (Sec. 74), we get

$$y'_x = \frac{1}{x'_y} = \frac{1}{\cos y}. \tag{80-3}$$

Since $\cos y > 0$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$, then taking into account (80-2), we obtain

$$\cos y = +\sqrt{1-\sin^2 y} = \sqrt{1-x^2} > 0, \qquad -1 < x < 1.$$

Consequently, on the basis of (80-3), we have

$$y'_x = \frac{1}{\sqrt{1-x^2}},$$

i.e.

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}. \tag{80-4}$$

From formula (80-4) it follows that curve (80-1) for $x = \pm 1$ has vertical tangents.

II. The derivative of arccos x. Let

$$y = \arccos x,$$

then

$$x = \cos y,$$

where $-1 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant \pi$.

According to the rule for differentiation of an inverse function (Sec. 74), we have

$$y'_x = \frac{1}{x'_y} = -\frac{1}{\sin y}.$$

Since $\sin y > 0$ for $0 < y < \pi$ we get

$$\sin y = +\sqrt{1-\cos^2 y} = \sqrt{1-x^2} > 0 \qquad (-1 < x < 1).$$

Therefore

$$y'_x = -\frac{1}{\sqrt{1-x^2}}.$$

Thus

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}. \tag{80-5}$$

Note. Formula (80-5) could also be obtained from the relation

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

III. The derivative of arctan x. Let

$$y = \arctan x \qquad \left(-\infty < x < +\infty,\ -\frac{\pi}{2} < y < \frac{\pi}{2}\right)$$

and, consequently

$$x = \tan y.$$

We have (Sec. 36, Chapter 6)

$$y'_x = \frac{1}{x'_y} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

Thus

$$(\arctan x)' = \frac{1}{1+x^2}.$$

IV. The derivative of arccot x. Let

$$y = \operatorname{arccot} x \qquad (-\infty < x < +\infty,\ 0 < y < \pi),$$

then

$$x = \cot y.$$

We have (see Sec. 46, Chapter 7)

$$y'_x = \frac{1}{x'_y} = -\frac{1}{\csc^2 y} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2},$$

i.e.

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}.$$

Example.

$$\left(\operatorname{arccot}\frac{1}{x}\right)' = -\frac{1}{1+\frac{1}{x^2}} \cdot \left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} \qquad (x \neq 0).$$

Sec. 81. The Derivative of a Function Represented Parametrically

It is sometimes convenient to represent the relation between the variables x and y by two equations

$$x = \varphi(t), \quad y = \psi(t), \tag{81-1}$$

where t is an auxiliary variable (parameter). This method is frequently applied in mechanics were the parameter t usually denotes time, and equations (81-1) represent the *parametric equations of the path* of a moving point $M(x, y)$.

Generally speaking, equations (81-1) define y as a composite function of x. Indeed, resolving the first equation of system (1) with respect to the parameter t (if it is possible), we have

$$t = \theta(x),$$

where θ is a function inverse to the function φ. Hence, eliminating the parameter t from equations (81-1), we get

$$y = \psi(\theta(x)). \tag{81-2}$$

Applying formula (81-2), we easily find the derivative y'_x as the derivative of a composite function.

But in practice elimination of the parameter t frequently involves many difficulties and sometimes turns out to be even impossible. Therefore we give a rule for finding the derivative y'_x which does not require elimination of the parameter.

Theorem. *If a function y of an argument x is represented parametrically: $x = \varphi(t)$, $y = \psi(t)$, where the functions $\varphi(t)$ and $\psi(t)$ are differentiable and $\varphi'(t) \neq 0$, then the derivative of this function is*

$$y'_x = \frac{y'_t}{x'_t}. \tag{81-3}$$

Proof. In the chain of the equations

$$y = \psi(t), \qquad t = \theta(x),$$

where $t = \theta(x)$ is an inverse function with respect to the function $x = \varphi(t)$, we shall regard the parameter t as an intermediate argument. Then, according to the rule for differentiating a composite function (Sec. 73), we have

$$y'_x = y'_t \cdot t'_x. \tag{81-4}$$

Applying now the rule for differentiation of an inverse function (Sec. 74), we get

$$t'_x = \frac{1}{x'_t}. \tag{81-5}$$

Consequently from formulas (81-4) and (81-5) we find

$$y'_x = y'_t : x'_t,$$

which was to be proved.

Note. Using the Leibniz notation, formula (81-3) can be rewritten in the form of an obvious equality

$$\frac{dy}{dx} = \frac{dy}{dt} : \frac{dx}{dt}.$$

This once again demonstrates the convenience of the Leibniz notation.

Example. Let

$$x = t^2, \quad y = t^3. \tag{81-6}$$

We have

$$x'_t = 2t, \quad y'_t = 3t^2.$$

Hence

$$y'_x = y'_t : x'_t = 3t^2 : 2t = \frac{3}{2}t.$$

The validity of the last formula can be verified directly by eliminating the parameter t from equalities (81-6).

Sec. 82. The Table of Differentiation Formulas

The derived rules and differentiation formulas for functions of one and the same independent variable x are listed below:

I. $c' = 0$.

II. $(u + v - w)' = u' + v' - w'$.

III. $(cu)' = cu'$.

IV. $(uv)' = uv' + vu'$.

XI. $(\tan x)' = \sec^2 x$.

XII. $(\cot x)' = -\csc^2 x$.

XIII. $(\log_a x)' = \frac{1}{x \ln a}$,

$(\ln x)' = \frac{1}{x}$.

V. $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$.

VI. $y'_x = y'_z \cdot z'_x$.

VII. $x'_y = \frac{1}{y'_x}$.

VIII. $(x^n)' = nx^{n-1}$, $x' = 1$.

IX. $(\sin x)' = \cos x$.

X. $(\cos x)' = -\sin x$.

XIV. $(a^x)' = a^x \ln a$, $(e^x)' = e^x$.

XV. $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.

XVI. $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$.

XVII. $(\arctan x)' = \frac{1}{1+x^2}$.

XVIII. $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$.

Sec. 83. Derivatives of Higher Orders

The derivative $f'(x)$ of the function $f(x)$ is called the *derivative of the first order* and represents a new function. It may happen that this function, in its turn, has a derivative. Then the derivative of the derivative of the first order is termed the *derivative of the second order* or the *second derivative* and is denoted $f''(x)$. Thus,

$$f''(x) = [f'(x)]'.$$

Similarly, the derivative of the derivative of the second order (provided it exists) is called the *derivative of the third order* or the *third derivative* and is denoted by $f'''(x)$, i.e.

$$f'''(x) = [f''(x)]'.$$

In similar manner we define the derivatives of the fourth order $f^{IV}(x)$, fifth order $f^{V}(x)$ and so forth (numbers are used instead of dashes to save space and Roman numerals are used to avoid confusion with exponents).

Example. Let $y = \sin x$. We then have in consecutive order $y' = \cos x$, $y'' = -\sin x$, $y''' = -\cos x$, $y^{IV} = \sin x$, . . .

Sec. 84. Physical Meaning of the Second Derivative

We saw (Sec. 65) that we could find the velocity of motion with the aid of the first-order derivative. Here we are going to show that in order to compute the acceleration of motion, we have to take advantage of the second-order derivative.

Let the law of motion of a point M along the x-axis be expressed by the equation $x = f(t)$. Let at the moment t the point M have the velocity v, and at the moment $t + \Delta t$ the velocity $v + \Delta v$. Thus during the time interval Δt the velocity changed by the

increment Δv. The ratio

$$\frac{\Delta v}{\Delta t}$$

is called the *average acceleration* of rectilinear motion for time interval Δt. The limit of this ratio as $\Delta t \to 0$, i.e.

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = v'(t),$$

is called the *acceleration of the point M* at a given moment t. Denoting the acceleration by j, we may write

$$j = v'(t).$$

But

$$v = f'(t).$$

Therefore

$$v'(t) = [f'(t)]' = f''(t).$$

Thus, we have

$$j = f''(t),$$

i.e. *the acceleration of rectilinear motion of a point is equal to the second derivative of the distance* with respect to the time.*

Exercises

Find the derivatives of the following functions:

1. (a) $y = 3x^2 - x + 5$;
   (b) $f(x) = \frac{1}{x^2}$. Evaluate $f'(1)$, $f'(-10)$.
2. $y = 2\sqrt{x} + 3\sqrt[3]{x^2} - \frac{1}{\sqrt[3]{x}}$.
3. $y = \frac{x^2}{2} + \frac{2}{x^2}$.
4. $y = \frac{ax+b}{a+b}$.
5. $y = x^2(2x-1)$.
6. $y = (x+1)\sqrt{x}$.
7. $y = x^2 \sin x$.
15. $y = \sqrt{1+x^2}$.
16. $y = \ln \ln x$.
17. $y = \ln^2 x$.
18. $y = \ln x^2$.
19. $y = \ln \tan \frac{x}{2}$.
20. $y = x^n + n^x$.
21. $y = e^{-x^2}$.
22. $y = e^{\frac{x}{2}}(x^2 - 4x + 8)$.
23. $y = \arcsin \frac{x}{2}$.
24. $y = \arccos \frac{1}{x}$.

* More precisely: of the abscissa of the moving point.

8. $y=\dfrac{1}{1+x^2}$.

9. $y=\dfrac{2x}{1-x^2}$.

10. $y=\dfrac{\sin x-\cos x}{\sin x+\cos x}$.

11. $y=\sin^2 x$.

12. $y=\sin x^2$.

13. $y=\cos^3\dfrac{x}{2}$.

14. $y=\cos\dfrac{x^3}{2}$.

25. $y=\arctan\dfrac{1}{x}$.

26. $y=(\tan x-\cot x)^2$.

27. $f(x)=x+\operatorname{arccot} x$.

28. $y=\ln\left(x+\sqrt{1+x^2}\right)$.

29. $y=\dfrac{x}{2}\sqrt{1-x^2}+\dfrac{1}{2}\arcsin x$.

30. $y=\dfrac{1}{4}\ln\dfrac{1+x}{1-x}+\dfrac{1}{2}\arctan x$.

Find the derivatives of differentiable implicit functions $y = y(x)$, represented by the following equations:

31. $x^2 + y^2 - xy = 1$.

32. $x^3 + y^3 - 3xy = 0$.

33. (a) $y = x + \ln y$;

(*b*) find y' at the point $M(1, 1)$, if

$$\frac{x}{y}+2xy=3.$$

34. Find the derivatives y'_x of the functions represented parametrically:

(a) $x = a\cos t$, $y = b\sin t$;

(*b*) $x = \omega t$, $y = te^{\alpha t}$;

(c) $x = e^{-t}\cos t$, $y = e^{-t}\sin t$ for $t = 0$.

Find the second derivatives of the following functions:

35. (*a*) $y = e^{-x^2}$; (b) $y = x + \sin 2x$.

36. $y = \ln x$.

37. $y = x^2e^{-x}$.

38. Find $f(0)$, $f'(0)$ and $f''(0)$, if $f(x) = \cos 3x$.

39. (a) Write the equation of the tangent line to the curve $y = x^2$ at its point $M(2, 8)$; (b) write the equation of the tangent line to the sinusoid $y = \sin x$ at the point $x = \pi$, (c) write the equation of the tangent line to the parabola $y^2 = 2x$ at the point (8, 4).

40. Write the equation of the tangent line to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at its point $M(x_1, y_1)$.

41. Write the equation of the tangent line to the ellipse

$$x = 10 \cos t, \quad y = 6 \sin t$$

at the point corresponding to $t = \frac{\pi}{3}$.

42. The normal to a curve at its given point $M(x_1, y_1)$ is a perpendicular to the tangent line to this curve at the point of tangency M.

Write the equations of the normals to the following curves at the given points:

(a) $y = \tan x$ at the point O (0, 0).
(b) $y^2 = 2x$ at the point M (8, 4).
(c) $x^2 + y^2 = 25$ at the point (3, — 4).

43. What is the rate of increase of the area of a circle at the moment when its radius $R = 10$ m if the radius of the circle increases at a rate of 2 m/s?

44. A man of height $h = 175$ cm moves off a street lamp hanging at an altitude $H = 5$ m at a rate of $v = 1.5$ m/s. What is the rate of growth of his shadow?

45. The law of motion of a point is

$$x = x_0 + \alpha t + \frac{\beta}{2} t^2.$$

Find the velocity and acceleration of the motion.

46. Knowing the equation of motion of a point

$$x = t - \sin t,$$

determine the velocity and acceleration of this point.

Chapter II

Applications of Derivatives

Sec. 85. The Theorem About Finite Increments of a Function and Its Corollaries

Lagrange's theorem about finite increments of a function. *The finite increment of a differentiable function is equal to the corresponding increment of the argument multiplied by the value of its derivative at some interior point, i.e. if $f(x)$ is a differentiable function in some interval $\langle a, b\rangle$ and x_1, x_2 $(x_1 < x_2)$ are any values from this interval, then*

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(\xi), \tag{85-1}$$

where

$$x_1 < \xi < x_2.$$

Proof. On the graph of the function $y = f(x)$ we draw a secant AB through the points $A(x_1, f(x_1))$ and $B(x_2, f(x_2))$ (Fig. 109). We shall then displace this secant parallel to its initial position until

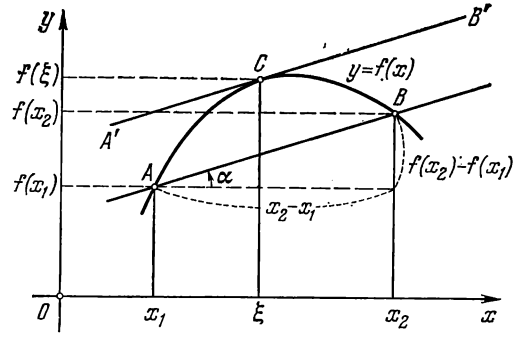


FIG. 109

it becomes a tangent $A'CB'$ to the graph of our function at one of its points $C(\xi, f(\xi))$*, where $x_1 < \xi < x_2$. According to our con-

* We omit here the proof of the fact that such a limiting position exists.

struction, the slopes of the secant AB and the tangent $A'CB'$ are equal to each other, therefore

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(\xi).$$

Hence

$$f(x_2)-f(x_1)=(x_2-x_1)f'(\xi),$$

which was to be proved.*

Corollary 1. *If the derivative of a function is equal to zero on some interval, then the function is identically constant on this interval.* Indeed, if, for instance,

$$f'(x)=0 \quad \text{for} \quad a<x<b,$$

then, putting in formula (85-1) $x_1=x_0$, where x_0 is a fixed value from the interval (a, b), and $x_2=x$, where x is any value from this interval, we have

$$f(x)-f(x_0)=(x-x_0)f'(\xi)=0.$$

Hence

$$f(x)=f(x_0)=\text{const},$$

if

$$a<x<b.$$

Corollary 2. *If two functions have equal derivatives on a certain interval, then these functions differ from each other on this interval utmost by a constant term.*

Indeed, if

$$f_1'(x)=f_2'(x)$$

for $x \in \langle a, b\rangle$, then on this interval we have

$$[f_1(x)-f_2(x)]'=f_1'(x)-f_2'(x)=0.$$

Consequently by Corollary 1, the function

$$f_1(x)-f_2(x)$$

is constant for $a<x<b$, i.e.

$$f_1(x)-f_2(x)=C$$

for all values of x belonging to the interval $\langle a, b\rangle$.

Example. As is known, for any value $-\infty<x<+\infty$ we have

$$(\arctan x)'=\frac{1}{1+x^2} \quad \text{and} \quad (-\operatorname{arccot} x)'=\frac{1}{1+x^2}.$$

* It is possible to prove that the Lagrange theorem remains true if the function is continuous on the closed interval $[a, b]$ and differentiable inside it.

Consequently,

$$\arctan x - (-\operatorname{arccot} x) \equiv C,$$

where C is a constant. Putting $x = 1$ in the last identity, we get

$$\frac{\pi}{4} + \frac{\pi}{4} = C, \quad \text{i.e.} \quad C = \frac{\pi}{2}.$$

Hence,

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2}.$$

Theorem about the roots of a function (Rolle's theorem). *Between every two zeros of a differentiable function there is at least one zero of its derivative.*

Proof. Indeed, if $f(x)$ is a differentiable function and

$$f(x_1) = f(x_2) = 0 \qquad (x_1 < x_2),$$

then from formula (85-1) we have

$$(x_2 - x_1) f'(\xi) = 0,$$

or, since $x_2 \neq x_1$, we get

$$f'(\xi) = 0,$$

where

$$x_1 < \xi < x_2.$$

Sec. 86. Increase and Decrease of a Function of One Argument

Definition. *A function is said to be increasing on an interval* $\langle a, b \rangle$ *if to greater values of the argument belonging to this interval there correspond greater values of the function.* In other words, $f(x)$ is an increasing function in an interval $\langle a, b \rangle$ if, whatever the values x_1 and x_2 from this interval, (Fig. 110, *a*), from the inequality

$$x_2 > x_1$$

there follows the inequality

$$f(x_2) > f(x_1).$$

Similarly, *a function is said to be decreasing on an interval* $\langle a, b \rangle$ *if to greater values of the argument in this interval there correspond smaller values of the function.* In other words, $f(x)$ is a decreasing function (Fig. 110, *b*) if from the inequality

$$x_2 > x_1$$

there follows the inequality

$$f(x_2) < f(x_1).$$

Theorem 1. Necessity test for the increase (decrease) of a function.

(1) *If a differentiable function is increasing in a certain interval, then the derivative of this function is non-negative in that interval.*

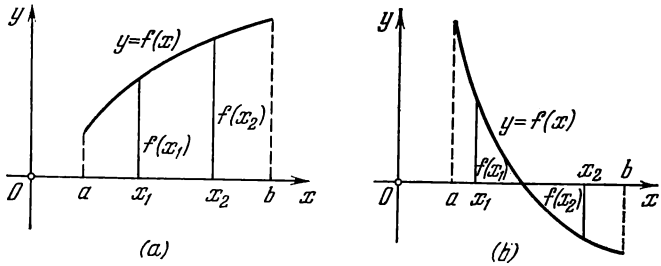


FIG. 110

(2) *If a differentiable function is decreasing in a certain interval, then its derivative is non-positive in that interval.*

Proof. (1) Let a differentiable function $f(x)$ be increasing in the interval $\langle a, b\rangle$. According to the definition of the derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

If the values x and $x + \Delta x$ belong to the interval $\langle a, b\rangle$, then, by virtue of the increase of the function $f(x)$, the sign of its increment

$$f(x + \Delta x) - f(x),$$

where $\Delta x \neq 0$, is the same as the sign of the increment of the argument x. Consequently, for sufficiently small (by absolute value) Δx we have

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} > 0.$$

Passing in the last inequality to the limit as $\Delta x \to 0$ and taking into account that the limit of a positive function cannot, obviously, be negative, we get (see Sec. 46, Chapter 7)

$$f'(x) \geqslant 0.$$

(2) The proof of the second part of the theorem is analogous to the proof of its first part.

Note. Geometrically, the statement of the theorem is reduced to that the tangents to the graph of an increasing differentiable function form *acute* angles α ($\tan \alpha = f'(x)$) with the positive direc-

tion of the x-axis or at some points A they are parallel to the x-axis (Fig. 111).

All the tangents to the graph of a decreasing differentiable function form *obtuse* angles with the positive direction of the x-axis or are parallel to it.

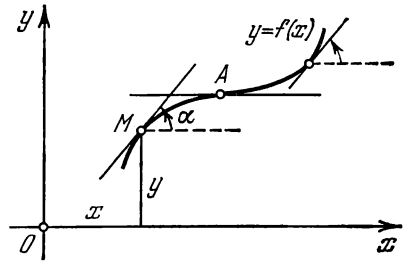


FIG. 111

Theorem 2. Sufficiency test for the increase (decrease) of a function.

(1) *If the derivative of a differentiable function is positive inside a certain interval, then the function is increasing on this interval.*

(2) *If the derivative of a differentiable function is negative inside a certain interval, then the function is decreasing on this interval.*

Proof. (1) Let, for instance, the differentiable function $f(x)$ is such that

$$f'(x) > 0 \quad \text{for} \quad a < x < b.$$

For any two values $a \leqslant x_1 < x_2 \leqslant b$ belonging to the interval $\langle a, b\rangle$ by virtue of the theorem about the finite increment of a function, we have

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(\xi),$$

where ξ is an intermediate value between x_1 and x_2 and, hence, lying inside the interval $\langle a, b\rangle$. Since

$$x_2 - x_1 > 0 \quad \text{and} \quad f'(\xi) > 0,$$

hence we get

$$f(x_2) - f(x_1) > 0$$

or

$$f(x_2) > f(x_1).$$

Hence, the function $f(x)$ is increasing on the interval $\langle a, b\rangle$.

(2) The proof of the second part of this theorem is absolutely analogous to the proof of its first part.

A function which in a given interval is increasing or decreasing is called *monotone* (in that interval). An interval of variation of the independent variable on which the function is increasing is called its *interval of increase*, and an interval on which it is decreasing is called its *interval of decrease*. Both types of intervals are referred to as *intervals of monotonicity* of this function.

Example. Test for an increase and decrease the function $f(x) = x^3 - 3x + 2$.

We find the derivative

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1).$$

The derivative vanishes for the values $x_1 = -1$ and $x_2 = 1$. These values divide the entire infinite x-axis into three intervals:

$$(-\infty, -1], \quad [-1, 1], \quad [1, +\infty),$$

inside each of which the derivative $f'(x)$ preserves a constant sign. Obviously, inside the first and third intervals the derivative $f'(x)$ is positive, and inside the second interval it is negative. This can be verified by taking points belonging to the corresponding intervals. Consequently, the function $f(x)$ is increasing in the first interval, decreasing in the second, and increasing once again in the third (see Fig. 112).

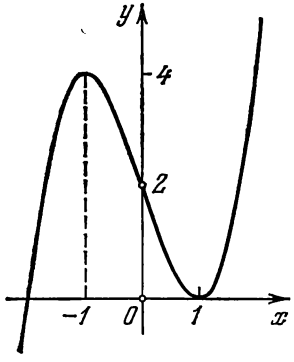


FIG. 112

Sec. 87. L'Hospital's Rule

Consider the ratio

$$f(x) = \frac{\varphi(x)}{\psi(x)}$$

where the functions $\varphi(x)$ and $\psi(x)$ are defined and differentiable in a neighbourhood U_a of a point a excluding, perhaps, the point a itself. It may happen that as $x \to a$ both functions tend to 0 or to ∞, i.e. both these functions are either *infinitely small* or *infinitely large* as $x \to a$. Then we say that at the point a the function $f(x)$ has an indeterminacy of the form

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}, \tag{87-1}$$

respectively. In this case, using the derivatives $\varphi'(x)$ and $\psi'(x)$, we can formulate a simple rule for finding the limit of the function $f(x)$ as $x \to a$, i.e. give a method for *evaluating certain indeterminacies* of the form (87-1). This rule is usually associated with the name of the French mathematician L'Hospital.*

* L'Hospital (also l'Hôpital, Lhospital), Guillaume Francois Antoine de (1661-1704), author of the first printed manual on differential calculus (1696), where the rule is formulated. In compiling this manual, L'Hospital made use of the manuscript of his teacher John Bernoulli. This rule is mentioned in the manuscript and so the name "L'Hospital's rule" is historically inaccurate.

Theorem. *The limit of the ratio of two infinitely small or infinitely large functions is equal to the limit of the ratio of their derivatives (finite or infinite) if the latter exists (in the indicated sense).*

Proof. We shall consider here only indeterminate forms $\frac{0}{0}$ and, for the sake of simplicity, assume that the functions $\varphi(x)$ and $\psi(x)$, as well as their derivatives $\varphi'(x)$ and $\psi'(x)$, are continuous at the point a and $\psi'(a) \neq 0$. The proof of another case of L'Hospital's rule referring to indeterminate forms $\frac{\infty}{\infty}$ is more complicated (see, for instance, a two-volume textbook by S. M. Nikolsky (Member, USSR Academy of Sciences) "A Course of Mathematical Analysis", Vol. 1, Mir Publishers, Moscow, 1977).

And so, let

$$\lim_{x \to a} \varphi(x) = \varphi(a) = 0 \tag{87-2}$$

and

$$\lim_{x \to a} \psi(x) = \psi(a) = 0. \tag{87-2'}$$

The difference $\varphi(x) - \varphi(a)$ may be regarded as the increment of the function $\varphi(x)$ at the point a corresponding to the increment of the argument $\Delta x = x - a$. Therefore

$$\lim_{x \to a} \frac{\varphi(x) - \varphi(a)}{x - a} = \varphi'(a), \tag{87-3}$$

analogously

$$\lim_{x \to a} \frac{\psi(x) - \psi(a)}{x - a} = \psi'(a) \neq 0. \tag{87-3'}$$

Taking into consideration formulas (87-2) and (87-2') for $x \neq a$, we get

$$\frac{\varphi(x)}{\psi(x)} = \frac{\varphi(x) - \varphi(a)}{\psi(x) - \psi(a)} = \frac{\dfrac{\varphi(x) - \varphi(a)}{x - a}}{\dfrac{\psi(x) - \psi(a)}{x - a}}.$$

Hence, passing to the limit as $x \to a$ and applying formulas (87-3) and (87-3'), we shall have

$$\lim_{x \to a} \frac{\varphi(x)}{\psi(x)} = \frac{\varphi'(a)}{\psi'(a)}. \tag{87-4}$$

But we assumed that the derivatives $\varphi'(x)$ and $\psi'(x)$ are continuous as $x \to a$, and $\psi'(a) \neq 0$, therefore

$$\lim_{x \to a} \frac{\varphi'(x)}{\psi'(x)} = \frac{\lim\limits_{x \to a} \varphi'(x)}{\lim\limits_{x \to a} \psi'(x)} = \frac{\varphi'(a)}{\psi'(a)}. \tag{87-5}$$

Comparing formulas (87-4) and (87-5), we obtain L'Hospital's rule

$$\lim_{x \to a} \frac{\varphi(x)}{\psi(x)} = \lim_{x \to a} \frac{\varphi'(x)}{\psi'(x)}. \qquad (87\text{-}6)$$

Note. The right-hand side of formula (87-6) is the limit of the ratio of the derivatives, but not of the derivative of the ratio of the functions.

Example 1. Find

$$\lim_{x \to 0} \frac{2^x - 1}{\sin x}.$$

Here, for $x = 0$, both the numerator and the denominator of the fraction are equal to zero, i.e. as $x \to 0$ we have an indeterminate form $\frac{0}{0}$. Applying L'Hospital's rule (87-6), we get

$$\lim_{x \to 0} \frac{2^x - 1}{\sin x} = \lim_{x \to 0} \frac{2^x \ln 2}{\cos x} = \frac{\ln 2}{1} = \ln 2.$$

Example 2. Find

$$\lim_{x \to +\infty} \frac{x^2}{e^x}$$

As $x \to +\infty$ we have indeterminate form of the type $\frac{\infty}{\infty}$ Applying L'Hospital's rule twice, we obtain

$$\lim_{x \to +\infty} \frac{x^2}{e^x} = \lim_{x \to +\infty} \frac{2x}{e^x} = \lim_{x \to +\infty} \frac{2}{e^x} = 0.$$

Thus, as $x \to +\infty$, the exponential function e^x is increasing more rapidly than the power function x^2.

Besides the above indicated types of indeterminacies $\frac{0}{0}$ and $\frac{\infty}{\infty}$, there exist other indeterminate forms. For instance, if

$$f(x) = \varphi(x)\,\psi(x),$$

where $\varphi(x) \to 0$ and $\psi(x) \to \infty$ as $x \to a$, then as $x \to a$ the function $f(x)$ represents an indeterminate form of the type $0 \cdot \infty$. Another example is the function

$$f(x) = \varphi(x) - \psi(x),$$

where $\varphi(x) \to +\infty$ and $\psi(x) \to +\infty$ as $x \to a$. Here as $x \to a$ we obtain an indeterminacy of the form $\infty - \infty$. Other indeterminate forms, that is forms of the type ∞^0, 1^∞ and 0^0, are also possible. To evaluate these forms, we try to reduce them to the principal forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ which are then evaluated using l'Hospital's rule. This is best explained by examples.

Example 3. Find

$$\lim_{x\to+0} x\ln x.$$

Here we have an indeterminate form of the type $0\cdot\infty$ Rewriting the given expression in the form

$$\lim_{x\to+0} x\ln x=\lim_{x\to+0}\frac{\ln x}{\frac{1}{x}},$$

we get an indeterminate form of the type $\frac{\infty}{\infty}$. Hence, applying L'Hospital's rule, we find

$$\lim_{x\to+0} x\ln x=\lim_{x\to+0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to+0}(-x)=0.$$

Example 4. Find

$$\lim_{x\to0}\left(\cot x-\frac{1}{x}\right)$$

The given expression represents an indeterminacy of the form $\infty-\infty$. Applying the formula $\cot x=\frac{\cos x}{\sin x}$, we shall have

$$\lim_{x\to0}\left(\cot x-\frac{1}{x}\right)=\lim_{x\to0}\left(\frac{\cos x}{\sin x}-\frac{1}{x}\right)=\lim_{x\to0}\frac{x\cos x-\sin x}{x\sin x}$$

Since an indeterminate form of the type $\frac{0}{0}$ is obtained, we use L'Hospital's rule

$$\lim_{x\to0}\left(\cot x-\frac{1}{x}\right)=\lim_{x\to0}\frac{\cos x-x\sin x-\cos x}{\sin x+x\cos x}=$$

$$=-\lim_{x\to0}\frac{x\sin x}{\sin x+x\cos x}=-\lim_{x\to0}\frac{\sin x}{\frac{\sin x}{x}+\cos x}=-\frac{0}{1+1}=0$$

The latter result has been obtained on the basis of the known limit: $\lim\limits_{x\to0}\frac{\sin x}{x}=1$ (see Sec. 54, Chapter 7). In finding the limit, it turned out here, as in many other cases, to be appropriate to combine L'Hospital's rule with some elementary methods.

For the function

$$f(x)=[\varphi(x)]^{\psi(x)}$$

in the cases:

$$\varphi(x) \to 0 \text{ and } \psi(x) \to 0 \quad \text{as } x \to a,$$
$$\varphi(x) \to \infty \text{ and } \psi(x) \to 0 \quad \text{as } x \to a,$$
$$\varphi(x) \to 1 \text{ and } \psi(x) \to \infty \quad \text{as } x \to a$$

we get indeterminate forms of the type 0^0, ∞^0; and 1^∞, respectively. Here it is advantageous to take the logarithm of the function $f(x)$.

Example 5. Find

$$A = \lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}. \tag{87-7}$$

Expression (87-7) represents an indeterminate form 1^∞. Taking the logarithm of this expression and making use of the continuity of a logarithmic function, we find

$$\ln A = \ln \left[\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x} \right] = \lim_{x \to \frac{\pi}{2}} [\ln (\sin x)^{\tan x}] =$$
$$= \lim_{x \to \frac{\pi}{2}} (\tan x \cdot \ln \sin x) = \lim_{x \to \frac{\pi}{2}} \frac{\ln \sin x}{\cot x}.$$

Applying L'Hospital's rule to the obtained indeterminate form $\frac{\infty}{\infty}$, we have

$$\ln A = \lim_{x \to \frac{\pi}{2}} \left(\frac{\frac{\cos x}{\sin x}}{-\frac{1}{\sin^2 x}} \right) = \lim_{x \to \frac{\pi}{2}} (-\sin x \cos x) = 0;$$

hence

$$A = 1.$$

Sec. 88. Taylor's Formula for a Polynomial

Let it be required to expand a given polynomial

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \tag{88-1}$$

in powers of the binomial $x - x_0$, where x_0 is some number. This problem can be solved in an elementary way by using the identity $x \equiv x_0 + (x - x_0)$. But we are going to consider a simpler method. Let

$$P(x) = A_0 + A_1 (x - x_0) + A_2 (x - x_0)^2 + \ldots + A_n (x - x_0)^n \tag{88-2}$$

be the desired expansion whose coefficients A_0, A_1, A_2, . . ., A_n are required to be found. Putting $x = x_0$ in identity (88-2), we get $P(x_0) = A_0 + 0$, hence

$$A_0 = P(x_0) \tag{88-3}$$

Differentiating identity (88-2), we have

$$P'(x) = A_1 + 2A_2(x - x_0) + \ldots + nA_n(x - x_0)^{n-1}.$$

Hence, putting $x = x_0$, we get

$$A_1 = P'(x_0).$$

On second differentiation we find

$$P''(x) = 2!\,A_2 + \ldots + n(n-1)\,A_n(x - x_0)^{n-2}$$

and for $x = x_0$ we have $P''(x_0) = 2!\,A_2$, i.e.

$$A_2 = \frac{P''(x_0)}{2!}. \tag{88-4}$$

For determining further coefficients of expansion (88-2) we can use the same method. It is rather obvious that there is a general formula

$$A_k = \frac{P^{(k)}(x_0)}{k!} \qquad (k = 0, 1, 2, \ldots, n), \tag{88-5}$$

where, by definition, we put $P^{(0)}(x) = P(x)$ and $0! = 1$. Formula (88-5) can be strictly proved by the method of mathematical induction.

Substituting coefficients (88-5) into expansion (88-2), we get *Taylor's formula* for a polynomial

$$P(x) = P(x_0) + P'(x_0)(x - x_0) + \\ + \frac{P''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{P^{(n)}(x_0)}{n!}(x - x_0)^n$$

or briefer

$$P(x) = \sum_{k=0}^{n} \frac{P^{(k)}(x_0)}{k!}(x - x_0)^k. \tag{88-6}$$

Note, that, as is easy to be convinced, the senior coefficients in expansions (88-1) and (88-2) coincide, i.e. $A_n = a_n$. Therefore the following equation holds

$$\frac{1}{n!}P^{(n)}(x_0) = a_n.$$

If we put $x_0 = 0$, then the right-hand member of equality (88-6) will be identically equal to the right-hand side of polynomial (88-1).

Therefore the following equalities hold

$$\frac{P^{(k)}(0)}{k!} = a_k \qquad (k = 0,\ 1,\ \ldots,\ n).$$

Example. Expand the polynomial

$$P(x) = 1 - 2x + 3x^2 - 4x^3$$

in powers of the binomial $x + 1$.

Here $x_0 = -1$. We have

$$P'(x) = -2 + 6x - 12x^2,$$

$$P''(x) = 6 - 24x,$$

$$P'''(x) = -24$$

and

$$P(-1) = 10, \quad P'(-1) = -20,$$

$$P''(-1) = 30, \quad P'''(-1) = -24.$$

Thus,

$$P(x) = 10 - 20(x + 1) + 15(x + 1)^2 - 4(x + 1)^3.$$

Sec. 89. Binomial Formula

Consider the function

$$f(x) = (a + x)^n, \tag{89-1}$$

where n is a natural number. Putting $x_0 = 0$ and using Taylor's formula (88-6) from the preceding section, we get

$$(a + x)^n = A_0 + A_1 x + \ldots + A_n x^n, \tag{89-2}$$

where

$$A_k = \frac{f^{(k)}(0)}{k!} \qquad (k = 0,\ 1,\ \ldots,\ n).$$

Since from (89-1) the following is obtained

$$f^{(k)}(x) = n(n - 1)\ldots[n - (k - 1)](a + x)^{n-k},$$

we get

$$f(0) = a^n$$

and

$$f^{(k)}(0) = n(n - 1)\ldots[n - (k - 1)]\,a^{n-k} \qquad (k = 1, 2, \ldots, n).$$

Thus

$$A_0 = a^n$$

and

$$A_k = \frac{n(n-1)\ \ldots\ [n-(k-1)]}{k!}\, a^{n-k} \qquad (k=1,\ 2,\ \ldots,\ n). \tag{89-3}$$

The numbers A_k are called *binomial coefficients* and conventionally denoted in the following manner:

$$A_k = C_n^k, \tag{89-4}$$

where C_n^k is called the number of combinations of n elements k at a time (the combinatorial meaning of the numbers C_n^k will be explained later on, see Sec. 239, Chapter 25).

Thus, on the basis of formula (89-2), we have *Newton's binomial formula*

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}\, a^{n-2}x^2 + \ldots \quad x^n. \tag{89-5}$$

In particular, for $a = 1$ we obtain

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}\, x^2 + \ldots + x^n. \tag{89-6}$$

Sec. 90. Taylor's Formula for a Function

Let the function $f(x)$ have a continuous derivative of the Nth order $f^{(N)}(x)$* in the interval (a, b) and $x_0 \in (a, b)$. Using the results of the preceding section, we construct *Taylor's polynomial* of degree n

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\, (x-x_0)^k \tag{90-1}$$

where $n \leqslant N$.

The polynomial $P_n(x)$ may be regarded as a certain *approximation* of a given function. Denoting by $R_n(x)$ the corresponding error (the so-called *remainder term* or, simply, remainder), we shall have

$$f(x) = P_n(x) + R_n(x). \tag{90-2}$$

Let us show that as $x \to x_0$ the remainder $R_n(x)$ will be an infinitesimal of the order higher than n (*Peano's theorem*). Indeed, consider the limit

$$\lim_{x\to x_0} \frac{R_n(x)}{(x-x_0)^n} =$$

$$= \lim_{x\to x_0} \frac{f(x) - \left[f(x_0) + f'(x_0)(x-x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n\right]}{(x-x_0)^n}. \tag{90-3}$$

* Hence, by sense of the operation of differentiation, we get that in the interval (a, b) there exist the continuous derivatives $f(x) = f^{(0)}(x)$, $f'(x)$, ..., $f^{(N-1)}(x)$.

Obviously, we have an indeterminate form of the type $\frac{0}{0}$. Applying L'Hospital's rule n times in succession and taking into account the continuity of the derivative $f^{(n)}(x)$, we find

$$\lim_{x\to x_0}\frac{R_n(x)}{(x-x_0)^n}=$$

$$=\lim_{x\to x_0}\frac{f'(x)-\left[f'(x_0)+\frac{f''(x_0)}{1!}(x-x_0)+\ldots+\frac{f^{(n)}(x_0)}{(n-1)!}(x-x_0)^{n-1}\right]}{n(x-x_0)^{n-1}}=$$

$$=\lim_{x\to x_0}\frac{f''(x)-\left[f''(x_0)+\ldots+\frac{f^{(n)}(x_0)}{(n-2)!}(x-x_0)^{n-2}\right]}{n(n-1)(x-x_0)^{n-2}}=\ldots$$

$$\ldots=\lim_{x\to x_0}\frac{f^{(n)}(x)-[f^{(n)}(x_0)]}{n(n-1)\ldots 1}=0.$$

Hence,

$$R_n(x)=o[(x-x_0)^n]. \tag{90-4}$$

Thus, we obtain the *local Taylor formula*:

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+o[(x-x_0)^n]^* \tag{90-5}$$

In a particular case, for $a<0<b$ and $x_0=0$, we shall have the so-called *local Maclaurin's formula* (which is a special case of the Taylor formula):

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k+o(x^n). \tag{90-6}$$

Example. Approximate the function $f(x)=\sin x$ in the neighbourhood of the point $x_0=0$ using the third Taylor polynomial $P_3(x)$.

We have

$$f(x)=\sin x,\quad f'(x)=\cos x,\quad f''(x)=-\sin x,\quad f'''(x)=-\cos x.$$

Hence

$$f(0)=0,\quad f'(0)=1,\quad f''(0)=0,\quad f'''(0)=-1.$$

By formula (90-6), we get

$$\sin x=x-\frac{x^3}{3!}+o(x^3). \tag{90-7}$$

---

* In the general case formula (90-5) turns out to be meaningful if x belongs to a sufficiently small neighbourhood U_{x_0} of the point x_0.

Formula (90-7) is frequently used for finding the sines of small angles x; it should be borne in mind that here x is expressed in radians.

Putting $x - x_0 = h$, $x = x_0 + h$ and taking into consideration that

$$f(x) - f(x_0) = f(x_0 + h) - f(x_0) = \Delta f(x_0),$$

formula (90-5) may be written in the form

$$\Delta f(x_0) = hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots + \frac{h^n}{n!} f^{(n)}(x_0) + o(h^n). \qquad (90\text{-}8)$$

Sec. 91. Maxima and Minima of a Function of One Variable

Definition. *We say that a function $f(x)$ has a maximum for the value x_1 of the argument x if, in a sufficiently close neighbourhood of the point x_1 the inequality*

$$f(x_1) > f(x) \qquad (x \neq x_1)$$

is fulfilled (Fig. 113).

Analogously, we say that a function $f(x)$ has a minimum for the value x_2 of the argument x if, within a sufficiently close neighbourhood of the point x_2 the following inequality

$$f(x_2) < f(x) \qquad (x \neq x_2)$$

takes place (Fig. 113).

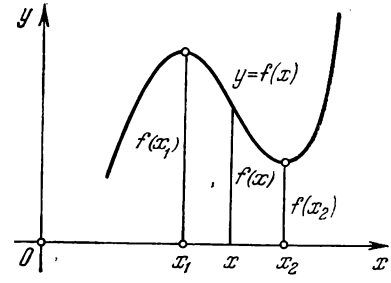


FIG. 113

The generic term for maximum and minimum is *extremum* (extreme value); and those values of the argument for which extrema of a function are obtained are called the *points of extremum of the function* (or: *points of maximum* or *points of minimum*).

It follows from the definition that an extremum of a function, generally speaking, has a *local character*, since this is the greatest or the least value of the function as compared with its *neighbouring values*. Therefore, the presence of an extremum of a function depends not at all on how the function behaves far from this value. From this point of view it is clear that a minimum of a function can be greater than its maximum, just as a depression in the mountains can have a greater elevation (the height above sea level) than a low summit.

Let the function $f(x)$ be defined on the closed interval $[a, b]$ and have an extremum at the point $x_0 \in [a, b]$. If x_0 is an interior

point of the closed interval, then the difference

$$f(x) - f(x_0) \qquad (x \neq x_0)$$

retains a constant sign in a certain *two-sided neighbourhood* $x_0 - h < x < x_0 + h$ $(x \neq x_0)$ of the point x_0. Such extremum is called *two-sided*. For instance, the function $f(x) = \sqrt{1 - x^2}$ has a two-sided maximum at $x_0 = 0$, since $f(x) < f(x_0) = 1$ for $-1 < x < 1$, $x \neq 0$. And if x_0 is an end-point of the closed interval $[a, b]$, say, $x_0 = a$, then $f(x) - f(x_0)$ preserves the sign only in a *one-sided neighbourhood* $a = x_0 < x < x_0 + h$ of the point x_0. Such an extremum is termed *one-sided* (or *boundary*). For instance, the function $f(x) = \sqrt{1 - x^2}$ has a one-sided minimum at $x_0 = -1$ and at $x_0 = 1$.

Henceforth, under the word 'extremum' we shall understand a *two-sided extremum*, i.e. we shall assume that for a point of extremum of a given function $f(x)$ there is a certain neighbourhood $0 < |x - x_0| < h$ of the point x_0 in which the difference $f(x) - f(x_0)$ preserves a constant sign.

1. Necessary condition for an extremum of a function.

Theorem. *If a differentiable function $f(x)$ has an extremum (two-sided) at a point x_0, then its derivative at this point is equal to zero.*

Proof. Let, for the sake of definiteness, x_0 is a point of minimum of the function $f(x)$. Consequently,

$$f(x_0 + \Delta x_0) > f(x_0),$$

if $\Delta x_0 \neq 0$ is sufficiently small by absolute value. Hence

$$\frac{f(x_0 + \Delta x_0) - f(x_0)}{\Delta x_0} > 0, \quad \text{if} \quad \Delta x_0 > 0,$$

and

$$\frac{f(x_0 + \Delta x_0) - f(x_0)}{\Delta x_0} < 0, \quad \text{if} \quad \Delta x_0 < 0.$$

Passing in these inequalities to the limit as $\Delta x_0 \to 0$ for the derivative at the point x_0 equal to

$$f'(x_0) = \lim_{\Delta x_0 \to 0} \frac{f(x_0 + \Delta x_0) - f(x_0)}{\Delta x_0},$$

we get respectively

$$f'(x_0) \geqslant 0, \quad \text{if} \quad \Delta x_0 > 0, \quad \text{and} \quad f'(x_0) \leqslant 0, \quad \text{if} \quad \Delta x_0 < 0.$$

Since the value of the derivative $f'(x_0)$ must not depend on how Δx_0 tends to zero, it follows from this fact that

$$f'(x_0) = 0, \tag{91-1}$$

which was to be proved.

Geometrical illustration. Geometrically, condition (91-1) means that at the point of extremum of a differentiable function $y = f(x)$ the tangent to its graph is parallel to the x-axis (Fig. 114, *a*).

Corollary. *A continuous function may have extrema only at the points where the derivative of the function is equal to zero or does not exist.*

Indeed, if at a point x_0 of an extremum of the function $f(x)$ there exists a derivative $f'(x_0)$, then, by the above proved theorem, this derivative is equal to zero: $f'(x_0) = 0$.

The fact that at a point of extremum of a continuous function the derivative may not exist is illustrated by the function whose graph has the shape shown in Fig. 114, *b*.

Those values of the argument x which for a given function $f(x)$ make its derivative $f'(x)$ vanish, or for which the derivative $f'(x)$

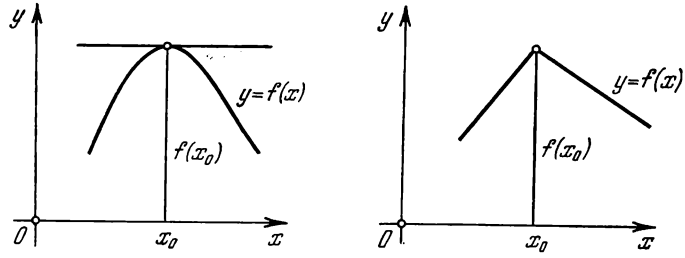


FIG. 114

does not exist (for instance, turns into infinity) are called the *critical values* of the argument.

2. Sufficient conditions for an extremum of a function.

From the fact that $f'(x_0) = 0$ it does not follow at all that the function $f(x)$ has an extremum at $x = x_0$.

Indeed, let $f(x) = x^3$. Then $f'(x) = 3x^2$, and, consequently, $f'(0) = 0$. But the value $f(0)$ is not an extremum of the given function, since the difference $f(x) - f(0)$ changes its sign with the change of the sign of the argument x (see Fig. 57).

Thus, not for any critical value of the argument of a function $f(x)$ an extremum of this function takes place. Therefore, along with the necessary condition we are going to give sufficient conditions for an extremum of a function.

Theorem 1 (the first rule). *If a differentiable function $f(x)$ is such that for a certain value x_0 of its argument x the derivative $f'(x)$ is equal to zero and changes its sign when the argument passes through this value*, then the number $f(x_0)$ is an extremum of the function*

* We say that a cerain function $F(x)$ changes its sign when passing through the value x_0 if there exists a sufficiently small positive ε such that

$F(x) < 0$ for $x_0 - \varepsilon < x < x_0$

and

$F(x) > 0$ for $x_0 < x < x_0 + \varepsilon$

or vice versa.

$f(x)$, *where*

(1) *the function* $f(x)$ *has a maximum at* $x = x_0$ *if the change of sign of the derivative* $f'(x)$ *takes place from plus to minus*;

(2) *the function* $f(x)$ *has a minimum at* $x = x_0$ *if the derivative changes its sign from minus to plus.*

Proof. (1) Let

$$f'(x_0) = 0,$$

where

$$f'(x) > 0 \quad \text{for} \quad x_0 - \varepsilon < x < x_0$$

and

$$f'(x) < 0 \quad \text{for} \quad x_0 < x < x_0 + \varepsilon,$$

where ε is a sufficiently small positive number.

Hence, from Theorem 2 of Sec. 86, it follows that the function $f(x)$ is increasing on the closed interval $[x_0 - \varepsilon, x_0]$ and decreasing on the closed interval $[x_0, x_0 + \varepsilon]$. Consequently, in immediate proximity to the value x we have

$$f(x_0) > f(x), \quad \text{if} \quad x < x_0,$$

and also

$$f(x_0) > f(x), \quad \text{if} \quad x > x_0.$$

In other words, at $x = x_0$ the function $f(x)$ has a *maximum.*

(2) The second part of the theorem is proved in an analogous manner.

Note. It is possible to prove that the theorem remains valid if the derivative $f'(x_0)$ does not exist at the critical point x_0, the function $f(x)$ being continuous at $x = x_0$.

Theorem 1'. *If for a differentiable function* $f(x)$ *its derivative* $f'(x)$ *vanishes at* $x = x_0$ *but retains its sign in passing through this value, then at* $x = x_0$ *the function* $f(x)$ *has no extremum.*

Proof. Indeed, if, for instance,

$$f'(x_0) = 0$$

and

$$f'(x) > 0 \quad \text{for} \quad x_0 - \varepsilon < x < x_0 + \varepsilon, \quad x \neq x_0,$$

then the function $f(x)$ is increasing both on the closed interval $[x_0 - \varepsilon, x_0]$ and on the closed interval $[x_0, x_0 + \varepsilon]$. Hence, at $x = x_0$ the function has neither a maximum, nor a minimum.

Using these theorems in investigating a differentiable function $f(x)$ for an extremum, we first find the critical values of the argu-

ment, i.e. those values of x_0 for which

$$f'(x_0) = 0,$$

and then, on choosing for each of such values x_0 a sufficiently small interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ such that it contains no other critical values (of course, if it is possible!), we check the character of this value according to the following scheme:

$f'(x_0 - h)$	$f'(x_0 + h)$	Conclusion
+	+	No extremum
+	−	Maximum
−	+	Minimum
−	−	No extremum

where the variable h runs through the interval

$$0 < h < \varepsilon.$$

Example. Investigate for an extremum the function

$$f(x) = x^3 - 6x^2 + 9x + 5.$$

Solution. We find the derivative

$$f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3).$$

Equating it to zero and solving the quadratic equation thus obtained, we get the roots of the derivative:

$$x_1 = 1 \quad \text{and} \quad x_2 = 3.$$

Hence,

$$f'(x) = 3(x - 1)(x - 3).$$

Let us investigate how $f'(x)$ changes its sign near the value $x = 1$. For any sufficiently small positive number h we have

x	$f'(x)$
$1-h$	+
$1+h$	−

Consequently, at $x = 1$ the function $f(x)$ has a maximum equal to $f(1) = 9$. Analogously, for the value $x = 3$ we get

x	$f'(x)$
$3-h$	−
$3+h$	+

Therefore, at $x = 3$ the function $f(x)$ has a minimum, and $f(3) = 5$.

The graph of the function

$$y = x^3 - 6x^2 + 9x + 5$$

is represented in Fig. 115.

Theorem 2 (the second rule). *If for a differentiable function $f(x)$ at some point x_0 its first derivative $f'(x)$ is equal to zero and its second derivative $f''(x)$ exists and is non-zero, i.e.*

$$f'(x_0) = 0, \quad f''(x_0) \neq 0,$$

then at this point the function $f(x)$ has an extremum; namely: (1) if $f''(x_0) > 0$, then $f(x_0)$ is a minimum of the function $f(x)$, and (2) if $f''(x_0) < 0$, then $f(x_0)$ is a maximum of the function $f(x)$.

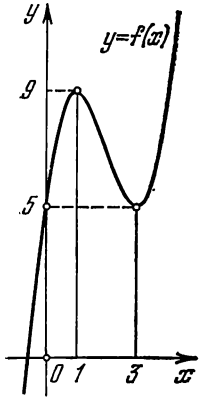


FIG. 115

Proof. (1) Let us first consider the case: $f'(x_0) = 0$ and $f''(x_0) > 0$. Let $x = x_0 + \Delta x_0$ be a point close to x_0. Since the second derivative $f''(x)$ is a derivative of the first derivative $f'(x)$, we have

$$f''(x_0) = \lim_{\Delta x_0 \to 0} \frac{f'(x_0 + \Delta x_0) - f'(x_0)}{\Delta x_0} = \lim_{x \to x_0} \frac{f'(x)}{x - x_0}$$

(here we have taken advantage of the fact that $f'(x_0) = 0$). Thus, the variable quantity $\frac{f'(x)}{x - x_0}$ tends to the limit $f''(x_0) \neq 0$, and, hence, beginning with some moment, this quantity has the sign of its limit (Sec. 49, lemma), i.e. in our case the plus sign. Therefore

$$\frac{f'(x)}{x - x_0} > 0 \quad \text{for} \quad 0 < |x - x_0| < \varepsilon,$$

where ε is a sufficiently small positive number. Hence we see that both the numerator and the denominator of the fraction $\frac{f'(x)}{x - x_0}$ have the same sign, and, consequently,

$$f'(x) < 0 \quad \text{for} \quad x_0 - \varepsilon < x < x_0$$

and

$$f'(x) > 0 \quad \text{for} \quad x_0 < x < x_0 + \varepsilon.$$

We see that the derivative $f'(x)$ changes its sign for plus when passing through the point x_0. According to Theorem 1, the number $f(x_0)$ is a *minimum of the function $f(x)$*.

(2) We can prove in a similar way that if $f'(x_0) = 0$ and $f''(x_0) < 0$ then $f(x_0)$ is a *maximum of the function $f(x)$*.

The theory of extrema of functions has numerous practical applications.

Problem. Given a triangle ABC whose base $AC = b$ and altitude $BL = h$ (Fig. 116). Find the rectangle of the largest area which can be inscribed in this triangle.

We denote the altitude KL of the required rectangle by x, and its base DE by y (Fig. 116). Then its area

$$U = xy.$$

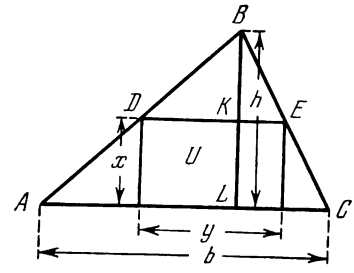


FIG. 116

The variables x and y are not independent, since they are connected by some relation. Indeed, from the similarity of the triangles DBE and ABC (taking into consideration that their altitudes BK and BL are proportional to the bases DE and AC), we have

$$\frac{BK}{BL} = \frac{DE}{AC},$$

or, since

$$BK = h - x, \quad DE = y, \quad BL = h, \quad AC = b,$$

then, consequently,

$$\frac{h-x}{h} = \frac{y}{b}.$$

Hence

$$y = \frac{b}{h}(h-x).$$

Eliminating y from the expression for U, we find

$$U = \frac{b}{h}(h-x)x = \frac{b}{h}(hx - x^2). \tag{91-2}$$

Now we are looking for a maximum of this function. Differentiating we get

$$U' = \frac{b}{h}(h-2x).$$

Equating the derivative U' to zero, we find

$$h - 2x = 0 \quad \text{or} \quad x = \frac{h}{2}.$$

It is easy to see that this value of x really yields a maximum of the function U. Indeed, writing the second derivative, we shall

have

$$U'' = -\frac{2b}{h} < 0.$$

Consequently, for $x = \frac{h}{2}$ the area U has a maximum, and from formula (91-2) we obtain

$$U_{\max} = \frac{bh}{4}.$$

Thus, the area of the largest rectangle inscribed in the triangle is equal to half the area of this triangle.

Sec. 92. Concavity and Convexity of the Graph of a Function. Points of Inflection

Definition. *The graph of a differentiable function* $y = f(x)$ ***is called concave upward**** (*or convex downward***) *in an interval* $\langle a, b \rangle$ *if the corresponding part of the curve*

$$y = f(x) \qquad (x \in \langle a, b \rangle) \tag{92-1}$$

lies above its tangent drawn at any of its points $M(x, f(x))$ (Fig. 117, *a*). *Analogously, the graph of a differentiable function* $y = f(x)$ *is said*

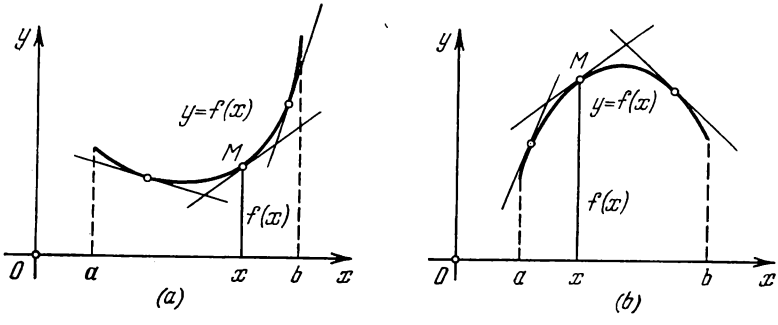


FIG. 117

to be convex upward (or concave downward) in an interval $\langle a, b \rangle$ *if the corresponding part of curve* (92-1) *lies below its tangent drawn at any of its points* $M(x, f(x))$ (Fig. 117, *b*).

Sufficient condition for concavity (convexity) of the graph of a function.

Theorem. (1) *If for a twice differentiable function* $y = f(x)$ *its second derivative* $f''(x)$ *is positive inside an interval* $\langle a, b \rangle$, *then the graph of this function is concave upward in the given interval.*

* That is in the positive direction of the y-axis.
** That is in the negative direction of the y-axis.

(2) *If the second derivative* $f''(x)$ *is negative inside an interval* $\langle a, b\rangle$, *then the graph of the function* $y = f(x)$ *is concave downward in this interval.*

Proof. (1) Let $f''(x) > 0$ for $a < x < b$ and x_0 be any point of the interval $\langle a, b\rangle$. We compare at point x the ordinate y of the curve $y = f(x)$ with the ordinate $\bar{y}$ of its tangent M_0N drawn at point $M_0(x_0, f(x_0))$ (Fig. 118). Since the slope of the tangent M_0N is equal to $f'(x_0)$, we have (see Sec. 16)

$$\bar{y} = f(x_0) + f'(x_0)(x - x_0).$$

Hence

$$\delta = y - \bar{y} = f(x) - f(x_0) - f'(x_0)(x - x_0). \qquad (92\text{-}2)$$

Using Lagrange's theorem (Sec. 85), we shall have

$$f(x) - f(x_0) = (x - x_0) f'(\xi),$$

where $\xi \in (x_0, x)$.

Therefore, from (92-2) we obtain

$$\delta = (x - x_0)[f'(\xi) - f'(x_0)]. \qquad (92\text{-}3)$$

Further, since $f''(x) = [f'(x)]' > 0$, then $f'(x)$ is an increasing function.

Let $x < x_0$, then, obviously $\xi < x_0$, and consequently, since $f'(x)$ is increasing, we have $f'(\xi) < f'(x_0)$. In this case from formula (92-3) we get $\delta > 0$.

If now $x > x_0$, then $\xi > x_0$ and therefore $f'(\xi) > f'(x_0)$. From formula (92-3) we derive once again: $\delta > 0$.

Thus, for $x \neq x_0$ we have

$$\delta = y - \bar{y} > 0, \quad \text{i.e.} \quad y > \bar{y}. \qquad (92\text{-}4)$$

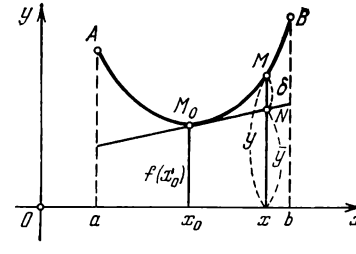


FIG. 118

Hence it follows that for $a < x < b$ the curve $y = f(x)$ lies *above* its tangents which means that the graph of the function $y = f(x)$ is concave *upward* on the interval $\langle a, b\rangle$.

(2) Similarly, it is proved that if for $a < x < b$ $f''(x) < 0$, then the graph of the function $y = f(x)$ is concave *downward* on the interval $\langle a, b\rangle$.

Definition. *The point of inflection of the graph of a differentiable function* $y = f(x)$ *is a point in which the curve changes its concavity for convexity or vice versa* (Fig. 119). *In other words, at such a point the direction of bending changes.*

Theorem. *If for a function* $y = f(x)$ *its second derivative* $f''(x)$ *vanishes at a certain point* x_0 *and, passing through this point, changes*

its sign, then the point $M\ (x_0, f\ (x_0))$ *is a point of inflection on the graph of the function.*

Proof. Suppose the second derivative $f''\ (x)$ vanishes at point M and changes its sign at this point, say, plus for minus. Then, to the left of the point M the second derivative of the function $f\ (x)$ is positive, and therefore for $x_0 - \varepsilon < x < x_0$ the graph of this function is concave upward; to the right of the point M the second derivative $f''\ (x)$ is negative and, consequently, for $x_0 < x < x_0 + \varepsilon$ the graph of the function $y = f\ (x)$ is convex upward. Hence, at the point M the curve $y = f\ (x)$ changes concavity for convexity, and, therefore, the point M is an inflection point of this curve.

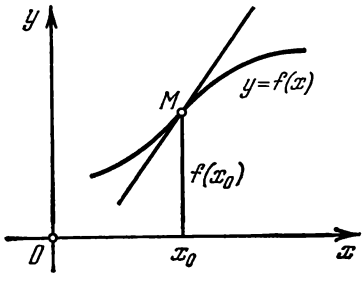


FIG. 119

Note. At a point of inflection x_0 of the function $y = f\ (x)$ the second derivative may also not exist; for instance, it may turn into infinity.

Example. Let there be given the *Gauss curve*

$$y = e^{-x^2}.$$

We have

$$y' = -2xe^{-x^2}$$

and

$$y'' = (4x^2 - 2)\, e^{-x^2} = 4\left(x^2 - \frac{1}{2}\right) e^{-x^2}.$$

The second derivative y'' vanishes if

$$x^2 - \frac{1}{2} = 0;$$

hence

$$x_1 = -\frac{1}{\sqrt{2}} \quad \text{and} \quad x_2 = \frac{1}{\sqrt{2}}.$$

The change of sign of the second derivative is characterized by the following table:

x	$-\infty < x < -\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}} < x < +\infty$
y''	+	−	+

Hence, the points $A\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$ and $B\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$ are

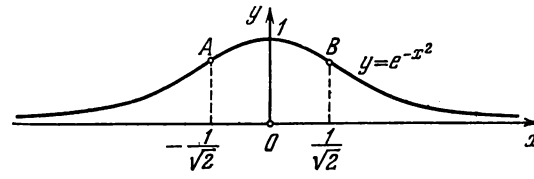


FIG. 120

points of inflection belonging to the given curve (Fig. 120).

Sec. 93. Approximate Solution of Equations

Consider the equation

$$f(x) = 0, \tag{93-1}$$

where the function $f(x)$ is defined and continuous on an interval $\langle a, b\rangle$. The value $\xi \in \langle a, b\rangle$ satisfying equation (93-1), i.e. such that $f(\xi) = 0$ is called the *root* of this equation (or *zero* of the function $f(x)$).

Geometrically, the roots of equation (93-1) represent the abscissas of the points of intersection of the graph of the function $y = f(x)$ with the x-axis (Fig. 121).

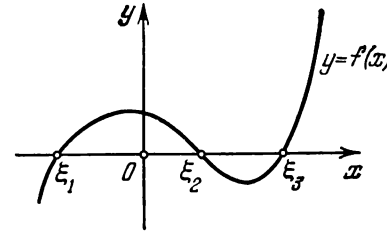


FIG. 112

For a geometrical solution of equation (93-1) it is sometimes advisable to replace it by an equivalent equation

$$\varphi(x) = \psi(x). \tag{93-2}$$

Then the roots of equation (93-1) are found as the *abscissas of the points of intersection* of the curves $y = \varphi(x)$ and $y = \psi(x)$.

Example 1. Solve graphically the equation

$$x + \log x = 2. \tag{93-3}$$

Obviously, we have

$$\log x = 2 - x.$$

Hence, the root of equation (93-3) represents the abscissa of the point of intersection of the logarithmic curve $y = \log x$ and a straight line $y = 2 - x$ (Fig. 122).

Constructing these lines on millimetre graph paper, we find approximately the root of equation (93-3): $\xi \approx 1.77$.

The following theorem seems to be geometrically obvious: *If a continuous function $f(x)$ takes on values of opposite signs at the end-points of a closed interval $[\alpha, \beta] \subset \langle a, b \rangle$* i.e. $f(\alpha) f(\beta) < 0$, then there is at least one zero of the function $f(x)$ inside the closed interval* $[\alpha, \beta]$ (i.e. there necessarily exists a root of the equation $f(x) = 0$).

This root will be unique if $f'(x)$ preserves its sign on (α, β) (in view of the fact that the function $f(x)$ is monotone).

Assuming that the equation $f(x) = 0$, where $f(x)$ is continuous on $\langle a, b \rangle$, has a unique root ξ inside the closed interval $[\alpha, \beta] \subset \langle a, b \rangle$, provided the condition $f(\alpha) f(\beta) < 0$ is fulfilled, we are going to indicate several simple methods for an approximate determination of this root.

A. Method of bisection. Let the function $f(x)$ be continuous on $[\alpha, \beta]$ and $f(\alpha) f(\beta) < 0$. We bisect the interval $[\alpha, \beta]$ and let γ be the mid-point of this interval. If $f(\gamma) = 0$, then γ is the desired root. If $f(\gamma) \neq 0$, then we denote by $[\alpha_1, \beta_1]$ one of the halves $[\alpha, \gamma]$

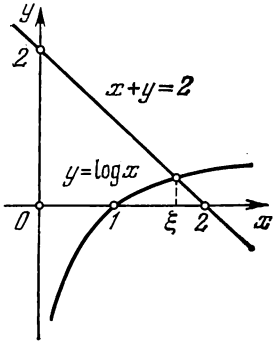


FIG. 122

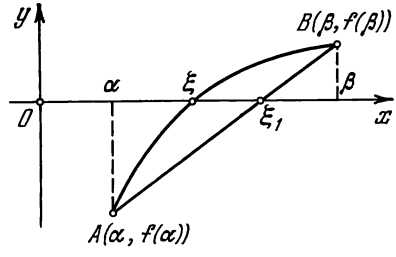


FIG. 123

or $[\gamma, \beta]$ at the end-points of which the function $f(x)$ has opposite signs. We then repeatedly apply this method. As a result, we obtain either the exact root of the equation $f(x) = 0$ or a convergent sequence of intervals

$$[\alpha, \beta] \supset [\alpha_1, \beta_1] \supset \ldots,$$

inside which the desired root ξ is contained (the "method of fork").

Since the length of the nth interval $[\alpha_n, \beta_n]$, equal to $\frac{\beta - \alpha}{2^n}$ tends to 0 as $n \to \infty$, repeating this method a sufficiently large

* The notation $[\alpha, \beta] \subset \langle a, b \rangle$ means that the segment $[\alpha, \beta]$ is contained in the interval $\langle a, b \rangle$.

number of times and putting $\xi \approx 1/2\,(\alpha_n + \beta_n)$, we can determine the sought-for root ξ with any preassigned accuracy.

B. Method of chords. We can make the method of bisection more precise by replacing the arc $\overset{\frown}{AB}$ of the curve $y = f(x)$ $(\alpha \leqslant x \leqslant \beta)$ by its chord $\overline{AB}$ passing through the end-points $A\,(\alpha,\ f(\alpha))$ and $B\,(\beta,\ f(\beta))$ and taking the abscissa ξ_1 of the point of intersection of the chord $\overline{AB}$ with the x-axis for an approximate value of the root ξ of the equation $f(x) = 0$ (Fig. 123).

If $f(\alpha)\,f(\beta) < 0$, then it means just the same as for the approximate value of the root ξ the point ξ_1 is taken which divides the interval $[\alpha,\ \beta]$ in the ratio $|f(\alpha)| : |f(\beta)|$ (the *method of proportional parts*).

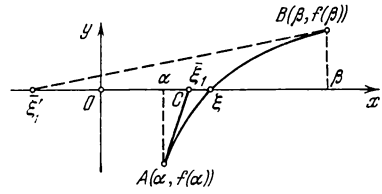


FIG. 124

The equation of the chord $\overline{AB}$ has the form (see Sec. 17)

$$y - f(\alpha) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}\,(x - \alpha). \tag{93-4}$$

Putting $y = 0$ in equation (93-4), we find the point of intersection $x = \xi_1$ of the chord AB with the x-axis (Fig. 123)

$$\xi_1 = \alpha - \frac{f(\alpha)}{f(\beta) - f(\alpha)}\,(\beta - \alpha). \tag{93-5}$$

The number ξ_1 is taken for the first approximation to the root ξ. If $f(\xi_1) \neq 0$, then formula (93-5) may be applied to that of the intervals $[\alpha,\ \xi_1]$ and $[\xi_1,\ \beta]$ at the end-points of which the function $f(x)$ attains values of opposite signs, and so on.

Example 2. Using the method of chords, determine the root of the equation

$$f(x) \equiv x^3 + x - 12 = 0. \tag{93-6}$$

To find an approximate value of the root of equation (93-6), we sketch the graphs of the functions $y = x^3$ and $y = 12 - x$. Trying on roughly, we reveal that the required root, that is the abscissa of the point of intersection of the graphs is contained in the interval (2, 3). Indeed,

$$f(2) = 8 + 2 - 12 = -2, \quad f(3) = 27 + 3 - 12 = +18.$$

Therefore, we may take $\alpha = 2$ and $\beta = 3$.

Applying formula (93-5), we get an approximate value of the root:

$$\xi_1 = 2 - \frac{-2}{18+2}(3-2) = 2.1.$$

We note that

$$f(\xi_1) = 9.261 + 2.1 - 12 = -0.639.$$

Therefore, to make the value ξ_1 more precise, formula (93-5) should be applied to the interval [2.1, 3].

C. Method of tangents. We replace now the arc $\overset{\frown}{AB}$ of the curve $y = f(x)$ by the tangent AC drawn at the point A $(\alpha, f(\alpha))$ (Fig. 124). Since the slope of the tangent AC is equal to $f'(\alpha)$, its equation has the form

$$y - f(\alpha) = f'(\alpha)(x - \alpha). \tag{93-7}$$

Hence, putting $y = 0$, we find for the root ξ its approximate value

$$\overline{\xi}_1 = \alpha - \frac{f(\alpha)}{f'(\alpha)} \tag{93-8}$$

(*Newton's formula*).

Note that if we construct on our drawing (Fig. 124) a tangent at point B $[\beta, f(\beta)]$, then the point of its intersection with the x-axis ($\overline{\xi}_1'$) will yield a poor approximation to the root ξ. Here it is advisable to stick to the following rule: *If the second derivative $f''(x)$ of a function retains its sign in an interval (α, β), then the tangent should be drawn at that end-point of the arc $\overset{\frown}{AB}$ for which the sign of the function coincides with the sign of its second derivative.*

Example 3. Using the method of tangents, determine the root of equation (93-6) contained within the interval (2, 3).

Here $f''(x) = 6x > 0$ for $2 \leqslant x \leqslant 3$, and $f(3) = +18$. Therefore, we put $\alpha = 3$ in formula (93-3). Since $f'(x) = 3x^2 + 1$ and $f'(3) = 28$, we have

$$\overline{\xi}_1 = 3 - \frac{18}{28} = 2.36.$$

It should be noted for a check that

$$f(\overline{\xi}_1) = 13.14 + 2.36 - 12 = +3.35.$$

Since in our case $\xi_1 < \xi < \overline{\xi}_1$, where $\xi_1 = 2.10$ and $\overline{\xi}_1 = 2.36$, we may put

$$\xi \approx \frac{1}{2}(\xi_1 + \overline{\xi}_1) = \frac{1}{2}(2.10 + 2.36) = 2.23.$$

Here

$$f(2.23) = 11.09 + 2.23 - 12 = 1.32.$$

To refine the root, we can apply a combined method of chords and tangents to the interval [2.10, 2.23], and so on.

Sec. 94. Construction of Graphs of Functions

In the foregoing sections it was shown how the general properties of a function are studied with the aid of the first and the second derivatives. Using the results of this study, we can conceive the character of a function and, in particular, successfully sketch its graph.

In simplest cases it is convenient to investigate a function $y = f(x)$ (which is assumed to be an elementary one) according to the following scheme.

(1) Analysing the properties of the function $f(x)$, determine the *domain of its existence.*

For the sake of simplicity, let us assume that it will be a certain interval $\langle a, b \rangle$. It is also useful to find out the character of symmetry of the graph (whether the function is even or odd, its periodicity, and so forth).

(2) Find the *points of discontinuity* of the function. Also study the behaviour of the function as $x \to a$ and $x \to b$, where a and b are boundary points of the domain of existence of the function.

(3) Solving the equation

$$f(x) = 0, \tag{94-1}$$

determine the roots (zeros) of the function. Find out the sign of the function as it takes on various values within the domain of definition, bearing in mind that an elementary function changes its sign only when passing through a zero or through a point of discontinuity.

(4) Solving the equation

$$f'(x) = 0, \tag{94-2}$$

find the *critical values of the argument for the function* $f(x)$. Studying then the sign of the derivative $f'(x)$ in each of the subintervals between two neighbouring critical values, determine the subintervals of increasing and decreasing of the function and find out the character of these critical values.

(5) Solving the equation

$$f''(x) = 0, \tag{94-3}$$

determine the critical values of the argument for the derivative $f'(x)$. Find out then the sign of the derivative $f''(x)$ in each of the subintervals between two neighbouring critical values of the argument for the derivative $f'(x)$, establish the subintervals on which the graph of the function $f(x)$ is convex and concave upward, and find the points of inflection.

In more complicated cases investigate also the points at which the derivatives $f'(x)$ and $f''(x)$ are nonexistent.

For the solution of equations (94-1), (94-2), and (94-3) we have, possibly, to apply the approximate methods set forth in the preceding section.

Tabulating, finally, the values of the function for its *characteristic points* (the boundary points of the domain of the function, points of discontinuity, points of intersection of the graph with the coordinate axes, points of extremum, points of inflection, and so on) and taking into account the results of the above carried investigation, we plot the graph of the function. A few intermediate points will suffice to get a curve of sufficient accuracy.

It should be noted that in some cases an incomplete investigation of a function may turn out to be sufficient for constructing its graph.

Example. Construct the graph of the function

$$y = x + \sqrt{1-x}.$$

We investigate the function according to the above scheme:
(1) The function is defined if

$$0 \leqslant 1 - x < +\infty.$$

Hence the domain of its existence:

$$-\infty < x \leqslant 1.$$

(2) There are no points of discontinuity, and

$$\lim_{x \to -\infty} y = -\infty$$

and

$$\lim_{x \to 1} y = y(1) = 1.$$

(3) Solving the equation

$$x + \sqrt{1-x} = 0,$$

we obtain the root of the function

$$x_0 = -\frac{1+\sqrt{5}}{2} \approx -1.62;$$

where $y < 0$, if $-\infty < x < x_0$, and $y > 0$, if $x_0 < x \leqslant 1$.

(4) We find the first derivative

$$y' = 1 - \frac{1}{2\sqrt{1-x}}.$$

Equating it to zero, we obtain a critical point $x_1 = \frac{3}{4}$. Besides, obviously, y' turns into infinity for $x = 1$. Therefore, $x_2 = 1$ will also be a critical point.

The intervals of monotonicity of the function are $\left(-\infty, \frac{3}{4}\right)$ and $\left(\frac{3}{4}, 1\right)$, and, as it is easy to be convinced by investigating the sign of the derivative, the function is increasing in the first subinterval and decreasing in the second. Consequently, x_1 is a point of maximum of the function. As is obvious, at point x_2 the function has a boundary minimum.

(5) We find the second derivative

$$y'' = -\frac{1}{4(1-x)^{3/2}}.$$

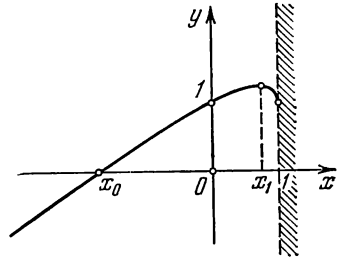


FIG. 125

Since the second derivative is everywhere negative, the graph of the function is concave downward and there is no point of inflection.

The results of our investigation are tabulated below:
An approximate graph of the function is represented in Fig. 125.

x	y	y'	y''	Note
$-\infty$	$-\infty$	$+$	$-$	The function is increasing
$x_0 \approx -1.68$	0			
0	1			
$x_1 = 0.75$	1.25	0		Maximum of the function
...	...	$-$		The function is decreasing
1	1	$-\infty$		

Exercises

1. Find the increment of the function

$f(x) = x^3$

on the segment $-1 \leqslant x \leqslant 2$ and check the theorem about the finite increment of a function.

2. Verify the theorem about the roots of a derivative for the function

$f(x) = (x-1)(x-2)(x-3)$

on the segments [1, 2] and [2, 3].

In Problems 3 to 5 determine the intervals of monotonicity for the given function $f(x)$:

3. $f(x) = x^2 - 4x - 1$.

4. $f(x) = 3x - x^3$.

5. $f(x) = xe^{-x}$.

In Problems 6 to 17 find the limits of the given functions:

6. $\lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$. **7.** $\lim\limits_{x\to 0} \dfrac{3^x - 2^x}{\ln(1+x)}$.

8. $\lim\limits_{x\to 0} \dfrac{x - \sin x}{\tan x - \sin x}$.

Derive the formula for approximate rectification of small arcs:

$x \approx \frac{1}{3}(2\sin x + \tan x)$.

9. $\lim\limits_{x\to 1} \dfrac{1-x^2}{\sin \pi x}$. **10.** $\lim\limits_{x\to+\infty} \dfrac{x^3}{1.01^x}$.

11. $\lim\limits_{x\to+\infty} \dfrac{\ln^2 x}{\sqrt[100]{x}}$. **12.** $\lim\limits_{x\to 1} (1-x)\tan\dfrac{\pi x}{2}$.

13. $\lim\limits_{x\to\infty} \left(x \arcsin \dfrac{2}{x}\right)$. **14.** $\lim\limits_{x\to 1} \left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right)$.

15. $\lim\limits_{x\to 0} x^x$. **16.** $\lim\limits_{x\to 1} x^{\frac{1}{1-x}}$.

17. $\lim\limits_{x\to 0} (\cot x)^{\sin x}$.

Evaluate the extrema of the following functions:

18. $y = x(a-x)\ (a > 0)$.

19. $y = \dfrac{x^3}{3} - 2x^2 + 3x - 1$.

20. $y=\frac{2x}{1+x^2}$. **21.** $y=\sqrt{2+x-x^2}$. **22.** $y=xe^{-2x}$.

23. In a semi-circle of radius a inscribe a rectangle of the largest area.

24. In an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

inscribe a rectangle of the largest area whose sides are parallel to the axes of the ellipse.

25. In a sphere of radius a inscribe a cone of the greatest volume.

26. At what dimensions has a box made of a square sheet of paper with side a a maximum capacity?

27. Determine the points of inflection of the graph of the function $y = x^2 - x^3$.

Construct the graphs of the following functions:

28. $y = 3x - x^3$.

29. $y=x^2(2-x)^2$. **30.** $y=\frac{x^2}{1+x^2}$.

31. $y=x+\frac{1}{x}$. **32.** $y=\frac{x}{1+x}$.

33. $y=\frac{8}{4-x^2}$. **34.** $y=x\sqrt{1-x}$.

35. $y=\sqrt{\frac{1-x}{x}}$. **36.** $y=\sin x+\cos x\ (0\leqslant x\leqslant 2\pi)$.

37. $y=x\ln x$*. **38.** $y=xe^{-x}$*. **39.** $y=x-\arctan x$.

* Pay attention to the hint given in Answers

Chapter 12

Differentials

Sec. 95. The Differential of a Function

Suppose we have a certain differentiable function

$$y = f(x).$$

The increment Δy of the function y serves as an important characteristic of the change of this function on a given closed interval $[a, b]$. But it is sometimes difficult to determine the increment of a function in a direct manner. Then we usually proceed in the following way: we subdivide the closed interval $[a, b]$ into a finite number of sufficiently small intervals $[x, x + \Delta x]$ and assume that on each of them the function increases according to the proportionality law (for instance, a small element of a curve is regarded as rectilinear; nonuniform motion of a point during a small interval of time is treated as uniform, and so on, where "smallness" is understood in a sense). In other words, we assume that on a sufficiently small interval $[x, x + \Delta x]$ the following equality takes place

$$\Delta y \approx k\,\Delta x,$$

where the proportionality factor k is independent of Δx but, generally speaking, depends on x. And if it turns out that at a proper choice of the proportionality factor the error

$$\Delta y - k\Delta x$$

is an infinitesimal of a higher order than Δx, i.e. the ratio

$$\frac{\Delta y - k\,\Delta x}{\Delta x} = \alpha \tag{95-1}$$

is infinitely small as $\Delta x \to 0$, then the quantity

$$dy = k\,\Delta x$$

is called the *differential of the function* y at a point x (here the letter d is the differential sign). In this case, as it follows from relation

(95-1), the following equality is true

$$\Delta y = k\Delta x + \alpha\,\Delta x, \tag{95-2}$$

where $\alpha \to 0$ as $\Delta x \to 0$.

In other words, (see Sec. 51, Chapter 7)

$$\Delta y = dy + 0\,(\Delta x).$$

Definition. *The magnitude proportional to the infinitesimal increment Δx of the argument and differing from the corresponding increment of the function by an infinitesimal of higher order than Δx is called the differential of that function.*

The first term $k\,\Delta x$ in formula (95-2) is frequently called the *principal linear part of the increment of a function* (or the *principal linear term of the increment*). Therefore, we may say: *The differential of a function represents the principal linear part of the infinitesimal increment of this function.*

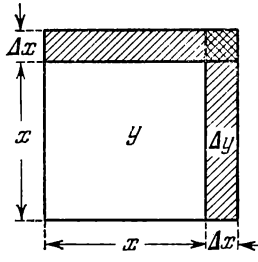


FIG. 126

Example 1. Let the function $y = x^2$ be the area of a square whose side is equal to x (Fig. 126). If we assign an increment Δx to the side x, then its new value will become equal to $x + \Delta x$ and, consequently, the area y of the square will receive increment

$$\Delta y = (x + \Delta x)^2 - x^2,$$

or

$$\Delta y = 2x\,\Delta x + (\Delta x)^2.$$

The first term of the sum constituting the right-hand member of the last equality is, obviously, the principal linear part of the function as $\Delta x \to 0$. Therefore,

$$dy = 2x\,\Delta x.$$

In Fig. 126 the increment Δy of the function is represented by the area of the entire hatched portion, whereas the differential dy of the function is represented by the area of the hatched portion less the area of the small square situated at the right-hand upper corner of the large square.

Uniqueness theorem. *A given function can have only one differential.*

Proof. Indeed, suppose the function

$$y = f(x)$$

has two differentials:

$$dy = k\,\Delta x \quad \text{and} \quad d_1 y = k_1\,\Delta x.$$

By definition of the differential, we have

$$\Delta y = k\,\Delta x + \alpha\,\Delta x$$

and

$$\Delta y = k_1 \Delta x + \alpha_1 \Delta x,$$

where α and α_1 are infinitesimals as $\Delta x \to 0$. Hence,

$$k \Delta x + \alpha \Delta x = k_1 \Delta x + \alpha_1 \Delta x$$

and, consequently, for $\Delta x \neq 0$ we have

$$k - k_1 = \alpha_1 - \alpha.$$

Passing to the limit as $\Delta x \to 0$ we obtain

$$k - k_1 = 0,$$

i.e. $k = k_1$. Thus, the differentials dy and $d_1 y$ coincide. Q.E.D.

From the definition of the differential it directly follows that *the differential of a function differs from the corresponding increment of this function by an infinitesimal of higher order as compared with the increment of the independent variable.* This circumstance is frequently made use of in carrying out approximate calculations.

Example 2. Let

$$y = x^3 - x^2 + 2x + 3.$$

Find Δy and dy for $x = 1$ and compare them with each other in the following three cases:

(a) $\Delta x = 1$; (b) $\Delta x = 0.1$ and (c) $\Delta x = 0.01$.

We have

$$\Delta y = [(x + \Delta x)^3 - (x + \Delta x)^2 + 2(x + \Delta x) + 3] - (x^3 - x^2 + 2x + 3).$$

Carrying out algebraic computation, we get

$$\Delta y = (3x^2 - 2x + 2) \cdot \Delta x + (3x - 1) \cdot (\Delta x)^2 + (\Delta x)^3.$$

The first term of the right-hand member of the last equality is, obviously, the principal linear part of the increment of the function. Hence,

$$dy = (3x^2 - 2x + 2) \cdot \Delta x.$$

Putting $x = 1$, we obtain the following table:

Δx	Δy	dy	$\frac{dy}{\Delta y}$ (in per cent)
1	6	3	50
0.1	0.321	0.3	93
0.01	0.030201	0.03	99

From this table it is clearly seen that the share of the differential dy in the increment Δy tends to 100% as $\Delta x \to 0$.

Sec. 96. Relation Between the Differential of a Function and Its Derivative. The Differential of the Independent Variable

Theorem 1. *If a function has a differential, then this function also has a derivative.*

Proof. Indeed, let there be given a certain function

$$y = f(x),$$

and let

$$dy = k\,\Delta x$$

be the differential of this function. As is known, the increment Δy may be written in the following form:

$$\Delta y = k\,\Delta x + \alpha\,\Delta x,$$

where α is an infinitesimal as $\Delta x \to 0$. Whence we derive

$$\frac{\Delta y}{\Delta x} = k + \alpha$$

and, hence,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = k,$$

i.e. the derivative y' exists and is equal to k.

Corollary. *The differential of a function is equal to the product of the derivative of this function by the increment of the independent variable*, i.e.

$$dy = y'\,\Delta x. \tag{96-1}$$

Theorem 2. *If a function has a derivative, then this function also has a differential.*

Proof. Let the function

$$y = f(x)$$

have a derivative

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = y'.$$

Hence we derive

$$\frac{\Delta y}{\Delta x} = y' + \alpha,$$

where α is an infinitesimal as $\Delta x \to 0$ and, consequently,

$$\Delta y = y' \Delta x + \alpha \Delta x. \tag{96-2}$$

In sum (96-2) the first term $y' \Delta x$, obviously, represents the principal linear part of the increment Δy and is thus the differential of the function y. Hence, the function has a differential

$$dy = y' \Delta x,$$

which is what we had to prove.

Note. Now it is clear why a function of one argument, having a derivative, is called *differentiable.*

Hitherto we have used the notion of the differential of a function. Let us now introduce the notion of the *differential of an independent variable.*

Definition. *The differential of an independent variable is understood as the differential of a function identical to the independent variable,* i.e. of the function $y = x$.

Since

$$y' = 1 \quad \text{for} \quad y = x, \tag{96-3}$$

then, by (96-1), we have

$$dx = \Delta x,$$

i.e. *the differential of an independent variable is equal to the increment of this independent variable.*

Taking advantage of this property, we may rewrite formula (96-1) in the following symmetrical form:

$$dy = y' dx. \tag{96-4}$$

Thus, *the differential of a function is equal to the product of the derivative of this function by the differential of the independent variable.*

Dividing both sides of the last formula by dx, we get

$$\frac{dy}{dx} = y'.$$

In other words: *The derivative of a function is equal to the ratio of the differential of this function to the differential of the independent variable.*

Hitherto the notation $\frac{dy}{dx}$ has been a symbol; now we may regard this expression as a common fraction with the numerator dy and the denominator dx.

Sec. 97. The Geometrical Meaning of the Differential

Let us find out the geometrical meaning of the differential of a function. Consider the graph of the function $y = f(x)$.

Let $M(x, y)$ and $M'(x + \Delta x, y + \Delta y)$ be two points on the given curve (Fig. 127). At the point M draw a tangent MT to the graph of the function (here T is the point of intersection of the tangent with $M'N$ which is parallel to the y-axis) and consider the triangle MTN with the legs $MN = \Delta x$ and NT (MN is parallel to the x-axis and NT to the y-axis). Denoting by φ the angle formed by the tangent MT with the positive direction of the x-axis, we have

$$NT = \Delta x \tan \varphi.$$

But from the geometrical meaning of the derivative it follows:

$$\tan \varphi = f'(x) = y'.$$

Therefore

$$NT = y' \Delta x = dy.$$

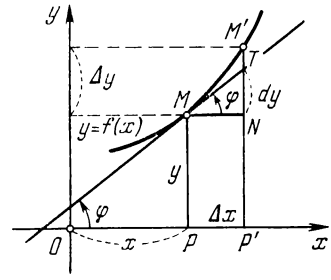


FIG. 127

Hence, we have the following theorem: *The differential of the function $y = f(x)$ at a given point x is equal to the increment of the ordinate of the tangent to the graph of the function at this point when x receives an increment Δx.*

Note. The increment of the function $\Delta y = NM'$ (Fig. 127) is, generally speaking, not equal to the differential $dy = NT$ of this function. In particular: (1) if the graph of the function is concave upward, then

$$\Delta y > dy;$$

(2) and if it is concave downward, then

$$\Delta y < dy.$$

Sec. 98. The Physical Meaning of the Differential

Suppose we know the law of motion of a point M along the x-axis:

$$x = f(t),$$

where x is the distance of the point M from the origin O and t is time, and we assume that the point M moves in one and the same direction. During an infinitesimal time interval dt the point M

will occupy the position M', thus covering the distance

$$\Delta x = f(t + dt) - f(t).$$

This is the *true increment of the distance.*

According to formula (96-4), the differential of the distance

$$dx = x'_t\, dt.$$

But x'_t, representing the derivative of the distance with respect to time, is the velocity v at the moment t. Therefore,

$$dx = v\, dt.$$

Thus, *the differential of distance is equal to the imagined increment of the distance which would have been obtained if we assume that, beginning with a given moment, the point performs uniform motion preserving the gained velocity.*

For instance, if a car speedometer reads 60 km/hr, then the driver, when calculating that for one minute the car will cover one kilometer, actually computes not the increment of the distance during one minute (which, due to a non-uniform motion, may be not equal to one kilometer!) but the *differential of the distance.*

Sec. 99. Approximate Calculation of Small Increments of a Function

If Δx is small by absolute value, then for a differentiable function $f(x)$ its increment

$$\Delta f(x) = f(x + \Delta x) - f(x)$$

differs from the differential

$$df(x) = f'(x)\, \Delta x$$

by a quantity infinitesimal with respect to Δx. Hence we have an approximate equality

$$f(x + \Delta x) - f(x) \approx f'(x)\, \Delta x \qquad (99\text{-}1)$$

or

$$f(x + \Delta x) \approx f(x) + f'(x)\, \Delta x. \qquad (99\text{-}1')$$

These equalities are very useful in carrying out approximate calculations. Note that formula (99-1′) represents the linear term of Taylor's formula (Sec. 90, Chapter 11).

Example 1. Find

$$\sqrt[3]{1.1}.$$

Setting in formula (99-1′)

$$f(x)=\sqrt[3]{x},\quad f'(x)=\frac{1}{3\sqrt[3]{x^2}},\quad x=1,\quad \Delta x=0.1,$$

we shall have

$$\sqrt[3]{1.1}=\sqrt[3]{1}+0.1\,\frac{1}{3\sqrt[3]{1^2}}=1+0.1\cdot\frac{1}{3}=1.033.$$

The value taken from the corresponding table is

$$\sqrt[3]{1.1}=1.032.$$

Let us consider one more problem which is of importance for approximate calculations.

Problem. For a given function

$$y=f(x)$$

the limiting absolute error of its argument x is equal to Δx, i.e.

$$|\Delta x|\leqslant\Delta_x.$$

What are the limiting absolute error Δ_y and the limiting relative error δ_y of the function y?

From formula (99-1) we have

$$|\Delta y|\approx|y'|\,|\Delta x|;$$

consequently, for $y'\neq 0$, we may take

$$\Delta_y=|y'|\,\Delta_x \tag{99-2}$$

and

$$\delta_y=\frac{\Delta_y}{|y|}=|(\ln y)'|\,\Delta_x.$$

Example 2. The angle $x=60°$ is determined to within 1°. What is in this case the error for $\sin x$?

Here

$$\Delta_x=\operatorname{arc}1°=\frac{\pi}{180}\approx\frac{1}{57}.$$

Therefore, the error for $y=\sin x$, on the basis of (99-2), where $y'=\cos x$, can reach the quantity

$$\Delta_y=\cos 60°\cdot\Delta_x\approx\frac{1}{114}\approx 0.01.$$

Sec. 100. Equivalence of the Increment and Differential of a Function

Let us introduce the notion of *equivalent* or *asymptotically equal* infinitesimal functions.

Definition. *Two infinitesimal functions $\alpha=\alpha(x)$ and $\beta=\beta(x)$ are called equivalent as $x\to a$ if the limit of their ratio is equal to unity,*

i.e. if

$$\lim_{x \to a} \frac{\alpha}{\beta} = 1.$$

To denote the fact that the infinitesimals α and β are equivalent to each other, we use the equivalence sign $\sim$, namely, we write $\alpha \sim \beta$.

Thus, for instance,

$$\sin \alpha \sim \alpha$$

as $\alpha \to 0$, since

$$\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1.$$

Note that *if two infinitesimals α and β are equiva lent to each other, then their difference is an infinitesimal of a higher order than either of them.*

Indeed, if

$$\frac{\alpha}{\beta} \to 1,$$

then we have

$$\frac{\alpha - \beta}{\beta} = \frac{\alpha}{\beta} - 1 \to 0,$$

i.e. $\alpha - \beta$ is of the order higher than β (see Sec. 51, Chapter 7). Analogous reasoning holds with respect to α as well.

Conversely, *if the difference between two infinitesimals α and β is an infinitesimal of higher order as compared with either of them, then these infinitesimals are equivalent to each other.*

Indeed, assuming that

$$\frac{\alpha - \beta}{\beta} = \frac{\alpha}{\beta} - 1 \to 0,$$

we get

$$\frac{\alpha}{\beta} \to 1$$

and, hence,

$$\alpha \sim \beta.$$

In particular, *rejecting from (or adding to) an infinitesimal an infinitesimal of higher order we get a quantity equivalent to the initial one.*

For instance, as $x \to 0$, we have

$$(x + 1000x^2) \sim x.$$

Here is an important property of equivalent infinitesimals.

Theorem 1. *When finding the limit of the ratio of two infinitesimals the given infinitesimals may be substituted by ones equivalent to them (assuming that the limit of the ratio of the latter, finite or infinite, exists).*

Proof. Indeed, let $\alpha(x) \sim \alpha_1(x)$ and $\beta(x) \sim \beta_1(x)$ as $x \to a$. We have

$$\frac{\alpha(x)}{\beta(x)} \equiv \frac{\alpha(x)}{\alpha_1(x)} \cdot \frac{\alpha_1(x)}{\beta_1(x)} \cdot \frac{\beta_1(x)}{\beta(x)}. \tag{100-1}$$

Passing to the limit in identity (100-1), we obtain

$$\lim_{x \to a} \frac{\alpha(x)}{\beta(x)} = \lim_{x \to a} \frac{\alpha(x)}{\alpha_1(x)} \cdot \lim_{x \to a} \frac{\alpha_1(x)}{\beta_1(x)} \cdot \lim_{x \to a} \frac{\beta_1(x)}{\beta(x)} =$$

$$= 1 \cdot \lim_{x \to a} \frac{\alpha_1(x)}{\beta_1(x)} \cdot 1 = \lim_{x \to a} \frac{\alpha_1(x)}{\beta_1(x)}.$$

Example. Since $\sin x \sim x$ as $x \to 0$ (Sec. 54, Chapter 7), we get

$$\lim_{x \to 0} \frac{2x + 5x^2}{\sin 3x} = \lim_{x \to 0} \frac{2x}{3x} = \frac{2}{3}.$$

Theorem 2. *The infinitesimal increment of a function is equivalent to the differential of this function for all values of the independent variable for which the derivative of the function is finite and non-zero.*

Proof. Indeed, if the function

$$y = f(x)$$

is differentiable, then from (96-2) we have

$$\Delta y = dy + \alpha \Delta x, \tag{100-2}$$

where α is an infinitesimal as $\Delta x \to 0$.

Since, according to the condition of the theorem, for $\Delta x \neq 0$ we have

$$dy = y' \Delta x \neq 0,$$

then

$$\frac{\Delta y}{dy} = 1 + \frac{\alpha}{y'}.$$

Consequently,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{dy} = 1,$$

i.e. the infinitesimals Δy and dy are equivalent as $\Delta x \to 0$.

Example. Let $f(x) = (1 + x)^{\alpha}$. We have

$$\Delta f(0) = f(x) - f(0) = (1 + x)^{\alpha} - 1$$

and

$$df(0) = f'(0)(x - 0) = \alpha x.$$

Therefore

$$(1 + x)^{\alpha} - 1 \sim \alpha x \quad \text{as} \quad x \to 0.$$

Note. In general, if the function $f(x)$ is differentiable at the point $x = 0$, then, as $x \to 0$, we have

$$\Delta f(0) = f(x) - f(0) \sim f'(0)\, x. \tag{100-3}$$

From formula (100-3), in particular, as $x \to 0$, we get

(a) $\sin x \sim x$;

(b) $a^x - 1 \sim x \ln a$ $(a > 0)$;

(c) $\ln(1 + x) \sim x$.

Sec. 101. Properties of the Differential

Let us now consider some properties of the differential which are analogous to those of the derivative.

In our further statements we shall assume, without mentioning this each time, that all the functions under consideration have derivatives, i.e. are differentiable.

I. The differential of a constant. *The differential of a constant is equal to zero.*

Setting in formula (96-4)

$$y = c \quad \text{and} \quad y' = 0,$$

we get

$$dc = 0.$$

II. The differential of a sum. *The differential of an algebraic sum of several differentiable functions is equal to the algebraic sum of their differentials.*

Indeed, if u, v, and w are differentiable functions with respect to an independent variable x, then, for instance, we have

$$(u + v - w)' = u' + v' - w'.$$

Multiplying both sides by dx, we get

$$(u + v - w)'\, dx = u'\, dx + v'\, dx - w'\, dx.$$

Hence, according to formula (96-4), we derive

$$d(u + v - w) = du + dv - dw.$$

III. *If two differentiable functions differ from each other by a constant term, then their differentials are equal to each other.*

We have

$$d(u + c) = du + dc.$$

Assuming here c to be constant and, hence, $dc = 0$, we get

$$d(u + c) = du.$$

IV. *A constant may be factored out in front of the differential sign.* Indeed, if c is a constant, then

$$(cu)' = cu'.$$

Multiplying both sides of this equality by dx, we obtain

$$(cu)'\,dx = c\,(u'\,dx)$$

or

$$d(cu) = c\,du.$$

V. The differential of a product. *The differential of a product of two functions is equal to the sum of the products of each of the functions by the differential of the other.*

Indeed, if u and v are differentiable functions of x, then we have

$$(uv)' = uv' + vu'.$$

Multiplying both sides by dx, we get

$$(uv)'\,dx = u\,(v'\,dx) + v\,(u'\,dx)$$

or

$$d(uv) = u\,dv + v\,du.$$

VI. The differential of a quotient (fraction). *The differential of a fraction is also a fraction whose numerator is the product of the denominator of the fraction by the differential of the numerator minus the product of the numerator by the differential of the denominator, and the denominator is the square of the denominator of the given fraction.*

We have

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}.$$

Multiplying both sides by dx, we get

$$\left(\frac{u}{v}\right)' dx = \frac{v\,(u'\,dx) - u\,(v'\,dx)}{v^2}.$$

Hence

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

VII. The differential of a composite function. *The differential of a composite function (function of function) is equal to the product of the derivative of this function with respect to the intermediate argument by the differential of this intermediate argument (both functions are differentiable).*

Let

$$y = f[\varphi(x)].$$

We put $\varphi(x) = u$ and, hence, $y = f(u)$. If $f(u)$ and $\varphi(x)$ are differentiable functions, then, according to the theorem about the derivative of a function of function, we may write

$$y'_x = y'_u u'_x.$$

Multiplying both sides of this equality by the differential dx of the independent variable x, we get

$$y'_x\,dx = y'_u(u'_x\,dx). \tag{101-1}$$

But

$$y'_x\,dx = dy \quad \text{and} \quad u'_x\,dx = du;$$

consequently, equality (101-1) may be rewritten as

$$dy = y'_u\,du. \tag{101-2}$$

Note. Formula (101-2) coincides by its form with formula (96-4), but they differ fundamentally from each other: in formula (96-4) x is an independent variable, and hence, $dx = \Delta x$, whereas in (101-2) u is a function of an independent variable x, and therefore, generally speaking, $du \neq \Delta u$.

From (101-2) there follows such a theorem:

VIII. Invariance of the form of the differential. *The differential of a function is equal to the product of the derivative of this function by the differential of the argument irrespective of whether the argument is an independent variable or a differentiable function of another independent variable.*

On the basis of the formulas for the derivatives (Sec. 82, Chapter 10), we get the corresponding table of the differentials, where u is an arbitrary differentiable function.

Table of the Differentials of Functions

I. $du^n = nu^{n-1}\,du.$

II. $da^u = a^u \ln a\,du \quad (a > 0),$

$de^u = e^u\,du.$

III. $d(\log_a u) = \dfrac{du}{u \ln a} \quad (a > 0,\ a \neq 1),\quad d(\ln u) = \dfrac{du}{u}.$

IV. $d(\sin u) = \cos u\,du.$

V. $d(\cos u) = -\sin u\, du$.

VI. $d(\tan u) = \dfrac{du}{\cos^2 u}$.

VII. $d(\cot u) = -\dfrac{du}{\sin^2 u}$.

VIII. $d(\arcsin u) = \dfrac{du}{\sqrt{1-u^2}}$.

IX. $d(\arccos u) = -\dfrac{du}{\sqrt{1-u^2}}$.

X. $d(\arctan u) = \dfrac{du}{1+u^2}$.

XI. $d(\operatorname{arccot} u) = -\dfrac{du}{1+u^2}$.

XII. $df(u) = f'(u)\, du$.

Sec. 102. Differentials of Higher Orders

Let x be an independent variable and

$$y = f(x)$$

is a differentiable function. According to (96-4), we have

$$df(x) = f'(x)\, dx; \tag{102-1}$$

thus, the differential of the function $f(x)$ is a function of two arguments: x and dx.

Henceforward we shall assume that dx (which is the differential of an independent variable x) has an arbitrary but a *fixed value* independent of the independent variable x and one and the same for all functions under consideration.

If dx is fixed, then $df(x)$ is a function of x proportional to the derivative $f'(x)$ with the proportionality factor equal to dx. It may happen that this function also has a differential*. In such a case the latter is called the *differential of the second order* (or the *second differential*) of the function $f(x)$, the differential defined by formula (102-1) being called more precisely the *differential of the first order* (or the *first differential*).

Definition. *The differential of the second order* (or the *second differential*) $d^2f(x)$ *of a function* $f(x)$ *is a differential of the differential of the first order of this function*, i.e.

$$d^2f(x) = d[df(x)]. \tag{102-2}$$

Analogously, the *differential of the third order* (or the *third differential*) $d^3f(x)$ of a function $f(x)$ is a differential of the differential of the second order of this function, i.e.

$$d^3f(x) = d[d^2f(x)].$$

In such a manner we successively define *differentials of higher orders*.

Let us now derive the formula for the differential of the second order of a function $f(x)$ of an independent variable x, assuming

* The sufficient condition for which is the existence of the second derivative $f''(x)$.

that this function is twice differentiable, i.e. it has a derivative of the second order. Since

$$df(x) = f'(x)\,dx,$$

by (102-2), we have

$$d^2f(x) = d[f'(x)\,dx]. \qquad (102\text{-}3)$$

If x is an independent variable, then dx equal to Δx is, obviously, independent of x, i.e. dx plays the role of a constant with respect to the variable x. Therefore, in formula (102-3) dx may be factored out in front of the differential sign:

$$d^2f(x) = dx \cdot d[f'(x)].$$

Since $f'(x)$ is, in its turn, a function of x, it follows from (102-1):

$$d[f'(x)] = [f'(x)]'\,dx = f''(x)\,dx.$$

Hence, we finally find

$$d^2f(x) = f''(x)\,dx^2, \quad \text{where} \quad dx^2 = (dx)^2. \qquad (102\text{-}4)$$

Thus we may formulate the following theorem:

The differential of the second order of a given function is equal to the product of the derivative of the second order of this function by the square of the differential of the independent variable.

Note that formula (102-4) is, generally speaking, incorrect if x is not an independent variable, since here we cannot regard dx as a factor independent of x (see S. M. Nikolsky, *A Course of Mathematical Analysis*, Vol. 1, Mir Publishers, Moscow, 1977).

Setting

$$f(x) = y,$$

we rewrite (102-4) as follows

$$d^2y = y''\,dx^2;$$

hence, we have

$$y'' = \frac{d^2y}{dx^2},$$

i.e. *the derivative of the second order of a given function is equal to the ratio of the differential of the second order of this function to the square of the differential of the independent variable.*

If x is an independent variable, then, similarly to formula (102-4), we have

$$d^3f(x) = f'''(x)\,dx^3, \quad d^4f(x) = f^{IV}(x)\,dx^4$$

and so on.

We put now in (102-4) and (102-5)

$f(x) \equiv x$.

Then

$f'(x) = 1, \quad f''(x) = 0, \quad f'''(x) = 0, \ldots$

Hence,

$d^2x = 0, \quad d^3x = 0, \ldots$

Thus we get the following theorem: *The differentials of higher orders of the independent variable are equal to zero.*

Exercises

In Problems 1 to 6 find the differential of the given functions:

1. $y = 3x^2$. **2.** $y = x \sin x + \cos x$. **3.** $y = \frac{x}{1-x^2}$.

4. $y = \sqrt{1-x^2}$. **5.** $y = \ln x$.

6. $y = x^2$, where $x = 2 - t + t^2$.

7. Let $y = x^3 - 2x^2 + 3x + 6$. Find Δy and dy and compare them with each other if (a) $x = 1$, $\Delta x = 1$; (b) $x = 1$, $\Delta x = 0.1$.

8. The edge of a cube $x = 10$ m. By how much will the volume of this cube increase if its edge receives an increment $\Delta x = 0.1$ m? Give an exact and an approximate solutions.

9. Substituting the increments of the given functions by the corresponding differentials, find approximately:

(a) $\sqrt[3]{0.95}$; (b) $\cos 60°20'$; (c) $\arctan 1.02$.

10. What is the limiting absolute error of the function $y = \tan x$, if $x = 60° \pm 1°$?

11. Find dy and d^2y, if

$y = e^{-x^2}$.

Chapter 13

Indefinite Integral

Sec. 103. Antiderivative. Indefinite Integral

The fundamental problem of differential calculus consists in finding the differential of a given function or its derivative. Integral calculus solves the *inverse problem*: given the differential, and hence the derivative of an unknown function $F(x)$, to find this function. In other words, having the expression

$$dF(x) = f(x)\,dx \tag{103-1}$$

or, correspondingly,

$$F'(x) = \frac{dF(x)}{dx} = f(x),$$

where $f(x)$ is a known function, we have to find the function $F(x)$. For the sake of simplicity, we shall assume that equality (103-1) is fulfilled on a certain finite or infinite interval.

The required function $F(x)$ is then called the *antiderivative* (or the *primitive*) with respect to the function $f(x)$. Thus, we may give the following definition of the antiderivative.

Definition. *An antiderivative (a primitive) of a given function $f(x)$ in a given interval is any function $F(x)$ whose derivative is equal to the given function $f(x)$ or whose differential is equal to $f(x)\,dx$ on the interval under consideration.*

For instance, one of the antiderivatives for the function $3x^2$ is x^3, since $(x^3)' = 3x^2$. The antiderivative thus obtained is not unique, since $(x^3 + 1)' = 3x^2$, $(x^3 - 5)' = 3x^2$, etc., and therefore the functions $x^3 + 1$, $x^3 - 5$, etc., are also antiderivatives for the function $3x^2$. Consequently, the given function possesses an *infinite number* of antiderivatives.

In the above example each two of our antiderivatives differed by an arbitrary constant term. We are going to show that it will take place in the general case as well.

Theorem. *Any two antiderivatives of one and the same function defined in a certain interval differ from each other in this interval by a constant term.*

Proof. Indeed, let $f(x)$ be a function defined on the interval $\langle a, b\rangle$, and $F_1(x)$, $F_2(x)$ its antiderivatives, i.e.

$$F_1'(x) = f(x) \quad \text{and} \quad F_2'(x) = f(x).$$

Hence,

$$F_1'(x) = F_2'(x).$$

But if two functions have the same derivative, then these functions differ from each other by a constant term (Sec. 85). Hence,

$$F_1(x) - F_2(x) = C,$$

where C is a constant, which was to be proved.

Geometrical illustration. If

$$y = F_1(x) \quad \text{and} \quad Y = F_2(x)$$

are antiderivatives of one and the same function $f(x)$, then the tangents to their graphs at the points with a common abscissa x are parallel to each other: $\tan\alpha = F_1'(x) = F_2'(x) = f(x)$ (Fig. 128). In this case the distance between these curves, as measured along the y-axis, remains constant:

$$F_2(x) - F_1(x) = C,$$

i.e. these curves are "parallel" to each other in a sense.

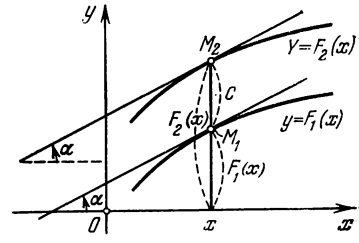


FIG. 128

Corollary. *Adding all possible constants C to any antiderivative $F(x)$ of a given function $f(x)$ defined on an interval $\langle a, b\rangle$, we get all antiderivatives for the given function $f(x)$.*

Indeed, on the one hand, if $F(x)$ is an antiderivative of the function $f(x)$, i.e. if $F'(x) = f(x)$, then the function $F(x) + C$, where C is any constant, by virtue of the fact that the derivative of a constant is equal to zero, will also be an antiderivative of the function $f(x)$, since

$$[F(x) + C]' = F'(x) + C' = f(x).$$

On the other hand, we proved that every antiderivative of the function $f(x)$ can be obtained from the function $F(x)$ by adding to it a properly chosen constant term C.

Consequently, the formula

$$F(x) + C, \tag{103-2}$$

where $-\infty < C < +\infty$, and $F(x)$ is an antiderivative of the function $f(x)$, exhausts the entire collection of antiderivatives for the given function $f(x)$.

Henceforward we shall assume, unless otherwise is stated, that the function $f(x)$ in question is defined and continuous on a certain finite or infinite interval $\langle a, b \rangle$.

Let us now introduce the fundamental notion of integral calculus—the concept of indefinite integral.

Definition. *A general expression for all antiderivatives of a given continuous function $f(x)$ is called the indefinite integral of the function $f(x)$ or of the differential expression $f(x)\,dx$ and is denoted by the symbol*

$$\int f(x)\,dx.$$

The function $f(x)$ is called the *integrand*, the expression $f(x)\,dx$ is the *element of integration*, and the variable x is the *variable of integration*.

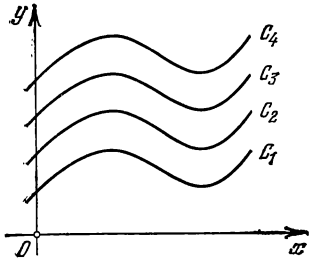


FIG. 129

Recalling the definition of the antiderivative, we may say that the indefinite integral

$$\int f(x)\,dx$$

on a given interval is a *function of the general form* whose differential is equal to the element of integration $f(x)\,dx$ and, hence, whose derivative with respect to the variable x is equal to the integrand $f(x)$ at all points of the interval under consideration.

Let $F(x)$ be a quite definite antiderivative for the function $f(x)$. As we saw, any other antiderivative of this function has the form

$$F(x) + C,$$

where C is a constant. According to the definition of the indefinite integral, we may write

$$\int f(x)\,dx = F(x) + C, \tag{103-3}$$

where $F'(x) = f(x)$ and the constant C may attain any value, and therefore is called an *arbitrary constant.*

Example. As we saw, one of the primitive functions for the function $3x^2$ is the function x^3.

Therefore,

$$\int 3x^2\,dx = x^3 + C.$$

Geometrically, the indefinite integral

$$y = F(x) + C$$

represents a family of "parallel" curves (Fig. 129).

From the definition of the indefinite integral it follows that if we have a *differential equation* (i.e. an equation containing differentials*) of the form

$$dy = f(x)\,dx,$$

where the function $f(x)$ is continuous in the interval (a, b), then the general solution of this equation for $a < x < b$ is given by the formula

$$y = \int f(x)\,dx.$$

Sec. 104. Basic Properties of the Indefinite Integral

Relying on formula (103-3), we are going to derive the basic properties of the indefinite integral.

I. *The differential of an indefinite integral is equal to the element of integration, and the derivative of an indefinite integral is equal to the integrand.*

This property directly follows from the definition of the indefinite integral.

Thus, we have

$$d\int f(x)\,dx = f(x)\,dx \tag{104-1}$$

and

$$\left[\int f(x)\,dx\right]' = f(x).$$

II. *The indefinite integral of the differential of a continuously differentiable function is equal to this function accurate to the constant term.*

Indeed, let

$$\int d\varphi(x) = \int \varphi'(x)\,dx, \tag{104-2}$$

where the function $\varphi'(x)$ is continuous. The function $\varphi(x)$, is obviously, an antiderivative for $\varphi'(x)$. Therefore, from (104-2) we have

$$\int d\varphi(x) = \varphi(x) + C.$$

Note. In formulas (104-1) and (104-2) the signs d and $\int$ following each other mutually annihilate each other (if the constant term is

* For more detail see Chapter 19.

ignored). In this respect, *differentiation and integration are mutually inverse mathematical operations.*

III. *A non-zero constant factor may be taken outside the sine of the indefinite integral*, i.e. if the constant $A \neq 0$, then

$$\int Af(x)\,dx = A\int f(x)\,dx. \tag{104-3}$$

Indeed, let $F(x)$ be an antiderivative of the function $f(x)$. By virtue of basic formula (103-3), we have

$$A\int f(x)\,dx = A[F(x)+C] = AF(x)+C_1, \tag{104-4}$$

where $C_1 = AC$, and C and C_1 are arbitrary constants for $A \neq 0$. But $AF(x)$ is an antiderivative of the function $Af(x)$, since

$$[AF(x)]' = AF'(x) = Af(x).$$

Therefore, from formula (104-4) we obtain the desired formula (103-3).

Note. For $A = 0$ formula (104-3) is false, since its left-hand member represents an arbitrary constant, while its right-hand member is identically equal to zero.

IV. *The indefinite integral of an algebraic sum of a finite number of continuous functions is equal to the algebraic sum of the indefinite integrals of these functions*, i.e. if, for instance, the functions $f(x)$, $g(x)$, and $h(x)$ are continuous in the interval (a, b), then

$$\int [f(x)+g(x)-h(x)]\,dx = \int f(x)\,dx+\int g(x)\,dx-\int h(x)\,dx \tag{104-5}$$

for $x \in (a, b)$.

Indeed, let $F(x)$, $G(x)$, and $H(x)$ be, respectively, antiderivatives of the functions $f(x)$, $g(x)$, and $h(x)$, i.e. $F'(x) = f(x)$, $G'(x) = g(x)$, $H'(x) = h(x)$ for $x \in (a, b)$. By (103-3), we have

$$\int f(x)\,dx+\int g(x)\,dx-\int h(x)\,dx = [F(x)+C_1]+[G(x)+C_2]-[H(x)+C_3] = [F(x)+G(x)-H(x)]+C, \tag{104-6}$$

where C_1, C_2, C_3 are arbitrary constants, and $C = C_1 + C_2 - C_3$ is, obviously, an arbitrary constant too. But the function $F(x) + G(x) - H(x)$ is an antiderivative of the function $f(x) + g(x) - h(x)$, since

$$[F(x)+G(x)-H(x)]' = F'(x)+G'(x)-H'(x) = f(x)+g(x)-h(x).$$

Hence,

$$\int [f(x)+g(x)-h(x)]\,dx = F(x)+G(x)-H(x)+C. \qquad (104\text{-}7)$$

From formulas (104-6) and (104-7) there follows identity (104-5).

Sec. 105. Table of Simplest Indefinite Integrals

Taking advantage of the fact that integration is an operation inverse to differentiation, we can easily obtain the table of simplest integrals. For this purpose we shall proceed from formula (104-3) which will now be rewritten in the following fashion: if

$$dF(x) = f(x)\,dx,$$

then

$$\int f(x)\,dx = F(x)+C.$$

Inverting the differentiation formulas, we obtain the following:

I. Since $d\left(\frac{x^{m+1}}{m+1}\right) = x^m\,dx$,

$$\int x^m\,dx = \frac{x^{m+1}}{m+1}+C \quad (m \neq -1).$$

II. Since $d(\ln|x|) = \frac{dx}{x}$,

$$\int \frac{dx}{x} = \ln|x|+C \quad (\text{for } x<0 \text{ and for } x>0).$$

III. Since $d(e^x) = e^x\,dx$,

$$\int e^x\,dx = e^x+C.$$

IV. Since $d\left(\frac{a^x}{\ln a}\right) = a^x\,dx$,

$$\int a^x\,dx = \frac{a^x}{\ln a}+C \quad (a>0,\ a\neq 1).$$

V. Since $d(\sin x) = \cos x\,dx$,

$$\int \cos x\,dx = \sin x+C.$$

VI. Since $d(-\cos x) = \sin x\,dx$,

$$\int \sin x\,dx = -\cos x+C.$$

VII. Since $d(\tan x) = \frac{dx}{\cos^2 x}$,

$$\int \frac{dx}{\cos^2 x} = \tan x+C.$$

VIII. Since $d(-\cot x) = \frac{dx}{\sin^2 x}$,

$$\int \frac{dx}{\sin^2 x} = -\cot x+C.$$

IX. Since $d(\arcsin x) = \frac{dx}{\sqrt{1-x^2}}$,

and

$$d(-\arccos x) = \frac{dx}{\sqrt{1-x^2}},$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C = -\arccos x + C_1.$$

X. Since $d(\arctan x) = \frac{dx}{1+x^2}$, and $d(-\operatorname{arccot} x) = \frac{dx}{1+x^2}$, $\int \frac{dx}{1+x^2} = \arctan x + C = -\operatorname{arccot} x + C_1$.

To make the above table more complete, we add here another two formulas whose validity can be verified directly by differentiation:

XI. $\int \frac{dx}{1-x^2} = \frac{1}{2} \ln \left| \frac{1+x}{1-x} \right| + C$;

$\int \frac{dx}{x^2-1} = \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + C$.

XII. $\int \frac{dx}{\sqrt{x^2+\alpha}} = \ln |x + \sqrt{x^2+\alpha}| + C$, where the constant $\alpha \neq 0$ (see Sec. 109).

Since* $d(\cosh x) = \sinh x\, dx$ and $d(\sinh x) = \cosh x\, dx$, we have two more useful formulas:

XIII. $\int \sinh x\, dx = \cosh x + C$.

XIV. $\int \cosh x\, dx = \sinh x + C$.

The integrals contained in this table will be called the *tabular* integrals, and they should be firmly memorized.

Example 1.

$\int x^3\, dx = \frac{x^4}{4} + C$.

Example 2.

$\int 2^x\, dx = \frac{2^x}{\ln 2} + C$.

Sec. 106. Independence of the Form of an Indefinite Integral of the Argument Chosen

In the basic table of integrals it was assumed that x is an independent variable. But this table completely preserves its validity and importance even if x is understood as any continuously differentiable function of an independent variable.

Indeed, let x be an independent variable, $f(x)$ a continuous function on a given interval, and $F(x)$ its antiderivative, i.e. $F'(x) =$

* See Sec. 78 (Chapter 10).

$= f(x)$. We have

$$\int f(x)\,dx = F(x) + C. \tag{106-1}$$

Let us put now

$u = \varphi(x)$,

where $\varphi(x)$ is a continuously differentiable function* and consider the integral

$$\int f(u)\,du = \int f(u)\,u'\,dx. \tag{106-2}$$

In this case the *composite function*

$$F(u) = F(\varphi(x)) \tag{106-3}$$

is an antiderivative of the integrand of integral (106-2). Indeed, since the differential of the first order is independent of the choice of the independent variable, we get

$$dF(u) = F'(u)\,du = f(u)\,du \tag{106-4}$$

and, hence,

$$\frac{d}{dx}[F(u)] = \frac{dF(u)}{du}\frac{du}{dx} = f(u)\,u'. \tag{106-4'}$$

Therefore

$$\int f(u)\,du = F(u) + C, \tag{106-5}$$

where

$F'(u) = f(u)$.

Thus, from the validity of formula (106-1) there follows the validity of formula (106-5). Here the latter formula is obtained from the former by formally substituting x for u. On the basis of this property, we get a *generalized table of simplest integrals*

$$\int u^m\,du = \frac{u^{m+1}}{m+1} + C\ (m \neq -1), \qquad \int \frac{du}{u} = \ln|u| + C \text{ and so on,}$$

where u is any continuously differentiable function of an independent variable. This table is an inversion of the generalized differentiation formulas (see Sec. 101).

By choosing the function u in different ways, we can essentially extend the table of simplest integrals.

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* That is we assume that the derivative $\varphi'(x)$ is continuous.

Example. From formula I it follows

$$\int x\,dx = \frac{x^2}{2} + C. \tag{106-6}$$

Substituting here x for $\sin x$, we get

$$\int \sin x\, d(\sin x) = \frac{\sin^2 x}{2} + C,$$

i.e.

$$\int \sin x \cos x\, dx = \frac{\sin^2 x}{2} + C.$$

Further, replacing, for instance, x by the function $\ln x$ in formula (106-6), we have

$$\int \ln x\, d(\ln x) = \frac{\ln^2 x}{2} + C$$

or

$$\int \frac{\ln x}{x}\, dx = \frac{\ln^2 x}{2} + C$$

and so on.

Hence, one can appreciate the importance of knowing how to reduce a given differential expression $f(x)\,dx$ to the form

$$f(x)\,dx = g(u)\,du,$$

where u is a certain function of x and g is a function simpler for integration than f.

Given below is a number of transformations of the differential helpful for the further studies:

(1) $dx = d(x+b)$, where b is a constant;

(2) $dx = \frac{1}{a}\, d(ax)$, where the constant $a \neq 0$;

(3) $dx = \frac{1}{a}\, d(ax+b)$ $(a \neq 0)$;

(4) $x\,dx = \frac{1}{2}\, d(x^2)$;

(5) $\sin x\,dx = -d(\cos x)$;

(6) $\cos x\,dx = d(\sin x)$.

In general

$$\varphi'(x)\,dx = d\varphi(x).$$

Using them, let us find some indefinite integrals.

Example 1.

$$\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{d(ax+b)}{ax+b} = \frac{1}{a}\ln|ax+b| + C \qquad (a \neq 0).$$

Example 2.

$$\int \sqrt{x-2}\,dx = \int (x-2)^{\frac{1}{2}}\,d(x-2) =$$

$$= \frac{(x-2)^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}(x-2)^{\frac{3}{2}} + C.$$

Example 3.

$$\int \sin 5x\,dx = \frac{1}{5}\int \sin 5x\,d(5x) = -\frac{1}{5}\cos 5x + C.$$

Example 4.

$$\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = \frac{1}{2}\ln(x^2+1) + C.$$

Example 5.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{d\cos x}{\cos x} = -\ln|\cos x| + C.$$

Example 6.

$$\int \frac{dx}{x^2+4} = \frac{1}{2}\int \frac{d\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2+1} = \frac{1}{2}\arctan\frac{x}{2} + C.$$

Example 7.

$$\int \frac{dx}{\sqrt{3-x^2}} = \int \frac{d\left(\frac{x}{\sqrt{3}}\right)}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}} = \arcsin\frac{x}{\sqrt{3}} + C.$$

Example 8.

$$\int \frac{dx}{x\sqrt{x^2-1}} = \mp\int \frac{dx}{x^2\sqrt{1-\frac{1}{x^2}}} = \mp\int \frac{d\left(\frac{1}{x}\right)}{\sqrt{1-\left(\frac{1}{x}\right)^2}} =$$

$$= \mp\arcsin\frac{1}{x} + C = -\arcsin\frac{1}{|x|} + C, \text{ where } |x| > 1.$$

Example 9.

$$\int xe^{x^2}\,dx = \frac{1}{2}\int e^{x^2}\,d(x^2) = \frac{1}{2}e^{x^2} + C.$$

Sec. 107. Basic Integration Methods

To compute a given integral, we must, if it is possible, using this or that method of integration, reduce it to a tabular formula and thus obtain the desired result. In our course we shall dwell only on most frequently used methods of integration and give simplest examples of their applications.

The most important methods of integration are: (1) *integration by expansion*, (2) *integration by substitution*, and (3) *integration by parts*.

1. Integration by expansion. Let

$$f(x) = f_1(x) + f_2(x);$$

then, according to property IV set forth in Sec. 104, we have

$$\int f(x)\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx.$$

As far as possible, the terms $f_1(x)$ and $f_2(x)$ are chosen so as to reduce their integration to the tabular formulas of Sec. 105 (so-called direct integration).

Example 1.

$$\int (1-\sqrt{x})^2\,dx = \int (1-2\sqrt{x}+x)\,dx =$$

$$= \int dx - \int 2\sqrt{x}\,dx + \int x\,dx = \int dx - 2\int x^{\frac{1}{2}}\,dx + \int x\,dx =$$

$$= x - 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^2}{2} + C = x - \frac{4}{3}x\sqrt{x} + \frac{x^2}{2} + C.$$

Note. There is no need to put an arbitrary constant after each term, since the sum of arbitrary constants is again an arbitrary constant which is written at the end of the whole expression.

Example 2.

$$\int \frac{x^4-6x^3-8x^2+9x-5}{x^2}\,dx = \int \left(x^2-6x-8+\frac{9}{x}-\frac{5}{x^2}\right)dx =$$

$$= \int x^2\,dx - 6\int x\,dx - 8\int dx + 9\int \frac{dx}{x} - 5\int x^{-2}\,dx =$$

$$= \frac{x^3}{3} - 6\cdot\frac{x^2}{2} - 8x + 9\ln|x| - 5\cdot\frac{x^{-1}}{-1} + C =$$

$$= \frac{x^3}{3} - 3x^2 - 8x + 9\ln|x| + \frac{5}{x} + C.$$

Example 3.

$$\int \tan^2 x\,dx = \int \left(\frac{1}{\cos^2 x}-1\right)dx = \int \frac{dx}{\cos^2 x} - \int dx = \tan x - x + C.$$

Example 4.

$$\int \sin^2 x\,dx.$$

Since

$$\sin^2 x = \frac{1}{2}(1-\cos 2x),$$

we get

$$\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx =$$

$$= \frac{1}{2}\int dx - \frac{1}{4}\int \cos 2x\,d(2x) = \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$$

Example 5.

$$\int \sin x \cos 3x\,dx.$$

Since

$$\sin x \cos 3x = \frac{1}{2}(\sin 4x - \sin 2x),$$

we have

$$\int \sin x \cos 3x\,dx = \frac{1}{2}\int (\sin 4x - \sin 2x)\,dx =$$

$$= \frac{1}{8}\int \sin 4x\,d(4x) - \frac{1}{4}\int \sin 2x\,d(2x) =$$

$$= -\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x + C.$$

2. Integration by substitution (or the method of introducing a new variable).

Let $f(x)$ be continuous on an open interval (a, b) and $x = \varphi(t)$ continuously differentiable on an open interval (α, β); the function φ mapping the interval (α, β) into the interval (a, b).

On the basis of the property of independence of the indefinite integral of the argument chosen (Sec. 106) and, taking into account that $dx = \varphi'(t)\,dt$, we obtain the *formula for changing the variable in an indefinite integral*

$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt. \qquad (107\text{-}1)$$

The integral constituting the right-hand member of equality (107-1) may turn out to be simpler than the integral standing in the left-hand side of this equality or even a tabular one.

Let us consider several examples.

Example 6.

$$\int x\sqrt{x-5}\,dx.$$

To get rid of the radical, we set

$$\sqrt{x-5}=t.$$

Hence

$$x = t^2 + 5$$

and, consequently, $dx = 2t\,dt$.

We substitute into the integral and obtain

$$\int x\sqrt{x-5}\,dx=\int (t^2+5)\,t\cdot 2t\,dt=\int (2t^4+10t^2)\,dt=$$

$$=2\int t^4\,dt+10\int t^2\,dt=2\cdot\frac{t^5}{5}+10\cdot\frac{t^3}{3}+C=$$

$$=\frac{2}{5}(x-5)^{\frac{5}{2}}+\frac{10}{3}(x-5)^{\frac{3}{2}}+C.$$

Example 7.

$$\int \sqrt{a^2-x^2}\,dx \qquad (a>0).$$

Here it is advisable to use a trigonometric substitution:

$$x = a\sin t,$$

whence

$$dx = a\cos t\,dt.$$

Hence,

$$\int \sqrt{a^2-x^2}\,dx=\int \sqrt{a^2-a^2\sin^2 t}\cdot a\cos t\,dt=a^2\int \cos^2 t\,dt=$$

$$=\frac{a^2}{2}\int (1+\cos 2t)\,dt=\frac{a^2}{2}\int dt+\frac{a^2}{4}\int \cos 2t\,d(2t)=$$

$$=\frac{a^2}{2}t+\frac{a^2}{4}\sin 2t+C.$$

Returning to the variable x, we shall have

$$\sin t=\frac{x}{a} \quad \text{and} \quad t=\arcsin\frac{x}{a}.$$

Then

$$\sin 2t=2\sin t\cos t=2\sin t\sqrt{1-\sin^2 t}=$$

$$=2\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}=\frac{2x}{a^2}\sqrt{a^2-x^2}.$$

Therefore, we finally get

$$\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C. \qquad (107\text{-}2)$$

It is sometimes useful to apply formula (107-1) from right to left:

$$\int f[\varphi(x)]\,\varphi'(x)\,dx = \int f[\varphi(x)]\,d\varphi(x) \qquad (107\text{-}3)$$

or

$$\int f[\varphi(x)]\,\varphi'(x)\,dx = \int f(t)\,dt,$$

where

$$t = \varphi(x).$$

For practical purposes it is desirable not to introduce a new variable t but to make use of formula (107-1). Simplest examples of this type were given in the preceding section. Here we are going to consider some more examples.

Example 8.

$$\int \frac{\sqrt[3]{1+\cot x}}{\sin^2 x}\,dx.$$

Setting

$$t = 1+\cot x, \quad dt = -\frac{dx}{\sin^2 x},$$

we shall have

$$\int \frac{\sqrt[3]{1+\cot x}}{\sin^2 x}\,dx = -\int (1+\cot x)^{\frac{1}{3}}\,d(1+\cot x) =$$

$$= -\int t^{\frac{1}{3}}\,dt = -\frac{3}{4}t^{\frac{4}{3}} + C = -\frac{3}{4}(1+\cot x)^{\frac{4}{3}} + C.$$

Example 9.

$$\int \left(\ln x + \frac{1}{\ln x}\right)\frac{dx}{x}.$$

Since

$$\frac{dx}{x} = d(\ln x),$$

we have

$$\int \left(\ln x + \frac{1}{\ln x}\right)\frac{dx}{x} = \int \ln x\,d(\ln x) + \int \frac{d(\ln x)}{\ln x} =$$

$$= \frac{1}{2}\ln^2 x + \ln|\ln x| + C.$$

Example 10.

$$\int \frac{e^x\,dx}{e^{2x}-1} = \int \frac{d(e^x)}{(e^x)^2-1} = \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C$$

(see Formula XI in Sec. 105).

3. Integration by parts. Let u and v be continuously differentiable functions of x. On the basis of the formula for the differential of a product (Sec. 101, Item V), we have

$$d(uv) = u\,dv + v\,du,$$

hence

$$u\,dv = d(uv) - v\,du.$$

Integrating we obtain

$$\int u\,dv = \int d(uv) - \int v\,du,$$

or, finally

$$\int u\,dv = uv - \int v\,du. \tag{107-4}$$

This is just the *formula for integration by parts.* The derived formula shows that the integral $\int u\,dv$ is reduced to the integral $\int v\,du$ which may turn out to be simpler than the initial one, or even a tabular one.

Let us consider a couple of examples.

Example 11.

$$\int \ln x\,dx.$$

Setting here

$$u = \ln x \quad \text{and} \quad dv = dx,$$

we obtain

$$du = d\ln x = \frac{dx}{x} \quad \text{and} \quad v = x.$$

Hence, by formula (107-4), we have

$$\int \ln x\,dx = x\ln x - \int x\cdot\frac{dx}{x} = x\ln x - x + C.$$

Example 12.

$$\int x\cos x\,dx.$$

Putting

$$u = x \quad \text{and} \quad dv = \cos x\,dx,$$

we have

$$du = dx \quad \text{and} \quad v = \int \cos x\, dx = \sin x.$$

Applying the formula for integration by parts (107-4), we get

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + C.$$

In practice, it is important to learn how to apply formula (107-4) without writing down the expression for the functions u and v.

Example 13.

$$\int x \arctan x\, dx = \frac{1}{2} \int \arctan x\, d(x^2+1) =$$

$$= \frac{1}{2}\left[(x^2+1)\arctan x - \int (x^2+1)\, d \arctan x\right] =$$

$$= \frac{x^2+1}{2}\arctan x - \frac{1}{2}\int (x^2+1)\frac{dx}{x^2+1} = \frac{x^2+1}{2}\arctan x - \frac{1}{2}x + C.$$

Sec. 108. Techniques for Integrating Rational Fractions with a Quadratic Denominator

The question is how to compute integrals of the form

$$\int \frac{P(x)}{ax^2+bx+c}\, dx,$$

where $P(x)$ is an integral polynomial and a, b, c are constants, $a \neq 0$. Dividing the numerator $P(x)$ by the denominator $ax^2 + bx + c$, we obtain a certain polynomial $Q(x)$ as a quotient and a linear binomial $mx + n$ as a remainder (since the power of the remainder is lower than that of the divisor); hence

$$\frac{P(x)}{ax^2+bx+c} = Q(x) + \frac{mx+n}{ax^2+bx+c}.$$

The integral of the polynomial $Q(x)$ is found directly; therefore we are going to show how to compute integrals of the form

$$\int \frac{mx+n}{ax^2+bx+c}\, dx. \tag{108-1}$$

First we derive two basic integrals

$$\text{I.} \quad \int \frac{dx}{x^2+a^2} = \frac{1}{a}\int \frac{d\left(\frac{x}{a}\right)}{\left(\frac{x}{a}\right)^2+1} = \frac{1}{a}\arctan\frac{x}{a} + C \qquad (a \neq 0),$$

i.e.

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + C \qquad (a \neq 0). \tag{108-2}$$

II. $\int \frac{dx}{x^2-a^2}$ $\qquad (a \neq 0)$.

We have

$$\frac{1}{x^2-a^2} = \frac{1}{(x+a)(x-a)} = \frac{1}{2a} \frac{(x+a)-(x-a)}{(x+a)(x-a)} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right).$$

Hence

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx = \frac{1}{2a}\left(\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right) =$$

$$= \frac{1}{2a}\left[\int \frac{d(x-a)}{x-a} - \int \frac{d(x+a)}{x+a}\right] =$$

$$= \frac{1}{2a}[\ln|x-a| - \ln|x+a|] + C = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C.$$

Thus (cf. XI from Sec. 105)

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C \qquad (a \neq 0). \tag{108-3}$$

The results yielded by (108-2) and (108-3) should be memorized.

To integrals I and II we add one more integral

III. $\int \frac{x\,dx}{x^2 \pm a^2} = \frac{1}{2} \int \frac{d(x^2 \pm a^2)}{x^2 \pm a^2} = \frac{1}{2} \ln|x^2 \pm a^2| + C.$

Example 1.

$$\int \frac{dx}{2x^2+3} = \frac{1}{\sqrt{2}} \int \frac{d(x\sqrt{2})}{(x\sqrt{2})^2+(\sqrt{3})^2} =$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}} \arctan \frac{x\sqrt{2}}{\sqrt{3}} + C = \frac{1}{\sqrt{6}} \arctan\left(x\sqrt{\frac{2}{3}}\right) + C.$$

Example 2.

$$\int \frac{dx}{x^2-5} = \frac{1}{2\sqrt{5}} \ln\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C.$$

The basic technique of computing integral (108-1) consists in the following: we complete the quadratic trinomial $ax^2 + bx + c$ to get a perfect square*. Then, if the coefficient $m = 0$, integral (108-1) is reduced either to integral I, or to integral II. And if $m \neq 0$, then integral (108-1) is reduced either to integrals I and III, or to integrals II and III. How it is done is illustrated by the following examples.

* We assume here that the quadratic trinomial is not an exact square.

Example 3.

$$\int \frac{dx}{x^2-10x+16} = \int \frac{dx}{(x^2-2\cdot 5x+25)+(16-25)} =$$

$$= \int \frac{d(x-5)}{(x-5)^2-3^2} = \frac{1}{2\cdot 3}\ln\left|\frac{(x-5)-3}{(x-5)+3}\right| + C = \frac{1}{6}\ln\left|\frac{x-8}{x-2}\right| + C.$$

Example 4.

$$\int \frac{dx}{x^2+3x+4} = \int \frac{dx}{\left(x^2+2\cdot\frac{3}{2}x+\frac{9}{4}\right)+\left(4-\frac{9}{4}\right)} =$$

$$= \int \frac{d\left(x+\frac{3}{2}\right)}{\left(x+\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2} = \frac{1}{\frac{\sqrt{7}}{2}}\arctan\frac{x+\frac{3}{2}}{\frac{\sqrt{7}}{2}} + C =$$

$$= \frac{2}{\sqrt{7}}\arctan\frac{2x+3}{\sqrt{7}} + C.$$

Example 5.

$$I = \int \frac{x\,dx}{x^2+x+1} = \int \frac{x\,dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}.$$

We set

$$x+\frac{1}{2} = t;$$

hence

$$x = t-\frac{1}{2} \quad \text{and} \quad dx = dt.$$

Consequently,

$$I = \int \frac{\left(t-\frac{1}{2}\right)dt}{t^2+\frac{3}{4}} = \int \frac{t\,dt}{t^2+\frac{3}{4}} - \frac{1}{2}\int \frac{dt}{t^2+\frac{3}{4}} =$$

$$= \frac{1}{2}\int \frac{d\left(t^2+\frac{3}{4}\right)}{t^2+\frac{3}{4}} - \frac{1}{2}\int \frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{1}{2}\ln\left(t^2+\frac{3}{4}\right) - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{t}{\frac{\sqrt{3}}{2}} + C =$$

$$= \frac{1}{2}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C.$$

Example 6.

$$\int \frac{x^4\,dx}{x^2+1}.$$

Dividing x^4 by x^2+1, we have

$$\frac{x^4}{x^2+1}=x^2-1+\frac{1}{x^2+1}.$$

Hence

$$\int \frac{x^4}{x^2+1}\,dx=\int\left(x^2-1+\frac{1}{x^2+1}\right)dx=$$

$$=\int x^2\,dx-\int dx+\int\frac{dx}{x^2+1}=\frac{x^3}{3}-x+\arctan x+C.$$

Note. If the quadratic trinomial ax^2+bx+c has real and different roots x_1 and x_2, then, as it is usually proved in similar courses of analysis, to compute integral (108-1), we may take advantage of the expansion of the integrand into partial fractions:

$$\frac{mx+n}{ax^2+bx+c}\equiv\frac{A}{x-x_1}+\frac{B}{x-x_2}, \tag{108-4}$$

where A and B are indefinite coefficients. The numbers A and B are found by reducing identity (108-4) to the integral form and equating the coefficients of equal powers of x in the left- and right-hand sides of the obtained equality.

Example 7. Find

$$I=\int\frac{x+2}{x^2+5x-6}\,dx.$$

Equating the denominator to zero, we obtain a quadratic equation

$$x^2+5x-6=0,$$

whence we find the roots: $x_1=1$ and $x_2=-6$. By (108-4), we have

$$\frac{x+2}{x^2+5x-6}\equiv\frac{A}{x-1}+\frac{B}{x+6}. \tag{108-5}$$

Hence, getting rid of the denominator and taking into account that

$$x^2+5x-6=(x-1)(x-6),$$

we obtain

$$x+2\equiv A(x+6)+B(x-1)$$

or

$$x+2\equiv(A+B)x+(6A-B). \tag{108-6}$$

Equating the coefficients of equal powers of x in the right- and left-hand sides of the last equality, we shall have

$$\left.\begin{aligned} A+B&=1,\\ 6A-B&=2. \end{aligned}\right\}$$

Hence, $A=\frac{3}{7}$, $B=\frac{4}{7}$.

Note that the coefficients A and B can be simply determined from identity (108-6). We first set $x = 1$, whence

$$3=A\cdot 7 \quad \text{and} \quad A=\frac{3}{7},$$

and then put $x = -6$, which yields

$$-4=B(-7) \quad \text{and} \quad B=\frac{4}{7}.$$

On the basis of expansion (108-5), we get

$$I=\frac{3}{7}\int\frac{d(x-1)}{x-1}+\frac{4}{7}\int\frac{d(x+6)}{x+6}=$$

$$=\frac{3}{7}\ln|x-1|+\frac{4}{7}\ln|x+6|+C=\frac{1}{7}\ln\{|x-1|^3(x+6)^4\}+C.$$

Sec. 109. Integration of Simplest Irrational Expressions

1. If the element of integration contains only the *linear* irrational $\sqrt[n]{ax+b}$ $(a\neq 0)$, then it is advisable to use the substitution $t=\sqrt[n]{ax+b}$.

Example 1. Find

$$I=\int\frac{x\,dx}{\sqrt[3]{x+1}}.$$

We put

$$t=\sqrt[3]{x+1};$$

whence

$$x=t^3-1 \quad \text{and} \quad dx=3t^2\,dt.$$

We have

$$I=\int\frac{(t^3-1)\cdot 3t^2\,dt}{t}=3\int(t^4-t)\,dt=$$

$$=\frac{3}{5}t^5-\frac{3}{2}t^2+C=\frac{3}{5}(x+1)^{\frac{5}{3}}-\frac{3}{2}(x+1)^{\frac{2}{3}}+C.$$

2. The integral of the simplest *quadratic* irrational

$$\int \frac{dx}{\sqrt{ax^2+bx+c}},$$

by completing the quadratic trinomial $ax^2 + bc + c$ to a perfect square, is reduced to one of the two integrals

$$\int \frac{dx}{\sqrt{\alpha \pm x^2}},$$

which are computed below.

1. $\int \frac{dx}{\sqrt{x^2+\alpha}}$ $\quad (\alpha \neq 0)$.

We apply here the *Euler substitution*

$$\sqrt{x^2+\alpha} = t - x,$$

where t is a new variable. Squaring this equality termwise, we have

$$x^2 + \alpha = t^2 - 2tx + x^2$$

or

$$\alpha = t^2 - 2tx.$$

Differentiating both sides of the last equality, we get

$$0 = 2t\,dt - 2x\,dt - 2t\,dx$$

or

$$t\,dx = (t - x)\,dt.$$

Hence

$$\frac{dx}{t-x} = \frac{dt}{t},$$

i.e.

$$\frac{dx}{\sqrt{x^2+\alpha}} = \frac{dt}{t}.$$

Thus, we have

$$\int \frac{dt}{\sqrt{x^2+\alpha}} = \int \frac{dt}{t} = \ln|t| + C.$$

Finally, substituting t by its expression in terms of x, we find tabular integral XII (see Sec. 105)

$$\int \frac{dx}{\sqrt{x^2+\alpha}} = \ln\left|x+\sqrt{x^2+\alpha}\right| + C \qquad (\alpha \neq 0). \tag{109-1}$$

This formula should be memorized.

Example 2.

$$\int \frac{dx}{\sqrt{x^2-6x+13}}.$$

Using formula (109-1), we have

$$\int \frac{dx}{\sqrt{x^2-6x+13}} = \int \frac{dx}{\sqrt{(x-3)^2+4}}.$$

Setting here

$$x - 3 = t,$$

we get in succession

$$\int \frac{dx}{\sqrt{x^2-6x+13}} = \int \frac{dt}{\sqrt{t^2+4}} = \ln\left(t+\sqrt{t^2+4}\right)+C.$$

Since,

$$t = x - 3,$$

we shall finally have,

$$\int \frac{dx}{\sqrt{x^2-6x+13}} = \ln\left[x-3+\sqrt{(x-3)^2+4}\right]+C =$$

$$= \ln\left(x-3+\sqrt{x^2-6x+13}\right)+C.$$

II. $$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{d\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \arcsin\frac{x}{a}+C \qquad (a>0).$$

Example 3.

$$\int \frac{dx}{\sqrt{1-x-x^2}} = \int \frac{d\left(x+\frac{1}{2}\right)}{\sqrt{\frac{5}{4}-\left(x+\frac{1}{2}\right)^2}} =$$

$$= \arcsin\frac{x+\frac{1}{2}}{\sqrt{\frac{5}{4}}}+C = \arcsin\frac{2x+1}{\sqrt{5}}+C.$$

Sec. 110. Integration of Trigonometric Functions

In applications an important role is played by the integrals

$$I = \int \sin^m x \cos^n x\, dx,$$

where m and n are non-negative integers.

Two cases are distinguished here:
(1) at least one of the exponents m or n is an odd number;
(2) both exponents m and n are even numbers.
In the first case the integral I is taken directly.

Example 1. Find

$$I = \int \sin^3 x \cos^2 x \, dx.$$

We set in succession

$$I = \int \sin^2 x \cos^2 x (\sin x \, dx) = -\int (1 - \cos^2 x) \cos^2 x \, d(\cos x) =$$
$$= -\int \cos^2 x \, d(\cos x) + \int \cos^4 x \, d(\cos x) =$$
$$= -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C.$$

In the second case the integral I is computed by means of formulas for a doubled argument

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x),$$

$$\sin x \cos x = \frac{1}{2} \sin 2x.$$

Example 2. Find

$$I = \int \sin^2 x \cos^4 x \, dx.$$

We have

$$I = \int (\sin x \cos x)^2 \cos^2 x \, dx = \int \left(\frac{\sin 2x}{2}\right)^2 \frac{1 + \cos 2x}{2} \, dx =$$
$$= \frac{1}{8} \int \sin^2 2x (1 + \cos 2x) \, dx =$$
$$= \frac{1}{8} \int \sin^2 2x \, dx + \frac{1}{8} \int \sin^2 2x \cos 2x \, dx =$$
$$= \frac{1}{8} \int \frac{1 - \cos 4x}{2} \, dx + \frac{1}{16} \int \sin^2 2x \, d(\sin 2x) =$$
$$= \frac{1}{16} \int dx - \frac{1}{16} \int \cos 4x \, dx + \frac{1}{16} \cdot \frac{\sin^3 2x}{3} =$$
$$= \frac{1}{16} x - \frac{1}{64} \sin 4x + \frac{1}{48} \sin^3 2x + C.$$

In the theory of the Fourier series (see Sec. 197, Chapter 21) an important role is played by the integrals

$$\int \sin mx \sin nx \, dx, \quad \int \cos mx \cos nx \, dx,$$

$$\int \sin mx \cos nx \, dx.$$

They are computed with the aid of the trigonometric formulas:

$$\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)],$$

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)],$$

$$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)].$$

Example 3.

$$\int \sin x\sin 5x\,dx=\frac{1}{2}\int(\cos 4x-\cos 6x)\,dx=$$
$$=\frac{1}{8}\sin 4x-\frac{1}{12}\sin 6x+C.$$

Sec. 111. Integration of Certain Transcendental Functions

The integral

$$\int P(x)\,e^{ax}\,dx,$$

where $P(x)$ is a polynomial is computed by repeated integration by parts.

Example.

$$\int x^2e^{2x}\,dx=\frac{1}{2}\int x^2d(e^{2x})=\frac{x^2}{2}e^{2x}-\frac{1}{2}\int e^{2x}\cdot 2x\,dx=$$
$$=\frac{x^2}{2}e^{2x}-\frac{1}{2}\int x\,d(e^{2x})=\frac{x^2}{2}e^{2x}-\frac{x}{2}e^{2x}+\frac{1}{2}e^{2x}\,dx=$$
$$=\frac{x^2}{2}e^{2x}-\frac{x}{2}e^{2x}+\frac{1}{4}e^{2x}+C=\left(\frac{x^2}{2}-\frac{x}{2}+\frac{1}{4}\right)e^{2x}+C.$$

A similar method is used to calculate integrals of the form

$$\int P(x)\sin ax\,dx \quad\text{and}\quad \int P(x)\cos ax\,dx,$$

where $P(x)$ is a polynomial.

Sec. 112. Cauchy's Theorem. Some Important Integrals Inexpressible in Terms of Elementary Functions

Up till now we were quite successful in finding for some continuous functions $f(x)$ their indefinite integrals

$$\int f(x)\,dx.$$

There arises a question whether this will always be so, i.e. (1) whether any continuous function $f(x)$ has an indefinite integral, and (2) how this integral can be found if it exists.

The first part of this question is answered by Cauchy's theorem which is the fundamental theorem of integral calculus.

Cauchy's theorem. *Any continuous function has an antiderivative.*

In other words, *for any function $f(x)$ continuous in an open interval (a, b) there exists a function $F(x)$ whose derivative in the open interval (a, b) is exactly equal to the given function $f(x)$*, i.e.

$$F'(x) = f(x),$$

which means that there exists the indefinite integral

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant.

We omit here the proof of this theorem since it is too complicated.

But the second part of our question remains unsolved: if we are given a continuous function $f(x)$, then how its indefinite integral is found. Cauchy's theorem does not state at all that the antiderivative of the given function can actually be found with the aid of a finite number of known operations and the answer can be expressed in elementary functions (algebraic, exponential, trigonometric, etc.). Moreover, there are continuous elementary functions whose integrals are not elementary functions. Such integrals are frequently called "inexpressible" thus implying that they cannot be expressed in terms of a finite number of elementary functions.

For instance, we can prove that the integrals

$$\int e^{-x^2}\,dx, \quad \int \frac{\sin x}{x}\,dx, \quad \int \frac{\cos x}{x}\,dx, \quad \int \frac{dx}{\ln x}$$

and a number of others cannot be reduced to a finite combination o elementary functions and, hence, are "inexpressible" in our sense of this word.

Exercises

Using the basic table of integrals, find the following indefinite integrals:

1. $\int (x^2 - 3x + 1)\,dx.$
2. $\int \left(\frac{x-1}{x}\right)^2 dx.$
3. $\int \frac{x - 2\sqrt{x} + 2}{x^2 \sqrt[3]{x}}\,dx.$
4. $\int (a^x + b^x)^2\,dx.$
5. $\int (\sin 3x + \cos 5x)\,dx.$
6. $\int \frac{dx}{\sin^2 x \cos^2 x}.$

Hint. Use the identity: $1 = \sin^2 x + \cos^2 x$.

7. $\int \frac{dx}{x^2+2}$.

8. $\int \frac{dx}{2-3x^2}$.

9. $\int \frac{dx}{\sqrt{2-3x^2}}$.

10. $\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}\,dx$.

11. $\int \tan^2 x\,dx$.

12. $\int \frac{x\,dx}{1+x^2}$.

13. $\int \frac{dx}{\sqrt{2-3x}}$.

Applying the method of expansion, find the following integrals:

14. $\int \frac{x^2}{1-x}\,dx$.

15. $\int \frac{x^2}{1+x^2}\,dx$.

16. $\int \frac{x^2}{1-x^2}\,dx$.

17. $\int \frac{(1+x)^2}{1+x^2}\,dx$.

18. $\int \frac{dx}{4-x^2}$.

19. $\int \frac{dx}{x^2-5x+6}$.

20. $\int \cos^2 5x\,dx$.

21. $\int \sin x \cos 7x\,dx$.

22. $\int \cos 3x \cos 5x\,dx$.

Using the indicated substitutions, find the following integrals:

23. $\int x\sqrt{x-1}\,dx \quad (\sqrt{x-1}=t)$.

24. $\int \frac{dx}{1+\sqrt{x}} \quad (\sqrt{x}=t)$.

25. $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} \quad (x=\sin t)$.

26. $\int \frac{dx}{1+2e^x} \quad (e^x=t)$.

Using the method of integration by parts, find the following integrals:

27. $\int \arctan x\,dx$.

28. $\int x \ln x\,dx$.

29. $\int x \sin x\,dx$.

30. $\int xe^x\,dx$.

31. $\int \frac{\ln x}{x^2}\,dx$.

32. $\int x^2 e^x\,dx$.

Find the integrals of the following rational fractions:

33. $\int \frac{dx}{3x^2+7}$.

34. $\int \frac{dx}{5x^2-2}$.

35. $\int \frac{dx}{x^2+2x+5}$.

36. $\int \frac{dx}{1+x+x^2}$.

37. $\int \frac{dx}{x^2-3x+1}$.

38. $\int \frac{dx}{x^2+8x-9}$.

39. $\int \frac{3x+5}{x^2+8x+15}\,dx$.

40. $\int \frac{x^2}{x^2+4x+5}\,dx$.

Find the integrals of the following irrational functions:

41. $\int \sqrt[3]{2x-3}\,dx$.

42. $\int \frac{x-1}{\sqrt{x+1}}\,dx$.

43. $\int \frac{dx}{\sqrt{2-x^2}}$.

44. $\int \frac{dx}{\sqrt{x^2-2}}$.

45. $\int \frac{dx}{\sqrt{x^2+x}}$.

46. $\int \frac{dx}{\sqrt{1+2x-x^2}}$.

47. $\int \frac{dx}{\sqrt{x^2+4x+3}}$.

Find the integrals of the following trigonometric functions:

48. $\int \sin x \cos^3 x\,dx$.

49. $\int \sin^2 x \cos^2 x\,dx$.

50. $\int \sin^3 x\,dx$.

51. $\int \cos^4 x\,dx$.

52. $\int \sin x \sin(x+\alpha)\,dx$.

53. $\int \frac{dx}{\cos^4 x}$.

Hint. Use the formula $\frac{dx}{\cos^2 x} = d(\tan x)$.

Find the integrals of the following transcendental functions:

54. $\int x^2 e^{-x}\,dx$.

55. $\int (x^2+2x+3)\,e^x\,dx$.

56. $\int (x-1)\sin 2x\,dx$.

57. $\int c^{-2x}\cos 3x\,dx$.

58. $\int x \operatorname{arccot} x\,dx$.

59. $\int \ln^2 x\,dx$.

60. $\int \arcsin x\,dx$.

Chapter 14

The Definite Integral

Sec. 113. The Concept of the Definite Integral

Let $f(x)$ be a function continuous on a given closed interval $[a, b]$, where $a < b$ or $a > b$, and $F(x)$ is one of its antiderivatives (see Sec. 103, Chapter 13), i.e. $F'(x) = f(x)$ for $x \in [a, b]$.

Definition. *The definite integral*

$$\int_a^b f(x)\,dx \tag{113-1}$$

of a given continuous function $f(x)$ on a given closed interval $[a, b]$ is understood as the corresponding increment of its antiderivative, i.e.

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{113-2}$$

(*the Newton-Leibniz formula*).

Besides, we assume that for any function $f(x)$ having sense at point a

$$\int_a^a f(x)\,dx = 0$$

(a any number). Thus, formula (113-2) holds for $a = b$ as well.

In expression (113-1) the numbers a and b are called the *limits of integration*: *lower limit* (a) and *upper limit* (b); $[a, b]$ the *interval of integration*, and $f(x)$ the *integrand*. (The expression is read "the integral of $f(x)$ with respect to x between the limits a and b".)

Formula (113-2) can be expressed in the form of a rule: *the definite integral is equal to the difference of the values of the antiderivative of the integrand for the upper and lower limits of integration.* Introducing

the following notation for the difference

$$F(b) - F(a) = F(x)\Big|_a^b,$$

where the symbol $\Big|_a^b$ is called the *sign of double substitution*; we can write formula (113-2) in the form

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b. \tag{113-3}$$

It should be remembered that in deciphering this sign we first substitute the *upper* limit of integration, and then the *lower* one. (It is unfortunate that the word "limit" is used in this connection. The limits of integration have nothing to do with the limits considered in Sec. 46, Chapter 7).

Example. Evaluate the integral of x^2 between the limits 2 and 4. Since $\frac{1}{3}x^3$ is an antiderivative for x^2, according to (113-3), we have

$$\int_2^4 x^2\,dx = \frac{x^3}{3}\Big|_2^4 = \frac{64}{3} - \frac{8}{3} = 18\frac{2}{3}.$$

Note that the same result could be obtained if we used another antiderivative for x^2, say, $\frac{x^3}{3} + 1$, or $\frac{x^3}{3} - 2$, and so on.

Theorem. *The definite integral of a continuous function is independent of the choice of the antiderivative for the integrand.*

Proof. Let $F(x)$ and $F_1(x)$ be two antiderivatives of a continuous on $[a, b]$ integrand function $f(x)$ of integral (113-1). By virtue of the basic theorem for an indefinite integral (Sec. 103, Chapter 13), we have

$$F_1(x) = F(x) + C,$$

where C is a constant. Hence,

$$F_1(x)\Big|_a^b = F_1(b) - F_1(a) = [F_1(b] + C] - [F(a) + C] =$$

$$= F(b) - F(a) = F(x)\Big|_a^b = \int_a^b f(x)\,dx,$$

which was to be proved.

Corollary.

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big|_a^b, \tag{113-4}$$

where $\int f(x)\,dx$ *is understood as one of the antiderivatives for the function* $f(x)$.

Formula (113-4) establishes relation between the definite and the corresponding indefinite integrals. The formal difference between them consists in that the definite integral represents a *number* and the indefinite integral a *function.*

According to Cauchy's theorem (see Sec. 112), any continuous on a segment function has an antiderivative. This yields the following theorem.

Theorem. *For any function continuous on a closed interval* $[a, b]$ *there exists an appropriate definite integral.*

Note. Let

$$y' = f(x),$$

i.e.

$$dy = f(x)\,dx. \tag{113-5}$$

Integrating equality (113-5) between the limits a and b, we have

$$y(b) - y(a) = \int_a^b f(x)\,dx. \tag{113-6}$$

The last formula is frequently used in practice.

The study of the indefinite and definite integrals and their applications constitute the subject of the *integral calculus.*

Sec. 114. A Definite Integral with a Variable Upper Limit

Let a function $f(x)$ be continuous on a closed interval $[a, b]$. Consider the integral

$$\int_a^x f(t)\,dt, \tag{114-1}$$

where $t \in [a, x] \subset [a, b]$ (to avoid confusion, the variable of integration is denoted by another letter).

If $F(x)$ is an antiderivative of the function $f(x)$, i.e.

$$F'(x) = f(x),$$

then, according to the Newton-Leibniz formula we have

$$\int_a^x f(t)\,dt = F(x) - F(a). \tag{114-2}$$

Hence,

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}[F(x) - F(a)] =$$

$$= F'(x) - [F(a)]' = f(x) - 0 = f(x).$$

Consequently, *the derivative of a definite integral with a variable upper limit with respect to this limit is equal to the value of the integrand for this limit*:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{114-3}$$

Thus, the integral

$$\Phi(x) = \int_a^x f(t)\,dt \tag{114-4}$$

is an *antiderivative* for the integrand $f(x)$. It should be noted that from formula (114-2) it follows that $\Phi(a) = 0$, i.e. $\Phi(x)$ is the *antiderivative of the function $f(x)$ which vanishes for $x = a$.*

Example. We have

$$\frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt = \sqrt{1+x^2}.$$

Consider now the definite integral with a variable lower limit

$$\int_x^b f(t)\,dt,$$

where $x \in [a, b]$.

By the Newton-Leibniz formula, we have

$$\frac{d}{dx}\left[\int_x^b f(x)\,dx\right] = \frac{d}{dx}[F(b) - F(x)] = [F(b)]' - F'(x) = -f(x).$$

Thus, *the derivative of a definite integral with a variable lower limit with respect to this limit is equal to the value of the integrand for this limit taken with an opposite sign.*

Note. If the function $f(x)$ is continuous on a closed interval $[a, b]$, then on the basis of the relation between the indefinite integral and

antiderivative (Sec. 103), we shall have

$$\int f(x)\,dx = C + \int_a^x f(t)\,dt$$

for $a \leqslant x \leqslant b$, where C is an arbitrary constant.

Sec. 115. Geometrical Meaning of the Definite Integral

Consider the area $S(x)$* of a variable curvilinear trapezoid (Fig. 130), bounded above by a continuous curve $Y=f(X)$ $(a \leqslant X \leqslant b,\ f(X) \geqslant 0)$, below by the axis OX $(Y = 0)$, from the left by a fixed vertical line $(X = a)$ and from the right by a moving vertical line $(X = x)$ $(a \leqslant x \leqslant b)$.

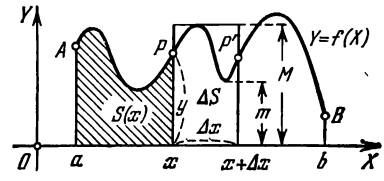

FIG. 130

Let us imagine that a flood develops along the axis OX and the vertical front of water moves from left to right.

Let x receive an increment Δx (for the sake of definiteness we assume that $\Delta x > 0$). Then the area will change by the quantity ΔS (Fig. 130) which represents the area of the strip bounded by the arc $\overset{\frown}{PP'}$ of the curve, the axis OX and two vertical lines $X = x$ and $X = x + \Delta x$. Let us set

$$m = \min_{x \leqslant X \leqslant x+\Delta x} f(X)$$

and

$$M = \max_{x \leqslant X \leqslant x+\Delta x} f(X).$$

Comparing the area ΔS with the areas of the rectangles with a common base Δx and the altitudes m and M, we have

$$m\Delta x \leqslant \Delta S \leqslant M\Delta x^{**}. \qquad (115\text{-}1)$$

Hence

$$m \leqslant \frac{\Delta S}{\Delta x} \leqslant M. \qquad (115\text{-}2)$$

Let now $\Delta x \to +0$. Since the function $f(X)$ is continuous, we have

$$m \to f(x) \quad \text{and} \quad M \to f(x). \qquad (115\text{-}3)$$

* For the concept of area see Note to the theorem of Sec. 121.

** Equality is not excluded here, since the function $f(X)$ can be constant.

Whence on the basis of the theorem about an intermediate variable (see Sec. 51, Chapter 7), we obtain

$$\lim_{\Delta x \to +0} \frac{\Delta S}{\Delta x} = f(x).$$

Analogously, as $\Delta x \to -0$, we shall have

$$\lim_{\Delta x \to -0} \frac{\Delta S}{\Delta x} = f(x).$$

Hence, there exists the limit

$$\lim_{\Delta x \to 0} \frac{\Delta S}{\Delta x} = \frac{dS}{dx} = f(x). \qquad (115\text{-}4)$$

Thus, *the derivative of the area of a variable curvilinear trapezoid for any value of the argument $X = x$ is equal to its terminal ordinate $y = f(x)$* (the Newton-Leibniz theorem).

From formula (115-4) we obtain

$$dS = f(x)\,dx. \qquad (115\text{-}5)$$

Let S be the total area of a curvilinear trapezoid (Fig. 130) bounded by the curve $Y = f(X)$, the axis OX, and two vertical lines $X = a$

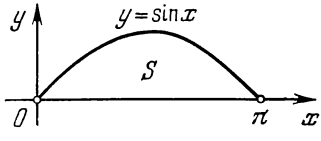

FIG. 131

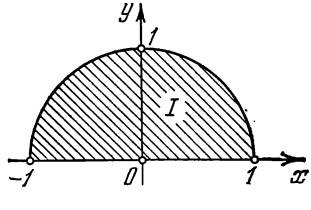

FIG. 132

and $X = b$. Integrating equality (115-5) between the limits a and b, and taking into consideration that $S(a) = 0$, by (113-6), we have

$$S = S(b) - S(a) = \int_a^b f(x)\,dx. \qquad (115\text{-}6)$$

Thus, *the definite integral* (115-6) *of a continuous nonnegative function for $a \leqslant b$ is equal to the area of the corresponding curvilinear trapezoid** (geometrical meaning of the definite integral).

Example 1. Find the area S under one half-wave of the sinusoid $y = \sin x$ $(0 \leqslant x \leqslant \pi)$ (Fig. 131).

* For what should be understood under the area of a curvilinear trapezoid in the case of an alternating-sign function $f(x)$ see Sec. 117 (Note to Item VIII)

By geometrical meaning of the definite integral, we have

$$S = \int_0^{\pi} \sin x\, dx = -\cos x \Big|_0^{\pi} = -(\cos \pi - \cos 0) = -(-1-1) = 2$$

(appropriate square units).

Example 2. Find out the geometrical meaning of the integral

$$I = \int_{-1}^{1} \sqrt{1-x^2}\, dx \tag{115-7}$$

and, taking advantage of this meaning, evaluate the integral.

Since $y = \sqrt{1-x^2}$ is the equation of the upper semi-circle $x^2 + y^2 = 1$, $y \geqslant 0$, the integral I represents the area of a semi-circle of radius 1 (Fig. 132).

Therefore, $I = \frac{\pi}{2}$; this result can also be obtained by direct calculation of integral (115-7).

Sec. 116. Physical Meaning of the Definite Integral

Problem. Knowing the velocity $v = v(t)$ of rectilinear motion of a point, find the distance covered by the point during the interval of time $0 \leqslant t \leqslant T$.

Assuming that the path of the point is the x-axis (Fig. 133) and

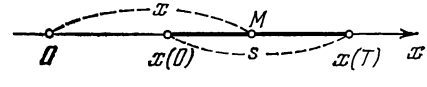

FIG. 133

$x = x(t)$ is the equation of motion, we shall have (Sec. 65)

$$v(t) = \frac{dx}{dt}. \tag{116-1}$$

hence

$$dx = v(t)\, dt. \tag{116-2}$$

Integrating equality (116-2) between the limits 0 and T we get the distance covered by the point during the time T:

$$s = x(T) - x(0) = \int_0^T v(t)\, dt\,*. \tag{116-3}$$

---

* More precisely, (116-3) yields the increment of the abscissa of a moving point, i.e. the *displacement of the point during the time T*. The distance covered by the point is obtained when the velocity $v(t)$ preserves the same sign, i.e. the point moves in one and the same direction (see Sec. 65).

Note. From (116-3) we obtain the equation of motion of the point

$$x = x_0 + \int_0^t v(t)\,dt,$$

where $x_0 = x(0)$.

Example. Determine the height reached by a rocket launched vertically upward by the end of the tenth second of its flight if its velocity changes according to the law

$$v = \left[2 + \frac{1}{(t+1)^2}\right] \frac{\text{km}}{\text{s}}.$$

Find the average velocity of the rocket flight during this interval of time.

The distance covered by the rocket for ten seconds is equal to

$$s = \int_0^{10} \left[2 + \frac{1}{(t+1)^2}\right] dt = \left(2t - \frac{1}{t+1}\right)\Big|_0^{10} =$$

$$= \left(20 - \frac{1}{11}\right) - (0 - 1) \approx 21.09 \text{ km}.$$

Therefore the corresponding average velocity of the rocket is

$$v_{\text{aver}} = \frac{21.09}{10} = 2.109 \frac{\text{km}}{\text{s}}.$$

Sec. 117. Basic Properties of the Definite Integral

In deriving the basic properties of the definite integral we shall proceed from the Newton-Leibniz formula (Sec. 113):

$$\int_a^b f(x)\,dx = F(b) - F(a), \tag{117-1}$$

where $f(x)$ is continuous on the closed interval $[a, b]$ and $F'(x) = f(x)$ for $x \in [a, b]$.

For a better systematization we are going to divide the properties of the definite integral into several groups.

A. General properties.

I. *The value of a definite integral does not depend on the designation of the variable of integration*, i.e.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt,$$

where x, t are any letters.

This property follows directly from formula (117-1).

We can mention here the following analogy: diplomatic documents of the same contents written in different languages are authentic.

II. *A definite integral with equal limits of integration is equal to zero* (on the basis of the above agreement).

It should be noted that this statement also corresponds to the Newton-Leibniz formula:

$$\int_a^a f(x)\,dx = F(a) - F(a) = 0.$$

III. *Interchanging the limits of a definite integral does not change its absolute value but only the sign.*

Indeed, interchanging the limits of integration, by formula (117-1), we have

$$\int_b^a f(x)\,dx = F(a) - F(b) = -[F(b) - F(a)] = -\int_a^b f(x)\,dx. \quad (117\text{-}2)$$

B. Additivity property.

IV. *If the interval of integration* $[a, b]$ *is split into a finite number of subintervals (partial intervals), then the definite integral taken over the interval* $[a, b]$ *is equal to the sum of definite integrals taken over all of its partial intervals.*

Indeed, let for instance, $[a, b] = [a, c] \cup [c, b]$, where $a \leqslant c \leqslant b$. Then, setting $F'(x) = f(x)$, we have

$$\int_a^b f(x)\,dx = F(b) - F(a) = [F(c) - F(a)] + [F(b) - F(c)] =$$

$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \quad (117\text{-}3)$$

Note. Formula (117-3) remains true if c lies outside the closed interval $[a, b]$ and the integrand $f(x)$ is continuous on the closed intervals $[a, c]$ and $[c, b]$.

C. Linearity properties.

V. *A constant factor can be taken outside the integral sign.*

Indeed, let $F(x)$ be an antiderivative for $f(x)$ on $[a, b]$ and A a constant, then $AF(x)$ is an antiderivative for $Af(x)$, since

$$[AF(x)]' = AF'(x) = Af(x).$$

We have

$$\int_a^b Af(x)\,dx = AF(x)\Big|_a^b = AF(b) - AF(a) =$$

$$= A[F(b) - F(a)] = A\int_a^b f(x)\,dx.$$

VI. *The definite integral of an algebraic sum of a finite number of continuous functions is equal to the algebraic sum of the definite integrals of these functions.*

Indeed, let us consider, for instance, the algebraic sum

$$f(x) + g(x) - h(x) \tag{117-4}$$

of three continuous functions $f(x)$, $g(x)$, $h(x)$, and let $F(x)$, $G(x)$, $H(x)$ be their antiderivatives, respectively, i.e.

$$F'(x) = f(x), \quad G'(x) = g(x), \quad H'(x) = h(x).$$

Then $F(x) + G(x) - H(x)$ is an antiderivative for the sum under consideration, since

$$[F(x) + G(x) - H(x)]' =$$

$$= F'(x) + G'(x) - H'(x) = f(x) + g(x) - h(x)$$

Hence, we have

$$\int_a^b [f(x) + g(x) - h(x)]\,dx = [F(x) + G(x) - H(x)]\Big|_a^b =$$

$$= [F(b) + G(b) - H(b)] - [F(a) + G(a) - H(a)] =$$

$$= [F(b) - F(a)] + [G(b) - G(a)] - [H(b) - H(a)] =$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \int_a^b h(x)\,dx.$$

D. Monotonicity properties.

VII. *If the integrand of a definite integral is continuous and non-negative, and the upper limit of integration is greater than, or equal to, the lower limit, then the definite integral is also non-negative.*

Indeed, let $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$. Since $F'(x) = f(x) \geqslant 0$, the antiderivative $F(x)$ is an increasing (or, more precisely, non-decreasing) function. In this case for $b \geqslant a$ we have

$$\int_a^b f(x)\,dx = F(b) - F(a) \geqslant 0.$$

VIII. *An inequality between continuous functions can be integrated termwise, provided the upper limit of integration is greater than the lower one.*

Indeed, let

$$f(x) \leqslant g(x) \quad \text{for} \quad a \leqslant x \leqslant b,$$

where $f(x)$ and $g(x)$ are continuous on the closed interval $[a, b]$. Since $g(x) - f(x) \geqslant 0$, then for $b > a$, by properties VI and VII, we have

$$\int_a^b [g(x) - f(x)]\, dx = \int_a^b g(x)\, dx - \int_a^b f(x)\, dx \geqslant 0,$$

hence,

$$\int_a^b f(x)\, dx \leqslant \int_a^b g(x)\, dx.$$

Note. Let $f(x)$ be an alternating-sign continuous function on the closed interval $[a, b]$ where $b > a$. For instance, $f(x) \leqslant 0$ for $a \leqslant x \leqslant \alpha$, $f(x) > 0$ for $\alpha < x < \beta$ and $f(x) \leqslant 0$ for $\beta \leqslant x \leqslant b$ (Fig. 134).

By the additivity property (IV), taking into account the geometrical meaning of the integral (Sec. 115), we have

$$\int_a^b f(x)\, dx = \int_a^\alpha f(x)\, dx + \int_\alpha^\beta f(x)\, dx + \int_\beta^b f(x)\, dx = -S_1 + S_2 - S_3, \qquad (117\text{-}5)$$

where S_1, S_2, S_3 are the areas of the respective curvilinear trapezoids.

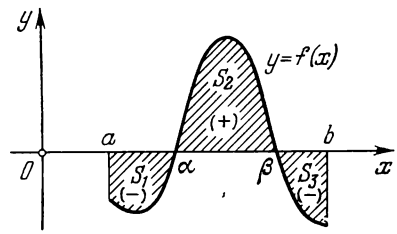

FIG. 134

Thus, *the definite integral, in the general case, for $a < b$ represents the algebraic sum of the areas of the corresponding curvilinear trapezoids, where the areas of the trapezoids above the x-axis are taken with the plus sign, and those of the trapezoids lying below the x-axis with the minus sign.*

If $b < a$, then everything is vice versa.

Note that the total area of the figure hatched in Fig. 134 is expressed by the integral

$$\int_a^b |f(x)|\, dx = S_1 + S_2 + S_3 \qquad (b > a).$$

Sec. 118. The Mean Value Theorem

Theorem. *The definite integral of a continuous function is equal to the product of the length of the interval of integration by the value of the integrand at some intermediate value of the argument**.

Proof. Indeed, by virtue of the Newton-Leibniz formula, we have

$$\int_a^b f(x)\,dx = F(b) - F(a), \tag{118-1}$$

where

$$F'(x) = f(x).$$

Applying to the difference of the antiderivatives the theorem about the finite increment of a function (Sec. 85, Chapter 11), we get

$$F(b) - F(a) = (b - a)\,F'(c) = (b - a)\,f(c),$$

where $a < c < b$. Hence,

$$\int_a^b f(x)\,dx = (b-a)\,f(c), \tag{118-2}$$

where $a < c < b$.

Note. We can give a simple geometrical illustration to formula (118-2) for $f(x) \geqslant 0$. Indeed, its left-hand side represents the area of the curvilinear trapezoid $AabB$, where AB has the equation $y = f(x)$, and a and b are the abscissas of the points A and B, respectively. The right-hand side of this formula expresses the area of the rectangle with the base $b - a$ and the altitude cC equal to $f(c)$ (Fig. 135).

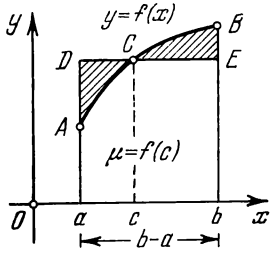

FIG. 135

Thus, geometrically formula (118-2) means that it is always possible to choose on the arc AB a point C with the abscissa c contained between a and b such that the area of the corresponding rectangle $aDEb$ with the altitude cC will be exactly equal to the area of the curvilinear trapezoid $aABb$.

And so, *the area of a curvilinear trapezoid bounded by a continuous line is equivalent to the area of a rectangle with the same "base" and altitude equal to some mean ordinate of the line.*

* It is assumed that the upper limit of integration is greater than the lower.

The number $f(c) = \mu$ is called the *mean value* of the function $f(x)$ on the interval $[a, b]$. From formula (118-2) we have

$$\mu = \frac{1}{b-a} \int_a^b f(x)\,dx. \tag{118-3}$$

Example 1. Intensity of alternating current is equal to

$$i = i_0 \sin \frac{2\pi t}{T},$$

where $i_0 > 0$ is the maximum value of the current, T the period, and t time.

Find the mean value of the square of current intensity during the period T.

By (118-3), we have

$$\mu = \overline{i^2} = \frac{1}{T} \int_0^T i^2\,dt = \frac{i_0^2}{T} \int_0^T \sin^2 \frac{2\pi t}{T}\,dt,$$

where the dash denotes the operation of averaging. Since

$$\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha),$$

we have

$$\overline{i^2} = \frac{i_0^2}{2T} \int_0^T \left(1 - \cos \frac{4\pi t}{T}\right) dt = \frac{i_0^2}{2T} \left(t - \frac{T}{4\pi} \sin \frac{4\pi t}{T}\right) \Bigg|_0^T = \frac{i_0^2}{2}. \tag{118-4}$$

The square root of the mean value of the squared current intensity is called the *root-mean-square intensity of current*, i.e.

$$i_{\text{rms}} = \sqrt{\overline{i^2}}.$$

On the basis of formula (118-4), we obtain the following result which is very important for electrical engineering:

$$i_{\text{rms}} = \frac{i_0}{\sqrt{2}}. \tag{118-5}$$

Corollary. Let

$$m = \min_{a \leqslant x \leqslant b} f(x) \quad \text{and} \quad M = \max_{a \leqslant x \leqslant b} f(x).$$

Since $m \leqslant f(x) \leqslant M$, for $a < b$ we have from (118-2)

$$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a). \tag{118-6}$$

Example 2. Estimate the integral

$$I=\int_0^{\frac{\pi}{2}}\frac{dx}{1+\sin^2 x}.$$

Since $\frac{1}{2}\leqslant\frac{1}{1+\sin^2 x}\leqslant 1$ for $0\leqslant x\leqslant\frac{\pi}{2}$, by (118-6) we have

$$\frac{\pi}{4}\leqslant I\leqslant\frac{\pi}{2}.$$

We may set approximately

$$I\approx\frac{1}{2}\left(\frac{\pi}{4}+\frac{\pi}{2}\right)\approx\frac{1}{2}(0.79+1.57)=1.18.$$

The exact value of the integral is

$$I=\frac{\pi}{2\sqrt{2}}\approx 1.12.$$

Sec. 119. Integration by Parts in the Definite Integral

Let $u=u(x)$ and $v=v(x)$ be continuously differentiable functions* on a closed interval $[a, b]$.

We have

$$d[u(x)\,v(x)]=v(x)\,du(x)+u(x)\,dv(x).$$

Integrating this equality between the limits a and b and taking into account that

$$du(x)=u'(x)\,dx \quad \text{and} \quad dv(x)=v'(x)\,dx,$$

we find

$$u(x)\,v(x)\Big|_a^b=\int_a^b v(x)\,u'(x)\,dx+\int_a^b u(x)\,v'(x)\,dx.$$

Hence we obtain the *formula for integration by parts in the definite integral*

$$\int_a^b u(x)\,v'(x)\,dx=u(b)\,v(b)-u(a)\,v(a)-\int_a^b v(x)\,u'(x)\,dx. \qquad (119\text{-}1)$$

For the sake of brevity we use the following notation

$$u(b)\,v(b)-u(a)\,v(a)=u(x)\,v(x)\Big|_a^b.$$

* That is having continuous derivatives $u'(x)$ and $v'(x)$.

Example. Find

$$\int_0^{2\pi} x \cos x \, dx.$$

Setting

$$u = x, \qquad dv = \cos x \, dx = d(\sin x),$$

we obtain

$$du = dx, \qquad v = \sin x.$$

Applying formula (119-1), we shall have

$$\int_0^{2\pi} x \cos x \, dx = x \sin x \Big|_0^{2\pi} - \int_0^{2\pi} \sin x \, dx =$$

$$= 2\pi \sin 2\pi - 0 \cdot \sin 0 + \cos x \Big|_0^{2\pi} = \cos 2\pi - \cos 0 = 0.$$

Sec. 120. Change of Variable in the Definite Integral (Integration by Substitution)

Let there be given a definite integral

$$\int_a^b f(x) \, dx, \tag{120-1}$$

where $f(x)$ is a continuous function on a closed interval $[a, b]$, and let, for some reasons, it be desirable to introduce a new variable t connected with the old one x by the relation

$$x = \varphi(t) \qquad (\alpha \leqslant t \leqslant \beta), \tag{120-2}$$

where $\varphi(t)$ is a continuously differentiable function on the segment $[\alpha, \beta]$. If in this case: (1) with a change of t from α to β the variable x changes from a to b, i.e.

$$\varphi(\alpha) = a, \quad \varphi(\beta) = b, \tag{120-3}$$

and (2) the composite function $f(\varphi(t))$ is defined and continuous on the interval $[\alpha, \beta]$*, then the following formula is valid

$$\int_a^b f(x) \, dx = \int_\alpha^\beta f(\varphi(t)) \, \varphi'(t) \, dt. \tag{120-4}$$

* If the values of $\varphi(t)$ do not fall outside the limits of the closed interval $[a, b]$, then condition (2) is unnecessary (see the theorem on the continuity of a composite function, Sec. 61).

Proof. Consider the composite function

$F(\varphi(t))$,

where $F(x)$ is an antiderivative of the function $f(x)$, i.e.

$F'(x) = f(x)$.

Applying the rule for differentiating a composite function, we get

$$\frac{d}{dt} F(\varphi(t)) = F'(\varphi(t)) \cdot \varphi'(t) = f(\varphi(t)) \cdot \varphi'(t),$$

consequently, the function $F(\varphi(t))$ is an antiderivative for the function $f(\varphi(t)) \cdot \varphi'(t)$. Hence, on the basis of the Newton-Leibniz formula and taking into account (120-3), we shall have

$$\int_{\alpha}^{\beta} f(\varphi(t)) \cdot \varphi'(t)\, dt = F(\varphi(t))\Big|_{\alpha}^{\beta} = F(\varphi(\beta)) - F(\varphi(\alpha)) =$$

$$= F(b) - F(a) = \int_{a}^{b} f(x)\, dx,$$

which was to be proved.

Note. In evaluating a definite integral by the method of substitution there is no necessity to return to the old variable, it is sufficient only to introduce new limits of integration when applying formulas (120-3).

Example. Evaluate the integral

$$\int_{0}^{3} x\sqrt{1+x}\, dx. \tag{120-5}$$

It is natural to set

$$t = \sqrt{1+x}, \tag{120-6}$$

hence

$x = t^2 - 1$ and $dx = 2t\, dt$.

The new limits of integration are determined from (120-6): putting $x = 0$, we get $t = 1$ and putting $x = 3$, we get $t = 2$. Consequently,

$$\int_{0}^{3} x\sqrt{1+x}\, dx = \int_{1}^{2} (t^2-1) \cdot t \cdot 2t\, dt = 2\int_{1}^{2} (t^4 - t^2)\, dt =$$

$$= 2\left(\frac{t^5}{5} - \frac{t^3}{3}\right)\Big|_{1}^{2} = 2\left(\frac{32-1}{5} - \frac{8-1}{3}\right) = \frac{62}{5} - \frac{14}{3} = 7\frac{11}{15}.$$

Sec. 121. The Definite Integral as the Limit of an Integral Sum

Let the function $f(x)$ be continuous on a segment $[a, b]$ and let, for definiteness, $f(x) \geqslant 0$ on $[a, b]$ where $a < b$. Then its definite integral

$$\int_a^b f(x)\,dx \tag{121-1}$$

geometrically represents the area S of a curvilinear trapezoid $aABb$ bounded by the given curve $y = f(x)$, the x-axis ($y = 0$), and two vertical lines $x = a$ and $x = b$ (Fig. 136).

As far back as ancient times (over two thousand years ago) Greek mathematicians used the following technique for computing the

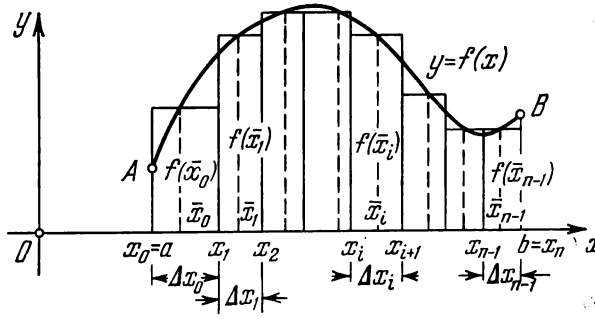

FIG. 136

area S: partition the figure S* into a large number of vertical strips bounded by the perpendiculars to the x-axis erected at the points

$$x_0 = a, \quad x_1, x_2, \ldots, x_n = b$$

$$(a = x_0 < x_1 < x_2 < \ldots < x_n = b).$$

Each of the strips thus obtained may be approximately regarded as a rectangle with the base $\Delta x_i = x_{i+1} - x_i$ ($i = 0, 1, 2, \ldots, n-1$) and a certain intermediate altitude $f(\bar{x}_i)$, where $x_i \leqslant \bar{x}_i \leqslant x_{i+1}$. Then the area of one such rectangle is, obviously, equal to

$$f(\bar{x}_i)\,\Delta x_i \qquad (i = 0, 1, 2, \ldots, n-1)$$

and, consequently, the area of the step-like figure consisting of n such rectangles will be

$$S_n = f(\bar{x}_0)\,\Delta x_0 + f(\bar{x}_1)\,\Delta x_1 + \ldots + f(\bar{x}_{n-1})\,\Delta x_{n-1} \tag{121-2}$$

* Here we denote the curvilinear trapezoid and its area by one and the same letter S. The difference between them is seen from the context.

or in a briefer notation

$$S_n = \sum_{i=0}^{n-1} f(\bar{x}_i)\,\Delta x_i, \tag{121-2'}$$

where the Greek letter Σ (sigma) is used as the summation sign.

Sum (121-2) or (121-2') is called the *integral sum* for the function $f(x)$*. Since, as $n \to \infty$ and $\max\limits_i |\Delta x_i| \to 0$, our strips turn into the ordinates of the graph of the function $y = f(x)$, it is natural to expect that

$$\lim_{n\to\infty} S_n = S = \int_a^b f(x)\,dx. \tag{121-3}$$

Indeed, the following theorem is valid.

Theorem. *If a function $f(x)$ is continuous on the closed interval $[a, b]$, then the limit of its integral sum S_n, as $n \to \infty$ and $\max\limits_i |\Delta x_i| \to 0$, is equal to the corresponding definite integral of this function*, i.e.

$$\lim_{\max|\Delta x_i|\to 0} \sum_{i=0}^{n-1} f(\bar{x}_i)\,\Delta x_i = \int_a^b f(x)\,dx^{**}. \tag{121-4}$$

Proof. Let

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

We set $S = \int_a^b f(x)\,dx$. By virtue of the additivity property, we have

$$S = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\,dx.$$

Hence, applying the mean-value theorem (Sec. 118), we shall have

$$S = \sum_{i=0}^{n-1} f(\xi_i)\,\Delta x_i, \tag{121-5}$$

where $\Delta x_i = x_{i+1} - x_i > 0$ and $\xi_i \in [x_i, x_{i+1}]$.

Consider the integral sum

$$S_n = \sum_{i=0}^{n-1} f(\bar{x}_i)\,\Delta x_i, \tag{121-6}$$

* The concept of integral sum (121-2') is, naturally, generalized for the case of an alternating-sign function.

** In this meaning the integral sign represents a stylized letter S (the summation sign), and the notation of the entire definite integral is a briefly written sum of an infinite number of infinitesimal terms of the form $f(x)\,\Delta x = f(x)\,dx$ on the interval $[a, b]$ (from our point of view, the limit of such a sum).

where $\bar{x}_i \in [x_i, x_{i+1}]$.

From formulas (121-5) and (121-6) we obtain

$$S - S_n = \sum_{i=0}^{n-1} [f(\xi_i) - f(\bar{x}_i)]\,\Delta x_i. \tag{121-7}$$

Hence

$$|S - S_n| \leqslant \sum_{i=0}^{n-1} |f(\xi_i) - f(\bar{x}_i)|\,|\Delta x_i|. \tag{121-8}$$

If ε is an arbitrary positive number, then for a sufficiently small $\max |\Delta x_i|$, by virtue of the continuity of the function $f(x)$, the following inequalities are ensured

$$|f(\xi_i) - f(\bar{x}_i)| < \varepsilon \tag{121-9}$$

(the property of uniform continuity of a function $f(x)$, see for instance, *A Course of Mathematical Analysis* by S. M. Nikolsky, Vol. 1, Chapter 9; Mir Publishers, Moscow, 1977). Therefore, from (121-9) and (121-8) we obtain

$$|S - S_n| < \sum_{i=0}^{n-1} \varepsilon \Delta x_i = \varepsilon (b - a), \tag{121-10}$$

where $(b - a)$ is the length of the interval $[a, b]$.

Since ε is an arbitrary number, it follows from inequality (121-10) that

$$\lim_{n \to \infty} S_n = S, \tag{121-11}$$

i.e. equality (121-3) is valid.

Note. If $y = f(x) \geqslant 0$ on $[a, b]$, then the area of the curvilinear trapezoid $aABb$ will be understood, by definition, as the number

$$S = \lim_{n \to \infty} S_n,$$

assuming that this limit exists.

Corollary. *If a function* $f(x) \geqslant 0$ *is continuous on* $[a, b]$, *then a curvilinear trapezoid* $\{a \leqslant x \leqslant b,\ 0 \leqslant y \leqslant f(x)\}$ *has a finite area*, i.e. it is a *squarable figure.*

Sec. 122. Approximate Evaluation of Definite Integrals

The definite integral

$$\int_a^b f(x)\,dx \tag{122-1}$$

of a given continuous function $y = f(x)$ is evaluated exactly by far not always. But, taking advantage of its geometrical meaning, we can give a number of approximate formulas with the aid of which this integral is found to any extent of accuracy. We are going to

consider here the simplest of them, the so-called *trapezoid formula*.

As is known, integral (122-1) represents the area (see Note on page 285) of a curvilinear trapezoid bounded by the line $y = f(x)$, the x-axis, and two ordinates $x = a$ and $x = b$ (Fig. 137).

Let us subdivide the interval $[a, b]$ into n equal parts, each of length $h = \frac{b-a}{n}$ (*the subdivision pitch*).

Let $x_0, x_1, \ldots, x_n$ ($x_0 = a$, $x_n = b$) be the abscissas of the division points and $y_0, y_1, \ldots, y_n$ the corresponding ordinates of the curve. We have the following computational formulas

$$x_i = x_0 + ih, \quad y_i = f(x_i)$$

($i = 0, 1, 2, \ldots, n$). As the result of the construction, our curvilinear trapezoid is separated into a number of vertical strips of one

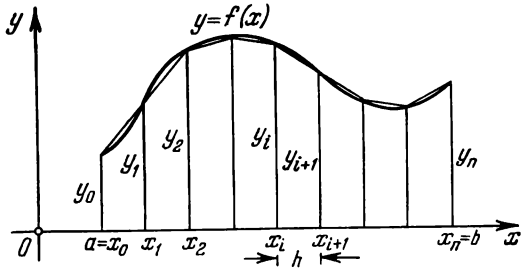

FIG. 137

and the same width h. Each of these strips may be taken for a trapezoid. Summing up the areas of the trapezoids thus obtained, we shall have

$$\int_a^b f(x)\,dx \approx \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \ldots + \frac{h}{2}(y_{n-1} + y_n)$$

or

$$\int_a^b y\,dx \approx h\left(\frac{y_0}{2} + y_1 + y_2 + \ldots + y_{n-1} + \frac{y_n}{2}\right) \tag{122-2}$$

(*the trapezoid formula*). Formula (122-2) may be briefly written in the form

$$\int_a^b y\,dx \approx h\sum_{i=0}^{n} \varepsilon_i y_i, \tag{122-3}$$

where $\varepsilon_i = \frac{1}{2}$ for $i = 0$ and $i = n$ and $\varepsilon_i = 1$ for $i = 1, 2, \ldots, n-1$.

The error

$$R_n = \int_a^b y\,dx - h\sum_{i=0}^{n} \varepsilon_i y_i$$

is called the *remainder term* of the trapezoid formula (122-3). It is proved that if the function $y = f(x)$ has a continuous second derivative $f''(x)$ on the closed interval $[a, b]$ then

$$|R_n| \leqslant \frac{b-a}{12} M_2 h^2,$$

where

$$M_2 = \max_{a \leqslant x \leqslant b} |f''(x)|$$

(see *Computational Mathematics* by B.P. Demidovich and I.A. Maron, Chapter 16, Mir Publishers, Moscow, 1977).

Example. Compute an approximate value of the integral

$$\int_0^1 \sqrt{1+x^2}\, dx = I.$$

We divide the interval of integration [0, 1] into ten equal parts ($n = 10$); consequently, the pitch $h = 0.1$.

The abscissas of the points of division x_i ($i = 0, 1, \ldots, 10$) and the corresponding ordinates $y_i = \sqrt{1 + x_i^2}$, computed with the aid of the table of square roots are tabulated below. For the sake of convenience, the ordinates y_i are multiplied by the factor ε_i such that $\varepsilon_i = \frac{1}{2}$ for $i = 0$ and $i = 10$ (marked with an asterisk), and $\varepsilon_i = 1$ for $i = 1, 2, \ldots, 9$.

i	x_i	$\varepsilon_i y_i$
0	0.0	0.5000*
1	0.1	1.0050
2	0.2	1.0198
3	0.3	1.0440
4	0.4	1.0770
5	0.5	1.1180
6	0.6	1.1662
7	0.7	1.2207
8	0.8	1.2806
9	0.9	1.3454
10	1.0	0.7071*
Σ		11.4838

By formula (122-3) we have

$I \approx 0.1 \cdot 11.4838 \approx 1.148.$

The exact value of the integral is

$$I = \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1+\sqrt{2}) \approx 1.1479.$$

Sec. 123. Simpson's Formula

A more accurate formula is obtained if we assume that the curve bounding a curvilinear strip is a parabola.

Consider the vertical strip bounded by a continuous curve $y =$

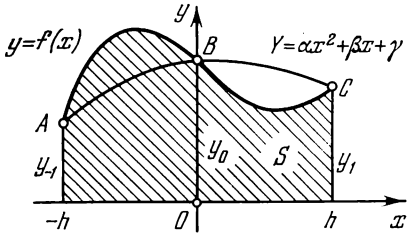

FIG. 138

$= f(x)$, the x-axis ($y = 0$) and two vertical lines: $x = -h$ and $x = h$ (Fig. 138).

If h is sufficiently small, then the curve $y = f(x)$ may be approximately replaced by the parabola

$$Y = \alpha x^2 + \beta x + \gamma, \tag{123-1}$$

passing through the points $A(-h, y_{-1})$, $B(0, y_0)$, and $C(h, y_1)$. Then

$$\int_{-h}^{h} y\,dx = I$$

will be approximately equal to

$$\int_{-h}^{h} (\alpha x^2 + \beta x + \gamma)\,dx = \left(\frac{\alpha x^3}{3} + \frac{\beta x^2}{2} + \gamma x\right)\Big|_{-h}^{h} =$$

$$= \frac{2\alpha}{3}h^3 + 2\gamma h. \tag{123-2}$$

Setting in formula (123-1) in succession $x = -h, 0, h$, we get

$$y_{-1} = \alpha h^2 - \beta h + \gamma, \quad y_0 = \gamma, \quad y_1 = \alpha h^2 + \beta h + \gamma. \tag{123-3}$$

Hence

$$\gamma = y_0, \quad \alpha = \frac{y_{-1} - 2y_0 + y_1}{2h^2}. \tag{123-4}$$

Substituting these values in formula (123-2), we shall have

$$\int_{-h}^{h} y\,dx \approx \frac{1}{3}h(y_{-1}-2y_0+y_1)+2y_0h=\frac{h}{3}(y_{-1}+4y_0+y_1) \qquad (123\text{-}5)$$

(*Simpson's formula*).

Example. Using Simpson's formula, find

$$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx.$$

Putting $h=\frac{\pi}{2}$, we have $y_{-1}=0$, $y_0=1$, $y_1=0$. Consequently,

$$I \approx \frac{\pi}{6}(0+4+0)=\frac{2}{3}\pi \approx 2.07$$

(the exact value is $I = 2$).

Translating the coordinate system, we can write Simpson's formula in the form

$$\int_a^b y\,dx=\frac{h}{3}\left[y(a)+4y\left(\frac{a+b}{2}\right)+y(b)\right], \qquad (123\text{-}5')$$

where $h=\frac{b-a}{2}$.

Note. To increase the accuracy of computation of the definite integral $\int_a^b y\,dx$, the interval of integration $[a, b]$ is usually subdivided into n subintervals, where n is a sufficiently large natural number, and Simpson's formula (123-5′) is applied to each of them, putting $h=\frac{b-a}{2n}$. By virtue of the additivity property, the given definite integral will approximately represent the sum of the results thus obtained (*the parabolic formula*) (see *Computational Mathematics* by B.P. Demidovich and I.A. Maron, Chapter 16, Mir Publishers, Moscow, 1977).

Sec. 124. Improper Integrals

When determining the integral

$$\int_a^b f(x)\,dx \qquad (124\text{-}1)$$

we assumed that: (1) the interval of integration $[a, b]$ is finite and (2) the integrand $f(x)$ is defined and continuous on the interval $[a, b]$. Such definite integral is called *proper* (the word 'proper' is usually omitted). If at least one of the two conditions is violated, then (124-1) will be called *improper definite integral*. Let us find out the meaning of this new concept for two simplest cases.

I. Let a function $f(x)$ be continuous for $a \leqslant x < +\infty$. Then, by definition, we set

$$\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx. \tag{124-2}$$

If limit (124-2) exists, then the *improper integral with an infinite limit of integration* standing in the left-hand side of equality (124-2) is called *convergent* and its value is determined by formula (124-2);

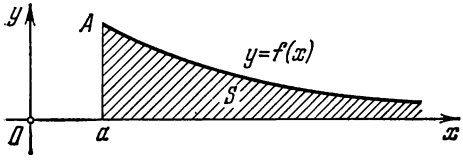

FIG. 139

otherwise equality (124-2) loses sense, the improper integral standing on the left is called *divergent* and no numerical value is assigned to it.

Geometrically, for a function $f(x)$ non-negative on $[a, \infty]$ improper integral (124-2) represents the area of a curvilinear figure bounded by the given curve $y = f(x)$, the x-axis, and a vertical line $x = a$ (Fig. 139).

Let $F(x)$ be an antiderivative for the integrand $f(x)$. By formula (124-2), we have

$$\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} [F(b) - F(a)].$$

Introducing the conventional notation

$$F(+\infty) = \lim_{b \to +\infty} F(b),$$

we obtain for a convergent improper integral with an infinite upper limit of integration a *generalized Newton-Leibniz formula*

$$\int_a^{+\infty} f(x)\,dx = F(+\infty) - F(a), \tag{124-3}$$

where

$$F'(x) = f(x).$$

Example 1.

$$\int_0^{+\infty} \frac{dx}{1+x^2} = \arctan x \Big|_0^{+\infty} = \arctan(+\infty) - \arctan 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

II. Let the function $f(x)$ be continuous for $a \leqslant x < b$ and have a point of discontinuity at $x = b$. Then the corresponding *improper integral of a discontinuous function* is determined by the formula

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to +0} \int_a^{b-\varepsilon} f(x)\,dx \tag{124-4}$$

and is called *convergent* or *divergent* depending on whether or not the limit of the right-hand member of equality (124-4) exists.

If there exists a function $F(x)$ continuous on $[a, b]$ and such that $F'(x) = f(x)$ for $a \leqslant x < b$ (a *generalized antiderivative*), then the Newton-Leibniz generalized formula

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to +0} \int_a^{b-\varepsilon} f(x)\,dx = \lim_{\varepsilon \to +0} [F(b-\varepsilon) - F(a)] =$$

$$= F(b) - F(a) = F(x)\Big|_a^b$$

is valid for improper integral (124-4).

Example 2. We have

$$\int_0^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big|_0^1 = 2 - 0 = 2.$$

Exercises

1. Calculate the definite integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx$$

and indicate its geometrical meaning.

In Problems 2 to 4 compute the definite integrals.

2. $\int_0^1 x^4\,dx$. 3. $\int_1^e \frac{dx}{x}$. 4. $\int_1^8 \frac{dx}{\sqrt[3]{x}}$.

5. Let

$$I = \int_x^y e^{t^2}\,dt.$$

Find: (a) $\frac{dI}{dy}$; (b) $\frac{dI}{dx}$.

6. Find the mean value of the function $f(x) = \sqrt{x}$ on the closed interval $[1; 9]$.

7. Integrating by parts, compute the following definite integrals:

(a) $\int_0^{2\pi} x \sin 2x\,dx$; (b) $\int_0^1 (x-1)\,e^{-x}\,dx$.

8. Using the indicated substitutions, compute the following definite integrals:

(a) $\int_0^2 \sqrt{4-x^2}\,dx$, $x = 2\sin t$; (b) $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$, $x = t^2$.

9. Evaluate approximately the integral

$$\int_0^1 \frac{dx}{1+x}$$

(a) by the trapezoid formula, (b) by Simpson's formula, subdividing the interval $[0, 1]$ into $n = 10$ equal parts. Compare the results thus obtained with the exact value of the integral equal to $\ln 2 \approx$ ≈ 0.69315.

10. Evaluate the following improper integrals:

(a) $\int_0^{+\infty} e^{-x}\,dx$; (b) $\int_{-\frac{1}{2}}^1 \frac{dx}{\sqrt{1-x^2}}$; (c) $\int_0^{+\infty} \frac{dx}{x^2+x+1}$.

Chapter 15

Applications of the Definite Integral

Sec. 125. Areas in Rectangular Coordinates

Problem 1. Find the area S of a curvilinear trapezoid $aABb$ bounded by a given continuous line $y = f(x)$, the segment $a \leqslant x \leqslant b$ of the x-axis, and two vertical lines $x = a$ and $x = b$ if $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$ (Fig. 140).

By the geometrical meaning of the definite integral, we have

$$S = \int_a^b y\,dx, \tag{125-1}$$

where $y = f(x)$ is the given function.

Note. Formula (125-1) can be justified in another way. We shall consider the area S as a variable quantity obtained as the result of displacement of the current ordinate $xM = y$ from the initial position aA to the terminal position bB. Assigning an increment $\Delta x = dx$ to the current abscissa x, we obtain the *increment of the area* ΔS which represents the area of the vertical strip $xMM'x'$ contained between the ordinates at the points x and $x' = x + dx$ (Fig. 140). The *differential of the area* dS is the principal linear part of the increment ΔS as $\Delta x \to 0$ and is, obviously, equal to the area of the rectangle with the base dx and altitude y*; therefore

$$dS = y\,dx \tag{125-2}$$

(*the element of area in rectangular coordinates*). Integrating equality (125-2) between the limits $x = a$ and $x = b$, we shall have formula (125-1):

$$S = \int_a^b y\,dx.$$

* It is possible to prove rigorously that for a continuous function $y = f(x)$ the area of the rectangle $y\,dx$ differs from the area of the strip ΔS by a quantity of a higher order of smallness than dx.

In this particular example we have shown the application of the so-called *method of differential* whose main point consists in that first, proceeding from elements, we make up the differential of a required quantity, and then, integrating between the corresponding limits,

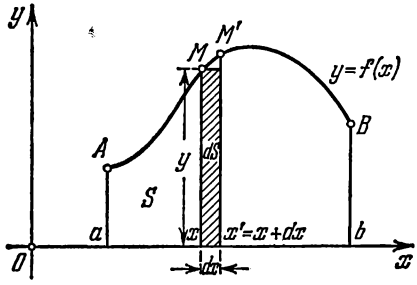

FIG. 140

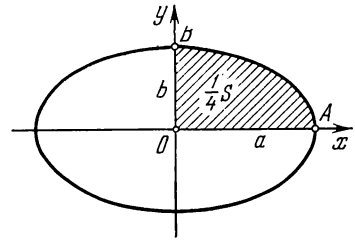

FIG. 141

find the value of this quantity. In more detail this method is developed in the theory of differential equations (see Chapter 22).

The two basic methods applied in the theory of the definite integral: (1) the method of integral sums (see Sec. 121) and (2) the method of differential, illustrated with ample examples, will be considered in the following sections.

Example 1. Find the area S of the region bounded by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Since the ellipse is a symmetrical figure, we may confine ourselves to computing a quarter of the area S (Fig. 141).

From the equation of the ellipse for the first quadrant we have

$$y=\frac{b}{a}\sqrt{a^2-x^2}.$$

Hence, by formula (125-1), we obtain

$$\frac{1}{4}S=\int_0^a y\,dx=\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx.$$

We use here the trigonometric substitution

$$x = a\sin t, \quad dx = a\cos t\,dt.$$

The new limits of integration $t = \alpha$ and $t = \beta$ are determined from the equations

$$0 = a\sin t, \quad a = a\sin t.$$

We may set

$$\alpha = 0 \quad \text{and} \quad \beta = \frac{\pi}{2}.$$

Consequently,

$$\frac{1}{4}S = \frac{b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2 t}\cdot a\cos t\,dt = ab\int_0^{\frac{\pi}{2}} \cos^2 t\,dt =$$

$$= \frac{ab}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2t)\,dt = \frac{ab}{2}\left(t+\frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{2}} = \frac{ab}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}ab$$

and

$$S = \pi ab.$$

In particular, setting $a = b$, we shall obtain the area of a circle

$$S = \pi a^2$$

of radius a.

Note. In more complicated cases we try to represent a given figure as a sum or a difference of curvilinear trapezoids.

Problem 2. Find the area of the region bounded by two continuous curves

$$y_1 = f_1(x) \quad \text{and} \quad y_2 = f_2(x) \quad (y_2 \geqslant y_1)$$

and two vertical lines: $x = a$ and $x = b$ (Fig. 142).

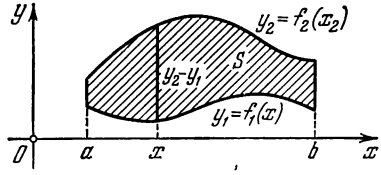

FIG. 142

We shall assume that $y_1 = f_1(x)$ and $y_2 = f_2(x)$ are non-negative functions on $[a, b]$. This can always be achieved by translating the x-axis.

The desired area S can be regarded as the difference between two curvilinear trapezoids bounded by the given lines. Therefore,

$$S = \int_a^b y_2\,dx - \int_a^b y_1\,dx$$

and, consequently,

$$S = \int_a^b (y_2 - y_1)\,dx, \tag{125-3}$$

where $y_1 = f_1(x)$ and $y_2 = f_2(x)$ are the given functions. Note that

$$y_2 - y_1 = f_2(x) - f_1(x)$$

represents the "thickness" of the area S at the point x.

Example 2. Determine the area S bounded by the parabola $y = x^2 + 1$ and the straight line $x + y = 3$ (Fig. 143).

Solving the system of equations of the parabola and the straight line:

$$\left.\begin{aligned} y &= x^2 + 1, \\ x + y &= 3, \end{aligned}\right\}$$

we find the abscissas of the points of intersection: $x_1 = -2$ and $x_2 = 1$.

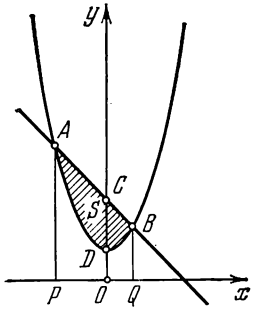

FIG. 143

Setting $y_2 = 3 - x$ and $y_1 = x^2 + 1$, by formula (125-3), we get

$$S = \int_{-2}^{1} [(3 - x) - (x^2 + 1)]\, dx =$$

$$= \int_{-2}^{1} (2 - x - x^2)\, dx = \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_{-2}^{1} =$$

$$= 2(1 + 2) - \frac{1}{2}(1 - 4) - \frac{1}{3}(1 + 8) = 6 + \frac{3}{2} - 3 = 4\frac{1}{2}.$$

Formula (125-1) is also applicable for computing areas of simple figures the equation of whose contour is represented parametrically.

Example 3. Find the area S bounded by the first arc of the cycloid

$$\left.\begin{aligned} x &= a(t - \sin t), \\ y &= a(1 - \cos t) \end{aligned}\right\} \qquad (125\text{-}4)$$

and the x-axis (see Sec. 33, Fig. 47).

We have

$$S = \int_{0}^{2\pi a} y\, dx.$$

We substitute the variables in this integral by taking the parameter t for an independent variable. From equation (125-4) we obtain

$$dx = a(1 - \cos t)\, dt,$$

and we have $t = 0$ for $x = 0$ and $t = 2\pi$ for $x = 2\pi a$. Hence,

$$S = \int_0^{2\pi} a(1-\cos t)\cdot a(1-\cos t)\,dt = a^2 \int_0^{2\pi} (1-2\cos t+\cos^2 t)\,dt =$$

$$= a^2\left[(t-2\sin t)\Big|_0^{2\pi} + \int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt\right] =$$

$$= a^2\left[2\pi + \frac{1}{2}\left(t+\frac{1}{2}\sin 2t\right)\Big|_0^{2\pi}\right] = a^2(2\pi+\pi) = 3\pi a^2.$$

Thus, we have obtained Galilei's* theorem: *The area bounded by an arc of a cycloid and its chord is equal to the tripled area of the generating circle.*

Sec. 126. Areas in Polar Coordinates

Problem. Find the area S of the sector OAB bounded by a given continuous line

$$\rho = f(\varphi)$$

and two rays $\varphi = \alpha$ and $\varphi = \beta$ where ρ and φ are polar coordinates (Fig. 144).

We are going to solve this problem using the *method of differential.*

Suppose the area S appeared as the result of angular displacement of the radius vector (or polar radius) $\rho = f(\varphi)$ for φ varying from $\varphi = \alpha$ to $\varphi = \beta$ (Fig. 144). If the current polar angle φ receives an increment $d\varphi$, then the increment of area ΔS = area OMM'. The differential dS represents the principal linear part of the increment ΔS as $d\varphi \to 0$ and is equal to the area of a circular sector OMN of radius ρ and central angle $d\varphi$**; therefore

$$dS = \frac{1}{2}\,\widehat{MN}\cdot OM = \frac{1}{2}\rho\,d\varphi\cdot\rho = \frac{1}{2}\rho^2\,d\varphi \qquad (126\text{-}1)$$

* Galileo Galilei (1564-1642). The founder of modern science, who said that the book of nature is written in mathematical symbols, was a mathematician, a physicist, an astronomer, a brilliant polemicist, and a talented writer. He formulated the principle of inertia and the laws of motion of the pendulum, of falling bodies, and of projectiles. He built some of the first telescopes and used them to discover the moons of Jupiter and Saturn, the phases of Venus, the rotation of the sun, and the ruggedness of the lunar landscape.

At the age of 70, Galileo was tried by the Inquisition for his *Dialogue on the Two Great World Systems.* He was forced to renounce the Copernican system and spent the rest of his life under house arrest.

** Indeed, by analogy with the physical meaning of the differential (Sec. 98), the differential of area dS is equal to the *imaginary increment* of the area S when the radius vector ρ is rotated through the angle $d\varphi$, provided it retains a constant length. Hence it is clear that dS is the area of a circular sector of a radius ρ and central angle $d\varphi$.

(*the element of area in polar coordinates*). Integrating equality (126-1) between the limits $\varphi = \alpha$ and $\varphi = \beta$, we get the sought-for area

$$S = \frac{1}{2} \int_{\alpha}^{\beta} \rho^2 \, d\varphi,$$

where $\rho = f(\varphi)$ is the given function.

Example. Find the area bounded by the cardioid

$$\rho = a(1 + \cos \varphi).$$

Compiling a table of the corresponding values, we obtain

φ	0	$\pm \frac{\pi}{6}$	$\pm \frac{\pi}{3}$	$\pm \frac{\pi}{2}$	$\pm \frac{2}{3}\pi$	$\pm \frac{5}{6}\pi$	$\pm \pi$
ρ	$2a$	≈ 1.9	$\frac{3}{2}a$	a	$\frac{a}{2}$	$\approx 0.1\,a$	0

Using the values of φ and ρ from our table, we plot seven points of the cardioid and join them with a smooth line to get the general idea of the shape of this curve (Fig. 145).

Since the cardioid is, obviously, symmetrical about the polar axis,

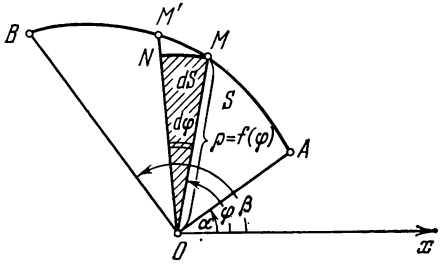

FIG. 144

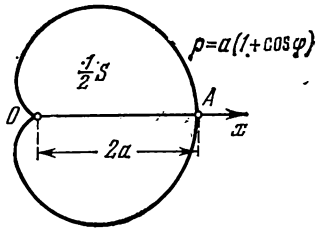

FIG. 145

it is sufficient to determine the upper half of the area and then to double it. Denoting by S the total area enclosed by the cardioid, we shall have

$$\frac{1}{2} S = \frac{1}{2} \int_{0}^{\pi} \rho^2 \, d\varphi.$$

Hence

$$S = \int_0^{\pi} \rho^2 \, d\varphi = a^2 \int_0^{\pi} (1 + \cos \varphi)^2 \, d\varphi =$$

$$= a^2 \left(\int_0^{\pi} d\varphi + 2 \int_0^{\pi} \cos \varphi \, d\varphi + \int_0^{\pi} \cos^2 \varphi \, d\varphi \right).$$

Or, since

$$\int_0^{\pi} d\varphi = \pi, \quad \int_0^{\pi} \cos \varphi \, d\varphi = \sin \varphi \Big|_0^{\pi} = 0,$$

$$\int_0^{\pi} \cos^2 \varphi \, d\varphi = \frac{1}{2} \int_0^{\pi} (1 + \cos 2\varphi) \, d\varphi = \frac{1}{2} \left(\varphi + \frac{1}{2} \sin 2\varphi \right) \Big|_0^{\pi} = \frac{\pi}{2},$$

we finally have

$$S = \frac{3\pi}{2} a^2.$$

Sec. 127. The Arc Length in Rectangular Coordinates

Definition. *The arc length of an arc AB is understood as the limit of the length of the polygonal line inscribed into this arc as the number of its segments increases indefinitely and the length of the greatest segment tends to zero.*

A curve is said to be *smooth* if this curve is continuous and has a tangent at each point which continuously changes its position from point to point. Obviously, a curve will be smooth if its equation can be written in the form

$$y = f(x) \qquad (a \leqslant x \leqslant b), \tag{127-1}$$

where the function $f(x)$ is continuous and has a continuous derivative $f'(x)$ on a given segment $[a, b]$.

Theorem. *Any smooth curve* (127-1) *has a definite finite arc length.*

Proof. We inscribe into the given smooth curve a polygonal line $M_0M_1M_2 \ldots M_n$ (Fig. 146), where $M_0 = A\,(a, f(a))$ and $M_n = B\,(b, f(b))$. Projecting the segments $\overline{M_{i-1}M_i}$ $(i = 1, 2, \ldots, n)$ of the polygonal line on the x-axis, we obtain the subdivision of the segment $[a, b]$ into a system of segments Δx_i. Let Δy_i be the increment of the given function $y = f(x)$ on the segment Δx_i (Fig. 146). By the Pythagorean theorem, we have

$$\overline{M_{i-1}M_i} = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}.$$

Applying Lagrange's theorem about the finite increment of a func-

tion (Sec. 85, Chapter 11), we get

$$\Delta y_i = \Delta x_i f'(\bar{x}_i),$$

where $\bar{x}_i$ is an intermediate point of the segment Δx_i*. Hence

$$\overline{M_{i-1}M_i} = \sqrt{1+f'^2(\bar{x}_i)}\,\Delta x_i,$$

and, consequently the length of the entire polygonal line $M_0M_1 \ldots M_n$ (i.e. its perimeter) is equal to

$$\Pi_n = \sum_{i=1}^{n} \sqrt{1+f'^2(\bar{x}_i)}\,\Delta x_i.$$

To find the length l of our curve (127-1) we have to pass to the lim-

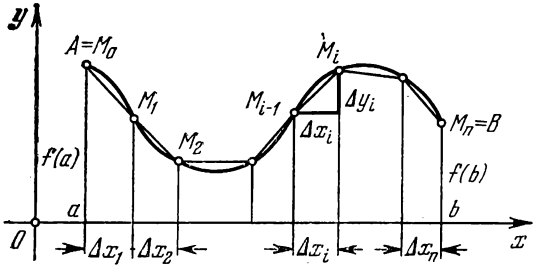

FIG. 146

it in the last expression assuming that $n \to \infty$ and $\max \Delta x_i \to 0$. Thus,

$$l = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{1+f'^2(\bar{x}_i)}\,\Delta x_i.$$

We have obtained the limit of an integral sum for the continuous function

$$F(x) = \sqrt{1+f'^2(x)}$$

(see Sec. 121, Chapter 14). Therefore

$$l = \int_a^b \sqrt{1+f'^2(x)}\,dx$$

or

$$l = \int_a^b \sqrt{1+y'^2}\,dx, \qquad (127\text{-}2)$$

where $y' = f'(x)$.

* Geometrically, $\bar{x}_i$ is the point of the segment Δx_i at which the tangent to the graph of the function $y = f(x)$ is parallel to its chord $\overline{M_{i-1}M_i}$.

The differential of an arc in rectangular coordinates. Let one point $A\,(a, h)$ of the curve be fixed, and the other $M\,(x, y)$ variable (Fig. 147). In such a case the arc length $l = AM$ is a certain function of the variable x. According to formula (127-2), we have

$$l = \int_a^x \sqrt{1+y'^2}\,dx.$$

Whence using the theorem about the derivative of a definite integral with a variable upper limit (Sec. 114), we get

$$l' = \sqrt{1+y'^2}$$

and, hence,

$$dl = l'\,dx = \sqrt{1+y'^2}\,dx.$$

Thus,

$$dl = \sqrt{1+y'^2}\,dx.$$

This is just the *formula for the differential of an arc in rectangular coordinates.*

Since

$$y' = \frac{dy}{dx},$$

we have

$$dl = \sqrt{(dx)^2+(dy)^2}. \qquad (127\text{-}3)$$

It is curious to note that the last formula represents the Pythagorean theorem for an infinitely small triangle MTP (Fig. 147).

Example 1. Let us compute the arc length of a segment of a *catenary* (the plane curve in which a uniform flexible cable hangs when suspended from two points).

Its equation in a properly chosen system of rectangular coordinates is

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right), \qquad (127\text{-}4)$$

where a is a certain positive number (*the parameter of the catenary*).

Equation (127-4) can be written in a simpler way:

$$y = a\cosh\frac{x}{a}, \qquad (127\text{-}4')$$

where cosh is hyperbolic cosine (see Sec. 78, Chapter 10).

The lowest point $A\,(0, a)$ of the catenary curve is called its *vertex* (Fig. 148).

Let us compute the length of the arc AB of the catenary assuming that the abscissa of the point B is equal to b, and its ordinate to h. Differentiating equation (127-4′), we have

$$y' = \sinh \frac{x}{a}.$$

Then we derive

$$1 + y'^2 = 1 + \sinh^2 \frac{x}{a} = \cosh^2 \frac{x}{a}.$$

Hence,

$$\sqrt{1 + y'^2} = \cosh \frac{x}{a}.$$

Whence, by (127-2), we obtain

$$\widehat{AB} = \int_0^b \cosh \frac{x}{a}\, dx = a \int_0^b \cosh \frac{x}{a}\, d\left(\frac{x}{a}\right) = a \sinh \frac{x}{a}\Big|_0^b =$$

$$= a\left(\sinh \frac{b}{a} - \sinh 0\right) = a \sinh \frac{b}{a}$$

(see formula XIV of Sec. 105, Chapter 13).

The formula for the arc length AB attains a simpler form if its

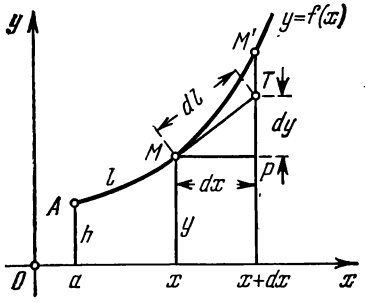

FIG. 147

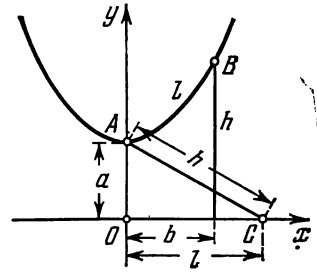

FIG. 148

right-hand member is expressed in terms of the ordinate h of the point B. Indeed, obviously,

$$h = a \cosh \frac{b}{a}.$$

By virtue of the identity

$$\sinh^2 \frac{b}{a} = \cosh^2 \frac{b}{a} - 1$$

we have

$$\widehat{AB} = a\sqrt{\cosh^2\frac{b}{a} - 1} = \sqrt{h^2 - a^2},$$

i.e. the arc AB is equal to the leg OC of the right-angled triangle OAC (Fig. 148) whose hypotenuse $AC = h$, the other leg being equal to $OA = a$.

Note. Let it be required to find the arc length l of a curve represented parametrically:

$$x = \varphi(t), \quad y = \psi(t) \qquad [t_0 \leqslant t \leqslant T],$$

where $\varphi(t)$ and $\psi(t)$ are continuously differentiable functions on $[t_0, T]$. We can prove that formula (127-3) for the differential of the arc dl will be true in this case as well. Since

$$dx = x'\,dt, \quad dy = y'\,dt,$$

we, therefore, have

$$dl = \sqrt{dx^2 + dy^2} = \sqrt{x'^2 + y'^2}\,dt.$$

Integrating this expression between the limits $t = t_0$ and $t = T$, we obtain the desired length of the arc

$$l = \int_{t_0}^{T} \sqrt{x'^2 + y'^2}\,dt. \tag{127-5}$$

Example 2. Find the length of the arc of the circle

$$x = a\cos t, \quad y = a\sin t$$

from $t = 0$ to $t = T$.

Here

$$dx = -a\sin t\,dt, \quad dy = a\cos t\,dt,$$

therefore

$$dl = \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt = a\,dt$$

and, consequently,

$$l = \int_0^T a\,dt = aT.$$

Example 3. Find the length of the curve of the astroid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

(Fig. 149). The equation of the astroid can be rewritten in the form

$$\left(x^{\frac{1}{3}}\right)^2+\left(y^{\frac{1}{3}}\right)^2=\left(a^{\frac{1}{3}}\right)^2.$$

It is natural to introduce a parameter t, putting

$$x^{\frac{1}{3}}=a^{\frac{1}{3}}\cos t, \quad y^{\frac{1}{3}}=a^{\frac{1}{3}}\sin t.$$

Hence, we get the *parametric equations of the astroid*

$$x = a\cos^3 t, \quad y = a\sin^3 t, \tag{127-6}$$

where $0 \leqslant t \leqslant 2\pi$.

Since (127-6) is a symmetrical curve, it is sufficient to find $\frac{1}{4}$ of

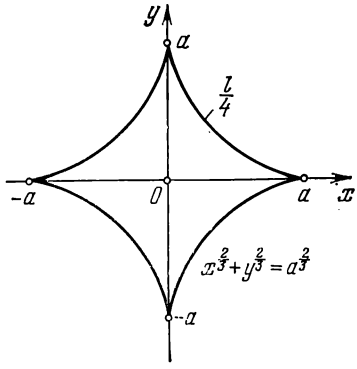

FIG. 149

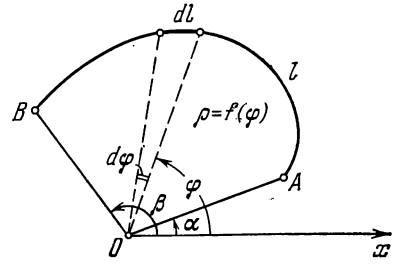

FIG. 150

the arc length l corresponding to the change of the parameter t from 0 to $\frac{\pi}{2}$. We have

$$dx = -3a\cos^2 t\sin t\,dt, \quad dy = 3a\sin^2 t\cos t\,dt.$$

Whence

$$dl=\sqrt{dx^2+dy^2}=3a\sin t\cos t\,dt.$$

Integrating this expression between the limits $t=0$ and $t=\frac{\pi}{2}$, we get

$$\frac{l}{4}=\int_0^{\frac{\pi}{2}} 3a\sin t\cos t\,dt=\frac{3a}{2}\int_0^{\frac{\pi}{2}}\sin 2t\,dt=\frac{3a}{4}\left(-\cos 2t\right)\Big|_0^{\frac{\pi}{2}}=$$

$$=\frac{3a}{4}(1+1)=\frac{3}{2}a.$$

Consequently the total length of the curve is

$$l = \frac{3}{2}\, a \cdot 4 = 6a.$$

Sec. 128. The Arc Length in Polar Coordinates

Let us first derive the differential dl of an arc in polar coordinates. By formula (127-3) from the preceding section, we have

$$(dl)^2 = (dx)^2 + (dy)^2,$$

where x and y are rectangular Cartesian coordinates of a point on the arc.

As is known, the formulas for passing from the polar coordinates ρ and φ to the rectangular coordinates x and y are:

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi.$$

Whence

$$dx = \cos \varphi\, d\rho - \rho \sin \varphi\, d\varphi, \quad dy = \sin \varphi\, d\rho + \rho \cos \varphi\, d\varphi.$$

Squaring these expressions, we get

$$(dx)^2 = \cos^2 \varphi\, (d\rho)^2 - 2\rho \cos \varphi \sin \varphi\, d\rho\, d\varphi + \rho^2 \sin^2 \varphi\, (d\varphi)^2,$$

$$(dy)^2 = \sin^2 \varphi\, (d\rho)^2 + 2\rho \sin \varphi \cos \varphi\, d\rho\, d\varphi + \rho^2 \cos^2 \varphi\, (d\varphi)^2.$$

Adding these equalities termwise, we shall have

$$(dl)^2 = (d\rho)^2 + \rho^2\, (d\varphi)^2.$$

Hence

$$dl = \sqrt{(d\rho)^2 + \rho^2\, (d\varphi)^2}$$

The last formula can be represented in the form

$$dl = \sqrt{\rho^2 + \rho'^2}\, d\varphi, \tag{128-1}$$

where

$$\rho' = \frac{d\rho}{d\varphi}.$$

Problem. Find the arc length l of the continuously differentiable function

$$\rho = f(\varphi)$$

between the points $A\ (\alpha,\ f(\alpha))$ and $B\ (\beta,\ f(\beta))$, where ρ and φ are the polar coordinates (Fig. 150). Integrating equality (128-1) between the limits $\varphi = \alpha$ and $\varphi = \beta$, we obtain the *arc length in polar*

coordinates

$$l = \int_{\alpha}^{\beta} \sqrt{\rho^2 + \rho'^2}\, d\varphi,$$

where $\rho = f(\varphi)$ is the given function and $\rho' = f'(\varphi)$ its derivative.

Example. Compute the complete arc length of the cardioid (Fig. 145)

$$\rho = a(1 + \cos\varphi).$$

We have

$$\rho' = -a \sin\varphi.$$

Therefore

$$\rho^2 + \rho'^2 = a^2(1 + 2\cos\varphi + \cos^2\varphi + \sin^2\varphi) =$$

$$= a^2(2 + 2\cos\varphi) = 2a^2(1 + \cos\varphi) = 4a^2\cos^2\frac{\varphi}{2}$$

and, consequently,

$$\sqrt{\rho^2 + \rho'^2} = 2a\left|\cos\frac{\varphi}{2}\right|.$$

Denoting the length of the upper half of the cardioid by $\frac{1}{2}\,l$, we get

$$\frac{1}{2}\,l = 2a\int_0^{\pi}\cos\frac{\varphi}{2}\,d\varphi = 4a\int_0^{\pi}\cos\frac{\varphi}{2}\,d\left(\frac{\varphi}{2}\right) = 4a\left(\sin\frac{\varphi}{2}\right)\Big|_0^{\pi} = 4a.$$

Since the cardioid is a symmetrical curve, we find its entire arc length l

$$l = 8a.$$

Sec. 129. Computing the Volume of a Solid by Known Cross Sections

Problem. Knowing the law of variation of the area of a cross section of a solid, find the volume V of this solid (Fig. 151).

Let Ox be a certain chosen direction and

$$S = S(x)$$

the area of a cross section perpendicular to the axis Ox at the point with the abscissa x. We shall assume that the function $S(x)$ is known and continuously varying with a change of x. Besides, we shall assume that in a sense the contour of the section changes also "continuously".

Projecting the solid on the axis Ox, we obtain a segment $[a, b]$ which yields the linear dimension of the solid in the direction of the axis Ox.

We then divide the segment $[a, b]$ into a large number of small parts Δx_i $(i = 1, 2, \ldots, n)$ and through the points of division pass

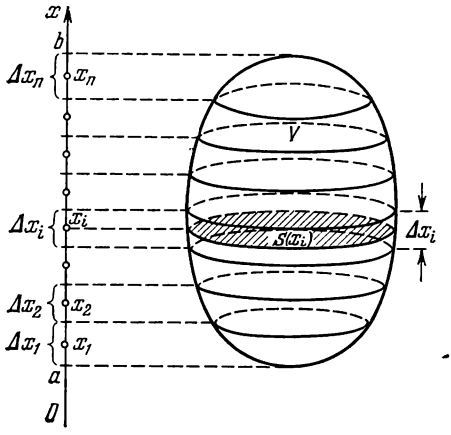

FIG. 151

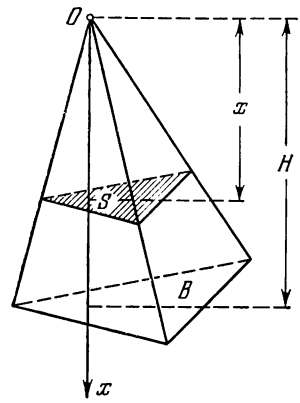

FIG. 152

planes perpendicular to the axis Ox. These planes separate our solid into n layers each of which may be taken for a cylinder. Since the volume of the ith layer is approximately equal to

$$S(x_i)\,\Delta x_i,$$

where x_i is some point of the segment Δx_i (see Fig. 151), we obtain the following expression for the volume V of the solid:

$$V \approx \sum_{i=1}^{n} S(x_i)\,\Delta x_i. \tag{129-1}$$

If $n \to \infty$ and $\max |\Delta x_i| \to 0$, then approximate equality (129-1) becomes more and more exact, until the following limit is obtained

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} S(x_i)\,\Delta x_i.$$

(129-1) represents an integral sum for the continuous function $S(x)$, and its limit is the corresponding definite integral (see Sec. 121). Therefore

$$V = \int_a^b S(x)\,dx. \tag{129-2}$$

Example 1. Find the volume V of a pyramid whose base is B and the altitude is H (Fig. 152).

For the axis Ox we shall take a straight line passing through the vertex O of the pyramid perpendicular to its base and directed from vertex to base.

Let S be the area of the section by the plane passing at a distance x from the vertex. Since the areas of parallel sections of a pyramid are to each other as the squares of their distances from the vertex, we have

$$\frac{S}{B}=\frac{x^2}{H^2};$$

hence

$$S=\frac{B}{H^2}x^2.$$

From formula (128-2) we obtain

$$V=\int_0^H S\,dx=\frac{B}{H^2}\int_0^H x^2\,dx=\frac{B}{H^2}\cdot\frac{H^3}{3}=\frac{1}{3}BH,$$

which is concordant with the known geometrical formula.

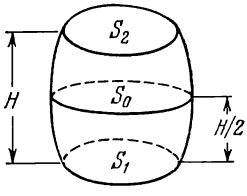

FIG. 153

Example 2. Let S_1 and S_2 be the areas of the lower and upper sections (respectively) of a "barrel-shaped" solid and S_0 the area of its mid-section (Fig. 153). Then, applying Simpson's formula to integral (129-2), we obtain

$$V=\int_0^H S(x)\,dx=\frac{H}{6}(S_1+S_2+4S_0),$$

where H is the altitude of the solid (*Simpson's volume formula*).

Sec. 130. The Volume of a Solid of Revolution

Problem. Find the volume V_x of a solid obtained by rotating about the x-axis a curvilinear trapezoid $aABb$ bounded by a given continuous line

$$y=f(x) \qquad (f(x)\geqslant 0),$$

the segment $a\leqslant x\leqslant b$ of the x-axis and two vertical lines $x=a$ and $x=b$ (Fig. 154).

This problem represents a particular case of the problem considered in the preceding section.

Here the area of the variable cross-section $S=S(x)$ corresponding to the abscissa x is a circle of radius y, therefore

$$S(x)=\pi y^2.$$

Hence, by formula (129-2), we have

$$V_x = \pi \int_a^b y^2\,dx, \tag{130-1}$$

where $y = f(x)$ is the given function.

Formula (130-1) can be obtained directly applying the differential method. The *element of volume* dV_x, obviously represents a cylinder with the base S and altitude dx*. Consequently,

$$dV_x = \pi y^2\,dx.$$

Hence, integrating between the limits $x = a$ and $x = b$, we get formula (130-1).

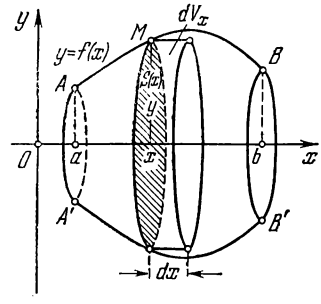

FIG. 154

Note. Let a curvilinear trapezoid $cCDd$ bounded by a single-valued continuous line $x = g(y)$, a segment $c \leqslant y \leqslant d$ of the y-axis, and two parallels $y = c$ and $y = d$ revolve about the y-axis (Fig. 155). Then the volume of the solid of revolution thus obtained, by analogy with formula obtained (130-1), will be equal to

$$V_y = \pi \int_c^d x^2\,dy, \tag{130-2}$$

where $x = g(y) \geqslant 0$ is the given function.

Example. Determine the volume of a solid bounded by a surface obtained by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{130-3}$$

about the major axis a (the x-axis) (Fig. 156).

Since the given ellipse is symmetrical about the coordinate axes, it is sufficient to find the volume obtained by rotating about the axis OX the area OAB which is equal to a quarter of the area of the ellipse depicted in Fig. 156 and to double the result obtained.

---

* In other words, dV_x is the principal linear part of the increment of the variable volume V_x as the cross-section $S(x)$ displaces by an infinitely small quantity dx.

Let us denote the volume of a solid of revolution by V_x; then by (130-1) we have

$$\frac{1}{2} V_x = \pi \int_0^a y^2 \, dx,$$

where 0 and a are the abscissas of the points B and A. From the equation of ellipse we find

$$y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right).$$

Whence

$$\frac{1}{2} V_x = \pi \int_0^a b^2 \left(1 - \frac{x^2}{a^2}\right) dx = \pi b^2 \int_0^a dx - \frac{\pi b^2}{a^2} \int_0^a x^2 \, dx =$$

$$= \pi b^2 \cdot x \Big|_0^a - \frac{\pi b^2}{a^2} \cdot \frac{x^3}{3} \Big|_0^a = \pi a b^2 - \frac{\pi b^2}{a^2} \cdot \frac{a^3}{3} = \frac{2}{3} \pi a b^2.$$

Consequently, we finally have

$$V_x = \frac{4}{3} \pi a b^2.$$

Analogously, when ellipse (130-3) is rotated about the minor axis

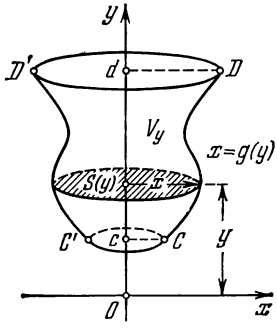

FIG. 155

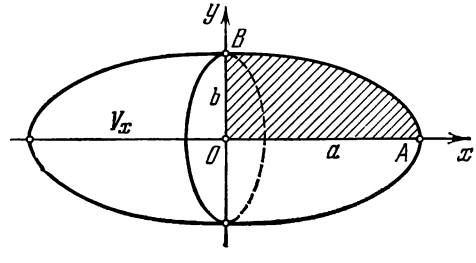

FIG. 156

b, the volume of the solid of revolution thus obtained is equal to

$$V_y = \frac{4}{3} \pi a^2 b.$$

Setting $a = b$, we get the volume of a sphere of radius a:

$$V = \frac{4}{3} \pi a^3.$$

Sec. 131. The Work of a Variable Force

Problem. Find the work A of a continuous variable force $F(x)$ applied to a material point M when the latter is displaced along the axis Ox from the position $x = a$ to the position $x = b$, assuming that the direction of the force coincides with the direction of displacement.

Suppose the point M is moved from the position x to the position $x + dx$ (Fig. 157). On the infinitesimal interval $[x, x + dx]$ of length

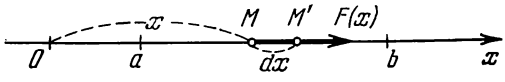

FIG. 157

dx the force $F(x)$ may be approximately regarded as constant. Therefore, the *element of work performed by the given force* is equal to

$$dA = F(x)\,dx. \tag{131-1}$$

Integrating expression (131-1) between the limits $x = a$ and $x = b$, we obtain the total work

$$A = \int_a^b F(x)\,dx. \tag{131-2}$$

Example. What amount of work should be spent to stretch a spring by 5 cm if a force of 100 N expands the spring by 1 cm?

According to the Hooke law, the elastic force F acting on the spring increases proportionally to the expansion x of the spring, i.e.

$$F = kx.$$

Here the displacement x is expressed in metres, and the force F in newtons. To determine the proportionality factor k, according to the initial condition of the problem, we set $F = 100$ N for $x = 0.01$ m. Hence

$$100 = k \cdot 0.01,$$

i.e. $k = 10{,}000$ and, consequently,

$$F = 10{,}000x.$$

The required work, by formula (131-2), is equal to

$$A = \int_0^{0.05} 10{,}000x\,dx = 5000x^2 \Big|_0^{0.05} = 12.5 \text{ J}.$$

Sec. 132. Other Applications of the Definite Integral in Physics

To illustrate the two principal methods used in the theory of the definite integral (1—the method of differential and 2—the method of integral sums), let us consider a few examples.

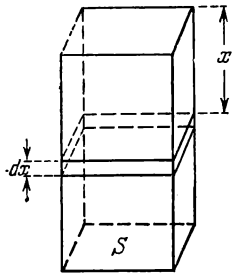

FIG. 158

Example 1. The concentration of a substance in water changes according to the law

$$C = \frac{10x}{x+1} \frac{\text{g}}{\text{m}^3}, \tag{132-1}$$

where x is the depth of the layer.

What amount of substance Q is contained in a vertical water column the area of whose cross-section $S = 1$ m^2 and the depth changes from 0 to 200 m?

Let us consider an infinitely thin layer of the water column with the cross-section S and thickness dx found at the depth x (Fig. 158). The amount of the substance contained in this layer is equal to

$$dQ = C \cdot S\, dx = \frac{10x}{x+1} dx. \tag{132-2}$$

Integrating this expression within the limits from 0 to 200, we obtain

$$Q = 10 \int_0^{200} \frac{x\, dx}{x+1} = 10 \int_0^{200} \frac{(x+1)-1}{x+1} dx =$$

$$= 10\,[x - \ln(x+1)]\Big|_0^{200} = 10\,[200 - \ln 201] = 10 \cdot (200 - 5.3) = 1947 \text{ g}.$$

Example 2. With what force does a uniform rod $0 \leqslant x \leqslant l$ of line density δ attract a material point P (a) $(a > l)$ of mass m (Fig. 159)?

According to Newton's law, an infinitely small element of the rod $[x, x + dx]$ of mass $\delta\, dx$ attracts the material point P with a force whose magnitude is equal to

$$dF = -k \frac{m\delta\, dx}{(a-x)^2}, \tag{132-3}$$

where k is a proportionality factor (the gravitational constant). Since these attractive forces act in one and the same direction, their magnitudes dF may be algebraically added and, consequently, in-

tegrated (since integral is the limit of an algebraic sum):

$$F = -km\delta \int_0^l \frac{dx}{(a-x)^2} = -km\delta \cdot \frac{1}{a-x}\Big|_0^l =$$

$$= -km\delta \left(\frac{1}{a-l} - \frac{1}{a}\right) = -\frac{km\delta l}{a(a-l)}.$$

Example 3. Determine the force with which water presses on a vertical circle of radius R whose centre is submerged into water to a depth H $(H > 2R)$.

We take a vertical line with the origin O coinciding with the centre of the circle as the x-axis (Fig. 160). Then we partition the given circle into n narrow horizontal strips of thickness Δx_i $(i = 1, 2, \ldots, n)$.

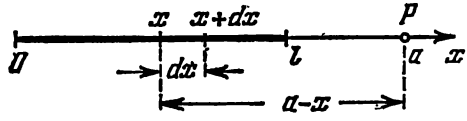

FIG. 159

Consider the ith strip $(AA'B'B)$ Δx_i thick situated at a distance x_i from the centre of the circle (Fig. 157). If Δx_i is a small quantity, then this strip may be approximately taken for a rectangle, and therefore its area

$$\Delta S_i \approx AB \cdot \Delta x_i =$$

$$= 2\sqrt{R^2 - x_i^2}\,\Delta x_i.$$

Assuming that the submersion level of the strip is equal to $H - x_i$, by the Pascal law, we obtain the magnitude of the pressure force exerted by the water upon this strip

$$\gamma (H - x_i)\,\Delta S_i =$$

$$= 2\gamma (H - x_i)\sqrt{R^2 - x_i^2}\,\Delta x_i,$$

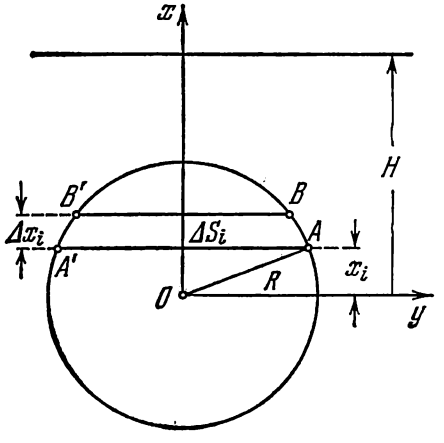

FIG. 160

where γ is the specific weight of water.

Summing up these expressions, we get the approximate value of the pressure force P exerted by the water on the entire plate

$$P \approx \sum_{i=1}^{n} 2\gamma (H - x_i)\sqrt{R^2 - x_i^2}\,\Delta x_i. \qquad (132\text{-}4)$$

The smaller Δx_i, the more precise is this formula. In the limit, as $n \to \infty$ and $\max \Delta x_i \to 0$, we shall obtain a precise formula for determining the water pressure

$$P = \lim_{n\to\infty} \sum_{i=1}^{n} 2\gamma (H - x_i) \sqrt{R^2 - x_i^2}\, \Delta x_i. \qquad (132\text{-}5)$$

(132-4) is an integral sum for the function

$$f(x) = 2\gamma (H - x) \sqrt{R^2 - x^2};$$

therefore its limit is the corresponding definite integral. Hence, from (132-5) we find

$$P = 2\gamma \int_{-R}^{R} (H - x) \sqrt{R^2 - x^2}\, dx = \pi\gamma R^2 H.$$

Exercises

In Problems 1 to 10 find the area of the figure bounded by the indicated lines:

1. Parabolas $y = x^2$ and $y^2 = x$.
2. Hyperbola $xy = a^2$ and straight lines $y = 0$, $x = b$, and $x = 2b$ $(b > 0)$.
3. Parabola $y = 2x - x^2$ and the x-axis.
4. Witch of Agnesi $y = \dfrac{a^3}{a^2 + x^2}$ $(a > 0)$ and the x-axis.
5. Curve $y = e^x$ and straight lines $x = 0$ and $y = e$.
6. Parabola $y = 2(x - 1)(3 - x)$ and the x-axis.
7. Parabola $y^2 = 2(x - 1)$ and a straight line $x = 3$.
8. Curve $y = 2^x$ and straight lines $y = 2$ and $x = 0$.
9. Parabolas $y = 1 - x^2$ and $y = x^2 - 7$.
10. Astroid $x = a\cos^3 t$, $y = a \sin^3 t$ $(0 \leqslant t \leqslant 2\pi)$.
11. Find the arc length of the semicubical parabola

$$y = \frac{2}{3} x^{\frac{3}{2}}$$

from the point $x_1 = 0$ to the point $x_2 = 8$.

12. Find the arc length of the curve

$$y = \frac{x^2}{4} - \frac{1}{2} \ln x$$

from the point $x_1 = 1$ to the point $x_2 = e$.

13. Find the arc length of one arc of the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t) \qquad (0 \leqslant t \leqslant 2\pi).$$

14. Construct the lemniscate

$$\rho^2 = a^2 \cos 2\varphi$$

and find the area of the entire region bounded by this line.

15. Construct the three-leafed rose $\rho = a \sin 3\varphi$ and find the area of the region bounded by this line.

16. Compute the arc length of the logarithmic spiral

$$\rho = ae^{m\varphi} \qquad (a > 0, \quad m > 0),$$

contained inside the circle

$$\rho = a.$$

17. Construct the curve

$$\rho = 2a \sin^2 \frac{\varphi}{2}$$

and find its arc length.

18. Find the volume of the obelisk whose bases are rectangles with sides A, B and a, b, and the altitude is equal to h.

19. By means of integration, find the volume of a ball of radius R.

In Problems 20 to 23 find the volume of the solid obtained by revolving about the x-axis a region bounded by the following lines:

20. $y = 2x - x^2$, $y = 0$.

21. One half-wave of the sinusoid $y = \sin x$ and $y = 0$.

22. $y = \sec x$, $y = 0$, $x = \pm \frac{\pi}{4}$.

23. $y = x^2$, $y = 2x$.

In Problems 24 and 25 find the volume of the solid generated by rotating about the y-axis a region bounded by the following lines:

24. $y = h\left(\frac{x}{a}\right)^2$, $y = h$ (paraboloid of revolution)

25. $y = e^{-x}$, $y = 0$ and $x = 0$.

26. The velocity of a point changes with time t according to the law

$$v = 0.01t^3 \text{ m/s}.$$

What distance will be covered by the point during 10 seconds? What is the average velocity of motion?

27. Specific heat of a body at a temperature $t°$ is equal to

$$c = 0.2 + 0.001t.$$

What amount of heat is required to warm 1g of the body from 0°C to 100°C?

28. Line density of a rod $0 \leqslant x \leqslant l$ of length l is equal to

$$q = q_0 \sin \frac{\pi x}{l},$$

where q_0 is a constant. Find the mass of the rod.

29. A cylinder of diameter 20 cm and length 80 cm is filled with vapour at a pressure of 100 N/cm^2. What work is required to reduce the volume of the vapour by half with temperature unchanged?

30. Find the pressure force exerted by water upon a vertical semicircular shield of radius a whose diameter lies on the water surface.

Chapter 16

Complex Numbers

Sec. 133. Arithmetic Operations on Complex Numbers

As is known, a *complex number* is defined as an expression of the form

$$z = x + iy \equiv x + yi \tag{133-1}$$

where x and y are any real numbers and i is the so-called imaginary unit ($i^2 = -1$).

The numbers of the form $x + i0 = x$ are identified with real numbers; in particular, $0 + i0 = 0$. The numbers of the form $0 + iy = iy$ are called *pure imaginary numbers.*

The real numbers x and y are called the *real* and the *imaginary part*, respectively, of the number z and are denoted in the following way

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z. \tag{133-2}$$

The *modulus* of the complex number z is understood as a non-negative number

$$|z| = [(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2]^{\frac{1}{2}} = \sqrt{x^2 + y^2} \geqslant 0. \tag{133-3}$$

The *conjugate number* $\bar{z}$ to number (133-1) is defined as the number

$$\bar{z} = x + i(-y) \equiv x - iy. \tag{133-4}$$

Thus,

$$\operatorname{Re} \bar{z} = \operatorname{Re} z, \quad \operatorname{Im} \bar{z} = -\operatorname{Im} z \tag{133-5}$$

and

$$|\bar{z}| = |z|. \tag{133-6}$$

On the set of complex numbers the equality of two numbers, as well as the operations of addition, subtraction, multiplication, and division, are defined in the following way.

I. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$z_1 = z_2 \Leftrightarrow \operatorname{Re} z_1 = \operatorname{Re} z_2, \quad \operatorname{Im} z_1 = \operatorname{Im} z_2.$$

In particular, $z = 0 \Leftrightarrow \operatorname{Re} z = 0, \ \operatorname{Im} z = 0$.

II. $z_1 \pm z_2 = (x_1 \pm x_2) + i\,(y_1 \pm y_2)$.
Hence it follows that

$$\operatorname{Re}(z_1 \pm z_2) = \operatorname{Re} z_1 \pm \operatorname{Re} z_2$$

and

$$\operatorname{Im}(z_1 \pm z_2) = \operatorname{Im} z_1 \pm \operatorname{Im} z_2.$$

III. $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)$.
Hence, in particular, we obtain an important relation

$$i^2 = (0 + i1)(0 + i1) = (0 - 1) + i\,(0 + 0) = -1. \qquad (133\text{-}7)$$

Note that this rule is obtained formally by multiplying the binomials $x_1 + iy_1$ and $x_2 + iy_2$, taking into account (133-7).

It is also obvious that for $z = x + iy$ and $\bar{z} = x - iy$ we have:

$$z\bar{z} = |z|^2 = x^2 + y^2.$$

IV. $$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{(x_1x_2 + y_1y_2) + i\,(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \qquad (z_2 \neq 0).$$

We can easily check the following properties:

(1) $\overline{(\bar{z})} = z$; (2) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$;

(3) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$; (4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ $(z_2 \neq 0)$;

(5) $\operatorname{Re} z = \frac{z + \bar{z}}{2}$, $\operatorname{Im} z = \frac{z - \bar{z}}{2i}$.

Sec. 134. The Complex Plane

Consider a plane with a rectangular coordinate system Oxy. Each complex number $z = x + iy$ can be represented by a point z in the xy-plane (Fig. 161), this correspondence being one-to-one. The plane on which such correspondence is established is called the *complex plane*, and instead of complex numbers we speak of *points of the complex plane*.

Located on the x-axis are real numbers: $z = x + 0i = x$; therefore it is called the *real axis* (or the *axis of reals*). Pure imaginary numbers $z = 0 + iy = iy$ lie on the y-axis, the latter being called the *imaginary axis*.

Note that $r = |z|$ represents the distance of the point z from the origin.

Each point z is associated with its radius vector $\overrightarrow{Oz}$. The angle formed by the radius vector of the point z with the x-axis is called the *argument* $\varphi = \operatorname{Arg} z$ of this point. Here $-\infty < \operatorname{Arg} z < +\infty$. For the point $z = 0$ the argument is arbitrary. The least (by modulus) value of $\operatorname{Arg} z$ is called its *principal value* and is denoted as $\arg z$:

$$-\pi < \arg z \leqslant \pi. \tag{134-1}$$

For the argument φ we have (Fig. 161)

$$\cos\varphi = \frac{x}{r}, \quad \sin\varphi = \frac{y}{r}, \quad \tan\varphi = \frac{y}{x}, \tag{134-2}$$

where $r = \sqrt{x^2 + y^2}$.

Example. (1) $\arg 2 = 0$; (2) $\arg(-1) = \pi$; (3) $\arg i = \frac{\pi}{2}$.

The modulus r and the argument φ of a complex number z can be regarded (see Fig. 161) as the polar coordinates of the point z. Hence we get

$$x = r\cos\varphi, \quad y = r\sin\varphi. \tag{134-3}$$

Thus, we have the trigonometric form of a complex number

$$z = x + iy = r\cos\varphi + ir\sin\varphi = r(\cos\varphi + i\sin\varphi), \tag{134-4}$$

where $r = |z|$, $\varphi = \operatorname{Arg} z$.

Theorem 1. *As complex numbers are added, their radius vectors are added (according to the rule of parallelogram).*

Indeed, if the number $z_1 = x_1 + iy_1$ corresponds to the point with the coordinates (x_1, y_1), and the number $z_2 = x_2 + iy_2$ to the point

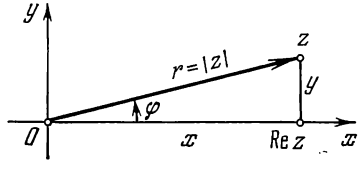

FIG. 161

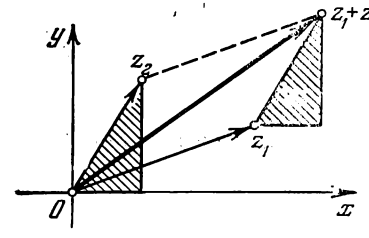

FIG. 162

with the coordinates (x_2, y_2), then to the number $z_1 + z_2$ there corresponds the point $(x_1 + x_2, y_1 + y_2)$. Since the hatched right triangles with the legs x_2 and y_2 are congruent (Fig. 162), the quadrilateral with the vertices 0, z_1, $z_1 + z_2$, z_2 is a parallelogram. Consequently, the radius vector of the point $z_1 + z_2$ is the sum of the radius vectors of the points z_1 and z_2 (cf. Sec. 146, Chapter 18).

Corollary. *Since z is the length of the vector $\overrightarrow{Oz}$,* we have

$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

Theorem 2. *As complex numbers are subtracted, their radius vectors are subtracted.*

Since $z_1 - z_2 = z_1 + (-z_2)$, the difference $z_1 - z_2$ is equal to the second diagonal of the parallelogram constructed on the vectors $\overrightarrow{Oz}$ and $\overrightarrow{Oz'}$ (Fig. 163), i.e. it is equal to the difference of the radius vectors of the points z_1 and z_2 (cf. Sec. 147).

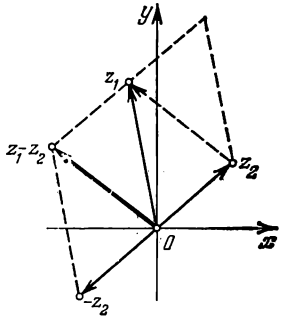

FIG. 163

Corollary. *The distance between the two points z and z' is equal to*

$$\rho(z', z) = |z' - z|.$$

Sec. 135. Theorems on the Modulus and Argument

Theorem 1. *The modulus of a product of complex numbers is equal to the product of the moduli of these numbers while the argument of the product is equal to the sum of the arguments of the cofactors.*

Indeed, if

$$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1), \quad z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2),$$

then we have

$$z_1z_2 = r_1r_2[(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + i(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2)] = r_1r_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)].$$

Whence

$$|z_1z_2| = r_1r_2 = |z_1| \cdot |z_2|$$

and

$$\operatorname{Arg} z_1z_2 = \varphi_1 + \varphi_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2, \tag{135-1}$$

where the values of many-valued functions Arg standing in the left- and right-hand sides of equality (135-1) should be chosen in a proper way. This remark should be taken into consideration for further reasoning as well.

Corollary. *The modulus of an integral power of a complex number is equal to the same power of the modulus of this number, while the argument of the power is equal to the argument of the number multiplied by*

the exponent, i.e.

$$|z^n| = |z|^n, \quad \operatorname{Arg} z^n = n \operatorname{Arg} z$$

(n a positive integer).

The proof follows directly from considering a product of equal cofactors.

Example. Construct the point $z' = iz$. We have

$$|z'| = |i| \cdot |z| = |z|,$$

$$\operatorname{Arg} z' = \operatorname{Arg} iz = \arg i + \arg z = \frac{\pi}{2} + \arg z.$$

Consequently, when multiplied by i, the vector $\overrightarrow{Oz}$ is turned by a right angle anticlockwise (Fig. 164).

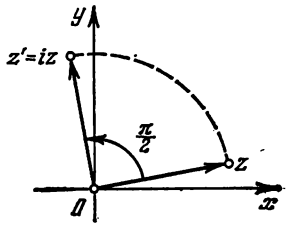

FIG. 164

Theorem 2. *The modulus of a quotient of two complex numbers is equal to the quotient of the moduli of these numbers, while the argument of the quotient is equal to the difference of the arguments of the divisor and dividend.*

Let

$$z_1 = r_1 (\cos \varphi_1 + i \sin \varphi_1), \quad z_2 = r_2 (\cos \varphi_2 + i \sin \varphi_2) \neq 0.$$

Since

$$\bar{z}_2 = r_2 (\cos \varphi_2 - i \sin \varphi_2) = r_2 [\cos (-\varphi_2) + i \sin (-\varphi_2)],$$

by Theorem 1 we have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \frac{\cos \varphi_1 + i \sin \varphi_1}{\cos \varphi_2 + i \sin \varphi_2} =$$

$$= \frac{r_1}{r_2} \cdot \frac{(\cos \varphi_1 + i \sin \varphi_1)[\cos (-\varphi_2) + i \sin (-\varphi_2)]}{(\cos \varphi_2 + i \sin \varphi_2)(\cos \varphi_2 - i \sin \varphi_2)} =$$

$$= \frac{r_1}{r_2} [\cos (\varphi_1 - \varphi_2) + i \sin (\varphi_1 - \varphi_2)] \quad (r_2 \neq 0).$$

Hence

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|},$$

$$\operatorname{Arg} \frac{z_1}{z_2} = \varphi_1 - \varphi_2 = \operatorname{Arg} z_1 - \operatorname{Arg} z_2.$$

Sec. 136. Taking the Root from a Complex Number

Let

$$\sqrt[n]{z} = \rho (\cos \psi + i \sin \psi), \tag{136-1}$$

where $z = r(\cos\varphi + i\sin\varphi)$. Then, on the basis of the preceding section, we have

$$z = [\rho(\cos\psi + i\sin\psi)]^n = \rho^n(\cos n\psi + i\sin n\psi). \qquad (136\text{-}2)$$

Hence we obtain

$$\rho^n = r, \quad n\psi = \varphi + 2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots). \qquad (136\text{-}3)$$

Thus,

$$\rho = \sqrt[n]{r} = \sqrt[n]{|z|}$$

and

$$\psi = \operatorname{Arg}\sqrt[n]{z} = \frac{\varphi + 2k\pi}{n} = \frac{\arg z + 2k\pi}{n}.$$

Note that here $\sqrt[n]{r}$ is understood as the arithmetic value of the root.

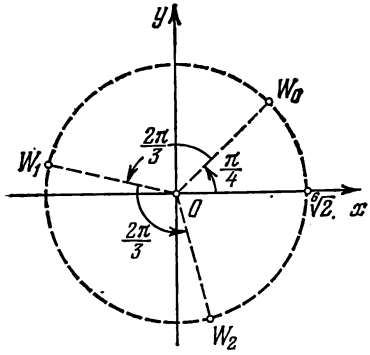

FIG. 165

Here it is sufficient to take for the number k only the values $k = 0, 1, 2, \ldots, n-1$, since all other values of k yield already found values of the root.

Hence, finally we have

$$\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\frac{\arg z + 2k\pi}{n} + i\sin\frac{\arg z + 2k\pi}{n}\right)$$

$$(k = 0, 1, 2, \ldots, n-1). \qquad (136\text{-}4)$$

It follows from (136-4) that the nth root of any complex number $z \neq 0$ has exactly n values.

Example. Find $W = \sqrt[3]{-1+i}$. Since

$$-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right),$$

by (136-4) we have

$$W=\sqrt[3]{\sqrt{2}}\left(\cos\frac{\frac{3\pi}{4}+2k\pi}{3}+i\sin\frac{\frac{3\pi}{4}+2k\pi}{3}\right)=$$

$$=\sqrt[6]{2}\left[\cos\left(\frac{\pi}{4}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{2k\pi}{3}\right)\right] \qquad (k=0,\ 1,\ 2).$$

Hence

$$W_0=\sqrt[6]{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right),$$

$$W_1=\sqrt[6]{2}\left[\cos\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)\right]=$$

$$=\sqrt[6]{2}\left(\cos\frac{11}{12}\pi+i\sin\frac{11}{12}\pi\right),$$

$$W_2=\sqrt[6]{2}\left[\cos\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)+i\sin\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)\right]=$$

$$=\sqrt[6]{2}\left(\cos\frac{19}{12}\pi+i\sin\frac{19}{12}\pi\right).$$

The points W_0, W_1, W_2 represent three equidistant points lying on a circle of radius $\sqrt[6]{2}$ (Fig. 165).

Sec. 137. The Concept of a Function of a Complex Variable

Suppose we are given two complex planes Oxy (plane z) and $O'uv$ (plane w).

Definition. *If, according to a certain law f, each point $z \in E$ (where E is a set of points of the plane z) is associated with a unique point $w \in E'$ (E' a set of points belonging to the plane w), then w is said to be a single-valued function of z:*

$$w = f(z) \qquad (137\text{-}1)$$

with the domain of definition E whose values belong to the set E'

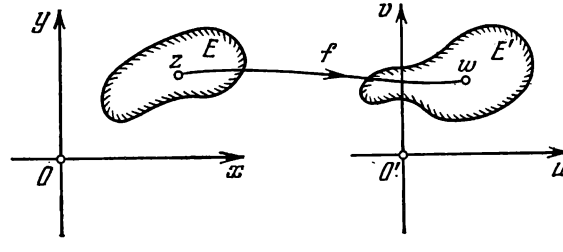

FIG. 166

(Fig. 166). If the set of values of the function $f(z)$ exhausts the entire set E', then E' is called the *set of values* (the *domain of variation*) of

the function $f(z)$. In this case we write

$$E' = f(E). \tag{137-2}$$

The sets E and E' can be represented on one complex plane.

Thus, *each complex function realizes a single-valued (in one direction) mapping of one set onto the other.* Due to this fact, complex functions

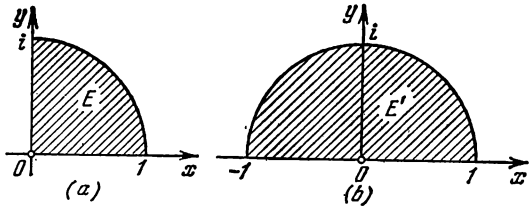

FIG. 167

find their applications in such fields of science as hydrodynamics and aerodynamics. With their aid it is convenient to describe a "history" of motion of a certain volume of liquid (or gas).

The branch of mathematics studying the properties of complex functions is named the *theory of functions of a complex variable.*

Example. Into what does the sector E

$$0 < \arg z < \frac{\pi}{2}, \qquad 0 < |z| < 1$$

(Fig. 167, *a*) go in the mapping $w = z^2$?

We have

$$\arg w = 2 \arg z < \pi$$

and

$$|w| = |z|^2 < 1.$$

Therefore, the mapped domain E' represents a semi-circle (Fig. 167, *b*).

Exercises

1. Find

$$z = \frac{(1+i)^{100}}{(\sqrt{3}-i)^{50}}.$$

2. Construct the following domains:

(a) $0 < \operatorname{Re} z < 1$; (b) $\operatorname{Im} z > 2$;

(c) $0 < \arg z < \frac{\pi}{4}$; (d) $|z| < 1$.

3. Into what does the square with the vertices 0, 1, $1 + i$, i belonging to the plane z go in the mapping $w = z^2$?

Chapter 17

Determinants of Second and Third Order

Sec. 138. Second-Order Determinants

A *determinant* of the second order is a square array of type

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \tag{138-1}$$

(see also Sec. 5, Chapter 1).

The numbers a_1, b_1, a_2, b_2 are called *elements* of the determinant; they are arranged in two rows and two columns. In our further reasoning we shall always assume that all the quantities under consideration are real.

Formula (138-1) gives the rule for "expanding" a determinant of the second order, namely: *a determinant of the second order is equal to the difference of products of its elements of the first and second diagonals.*

Second-order determinants are useful for solving simultaneous linear systems of two equations in two unknowns:

$$\left.\begin{aligned} a_1 x + b_1 y &= c_1, \\ a_2 x + b_2 y &= c_2. \end{aligned}\right\} \tag{138-2}$$

Such a linear system with the constant terms in the right-hand side we shall call, for definiteness, a *standard system.*

Any pair of numbers (x, y) reducing this system to an identity will be called the *solution* of simultaneous system (138-2). If there exists only one such pair, then the solution is said to be *unique.* The concept of solution of a simultaneous system in n unknowns ($n = 1, 2, 3, \ldots$) is introduced analogously.

To find the solutions of system (138-2), we use the following method. Multiplying the first equation of the system by b_2 and the second by $-b_1$, and then adding, we shall have

$$(a_1 b_2 - a_2 b_1)\, x = c_1 b_2 - c_2 b_1. \tag{138-3}$$

Analogously, multiplying the first equation of system (138-2) by $-a_2$ and the second by a_1, and adding, we shall obtain

$$(a_1b_2 - a_2b_1)\,y = a_1c_2 - a_2c_1. \tag{138-4}$$

Let us introduce the *determinant of the system*

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

and also the additional determinants

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Note that the additional determinants D_x and D_y are obtained from the determinant of the system D by replacing the coefficients of the indicated unknown by the corresponding constant terms.

Equations (138-3) and (138-4) attain the form

$$Dx = D_x, \quad Dy = D_y. \tag{138-5}$$

If $D \neq 0$, then system (138-2) has a unique solution

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D} \tag{138-6}$$

(*Cramer's formulas*). The fact that system of numbers (138-6) is the solution of system (138-2) can be verified by substituting them into system (138-2).

Note. If the determinant $D = 0$, then system (138-2) has either no solution (it is incompatible), or an infinite number of solutions (i.e. the system is indefinite).

Example. Solve the system

$$\left.\begin{aligned} 7x - 6y &= 5, \\ 8x - 7y &= -10. \end{aligned}\right\} \tag{138-7}$$

We have

$$D = \begin{vmatrix} 7 & -6 \\ 8 & -7 \end{vmatrix} = -49 + 48 = -1,$$

$$D_x = \begin{vmatrix} 5 & -6 \\ -10 & -7 \end{vmatrix} = -35 - 60 = -95,$$

$$D_y = \begin{vmatrix} 7 & 5 \\ 8 & -10 \end{vmatrix} = -70 - 40 = -110.$$

Hence, by Cramer's formulas, we obtain

$$x=\frac{D_x}{D}=\frac{-95}{-1}=95,$$

$$y=\frac{D_y}{D}=\frac{-110}{-1}=110.$$

Geometrically, the solution (95, 110) represents the point of intersection of straight lines (138-7).

Sec. 139. A System of Two Homogeneous Equations in Three Unknowns

Consider the following homogeneous system

$$\left.\begin{aligned} a_1x+b_1y+c_1z=0,\\ a_2x+b_2y+c_2z=0. \end{aligned}\right\} \tag{139-1}$$

This system is always compatible, since, obviously, it has a zero solution: $x = 0$, $y = 0$, $z = 0$. But we are interested in finding *non-zero* solutions (x, y, z) of system (139-1). Let, for instance, $z \neq 0$. Then we can rewrite our system as follows

$$\left.\begin{aligned} a_1\frac{x}{z}+b_1\frac{y}{z}=-c_1,\\ a_2\frac{x}{z}+b_2\frac{y}{z}=-c_2. \end{aligned}\right\} \tag{139-2}$$

Hence, assuming that

$$D=\begin{vmatrix} a_1 & b_1\\ a_2 & b_2 \end{vmatrix}\neq 0,$$

we obtain

$$\frac{x}{z}=\frac{1}{D}\begin{vmatrix} -c_1 & b_1\\ -c_2 & b_2 \end{vmatrix}=\frac{1}{D}\begin{vmatrix} b_1 & c_1\\ b_2 & c_2 \end{vmatrix}, \tag{139-3}$$

$$\frac{y}{z}=\frac{1}{D}\begin{vmatrix} a_1 & -c_1\\ a_2 & -c_2 \end{vmatrix}=-\frac{1}{D}\begin{vmatrix} a_1 & c_1\\ a_2 & c_2 \end{vmatrix}. \tag{139-4}$$

Let us introduce into consideration the *matrix of the coefficients* of system (139-1)

$$\left\|\begin{matrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2 \end{matrix}\right\|. \tag{139-5}$$

The second-order determinants D_1, D_2, and D_3 which are obtained from this matrix by deleting the corresponding column are called its

minors. Thus, we have

$$D_1 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = D.$$

Using these notations we may rewrite equations (139-3) and (139-4) in the following form:

$$\frac{x}{z} = \frac{D_1}{D_3}, \quad \frac{y}{z} = -\frac{D_2}{D_3}.$$

Whence we get

$$\frac{x}{D_1} = \frac{y}{-D_2} = \frac{z}{D_3}. \tag{139-6}$$

Equalities (139-6) are obviously valid for a zero solution as well.

And so, we have the following rule: *The unknowns of a homogeneous system* (139-1) *are proportional to the corresponding minors of the matrix of its coefficients taken with proper signs.*

Denoting by t the proportionality factor for ratios (139-6), we get the complete system of solutions of system (139-1):

$$x = D_1 t, \quad y = -D_2 t, \quad z = D_3 t \qquad (-\infty < t < +\infty). \tag{139-7}$$

In deriving the above formulas, we proceed from the assumption that $D = D_3 \neq 0$. But, as it is easy to get convinced, formulas (139-7) will be true if any (at least one) of the minors D_1, D_2, D_3 is non-zero.

Note. If all the minors (D_1, D_2, D_3) are equal to zero, then system (139-1) requires a special consideration.

Example. Solve the system

$$\left.\begin{aligned} x - 2y + 3z &= 0, \\ 4x + 5y - 6z &= 0. \end{aligned}\right\} \tag{139-8}$$

Compiling the matrix of the coefficients

$$\begin{Vmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \end{Vmatrix},$$

we find its minors

$$D_1 = \begin{vmatrix} -2 & 3 \\ 5 & -6 \end{vmatrix} = 12 - 15 = -3,$$

$$D_2 = \begin{vmatrix} 1 & 3 \\ 4 & -6 \end{vmatrix} = -6 - 12 = -18,$$

$$D_3 = \begin{vmatrix} 1 & -2 \\ 4 & 5 \end{vmatrix} = 5 + 8 = 13.$$

On the basis of formula (139-7), the complete set of solutions of system (139-8) has the form

$$x = -3t, \quad y = +18t, \quad z = 13t,$$

where $-\infty < t < +\infty$.

The simplest non-zero solution of system (139-1) obtained for $t = 1$ is: $x = -3$, $y = 18$, $z = 13$.

Sec. 140. Third-Order Determinants

Definition 1. *The expression*

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \quad (140\text{-}1)$$

is called the determinant of the third order.

The numbers a_i, b_i, c_i ($i = 1, 2, 3$) are called *elements of the determinant*. They are arranged in three rows and three columns.

Expanding the second-order determinants (minors) in formula (140-1) and collecting terms with the same sign, we see that a determinant of the third order represents an alternating sum of six terms

$$D = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1, \quad (140\text{-}2)$$

of which three are taken with plus, and three with minus.

Example. Compute

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}. \quad (140\text{-}3)$$

Using formula (140-1), we have

$$D = 1 \cdot \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} =$$

$$= 1 \cdot (-1) - 2 \cdot (-2) + 3 \cdot (-1) = 0.$$

Later on we shall introduce more convenient methods for evaluating determinants of the third order.

Definition 2. *A minor of any element of a determinant of the third order is the determinant of the second order obtained from the given determinant by deleting the row and column in which the element stands.*

For example, the minor of element 2 of determinant (140-3) (this element is contained in its second row and first column) is the determinant

$$M=\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}=-2.$$

Henceforward, for brevity, we shall state that an element of a determinant of the third order occupies an *even place* if the sum of the position numbers of the row and the column in which the element stands is an even number, and an *odd place* if this sum is an odd number.

Definition 3. *The cofactor (or the signed minor, or the algebraic adjunct) of an element of a determinant of the third order is the minor of this element taken with the plus sign if the element occupies an even place, and with the minus sign if its place is odd.*

Thus, if M denotes the minor of an element of a determinant, and i and j the number of row and the number of column, respectively, at the intersection of which the given element stands, then its cofactor is

$$A=(-1)^{i+j}M.$$

For instance, for the element c_2 of determinant (140-1) contained in the second row and in the third column its cofactor is

$$C_2=(-1)^{2+3}\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}=-\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}.$$

The signs prefixed in this event to the minor of elements of determinants can be specified by the following table

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

Let us agree to denote the cofactors of elements of a determinant whose elements are expressed by letters by the corresponding capital letters.

Expansion theorem. *A determinant of the third order is equal to the sum of the products of the elements of some row by their cofactors.*

Thus, for determinant (140-1) the following six expansions are valid:

$$\left.\begin{aligned} D&=a_1A_1+b_1B_1+c_1C_1, \\ D&=a_2A_2+b_2B_2+c_2C_2, \\ D&=a_3A_3+b_3B_3+c_3C_3 \end{aligned}\right\} \qquad (140\text{-}4)$$

and

$$\left.\begin{aligned} D &= a_1A_1 + a_2A_2 + a_3A_3, \\ D &= b_1B_1 + b_2B_2 + b_3B_3, \\ D &= c_1C_1 + c_2C_2 + c_3C_3. \end{aligned}\right\} \tag{140-5}$$

It is easy to check that formulas (140-4) and (140-5) yield one and the same expression (140-2) taken for the definition.

Note. With the aid of formulas of type (140-4) or (140-5) it is possible to introduce determinants of higher orders.

Sec. 141. Basic Properties of Determinants

In formulating the properties we shall not indicate the order of determinants, since they are valid for determinants of any order.

I. *The magnitude of a determinant does not change if each of the rows is substituted by a column of the same position number*, i.e.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{141-1}$$

Indeed, expanding the first determinant in terms of the elements of the first row, and the second in terms of the elements of the first column, by the expansion theorem (Sec. 140), we shall obtain one and the same result.

II. *If any two rows or any two columns are interchanged, the absolute value of a determinant remains unaltered, while the sign is reversed.*

Let, for instance, in the determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{141-2}$$

the first and second rows be interchanged: then we shall have the determinant

$$\widetilde{D} = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{141-2'}$$

Expanding the determinant $\widetilde{D}$ in terms of the elements of the second row and taking into account that the parity of positions of these elements has changed due to the interchange of the rows, we

shall have:

$$\widetilde{D} = a_1(-A_1) + b_1(-B_1) + c_1(-C_1) = -D.$$

Similar results are obtained in other cases.

Corollary 1. *A determinant having two identical rows or columns is equal to zero.*

Indeed, let, for instance,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Interchanging the first and the second rows of the determinant, due to the theorem, we get the determinant $-D$. But obviously, this operation does not change the determinant D; therefore $-D = D$ and, hence, $D = 0$.

Corollary 2. *The sum of the products of the elements of some row or column of a determinant by the cofactors of the corresponding elements of a parallel row or column is equal to zero*, i.e. for determinant (141-2) we have

$$\left.\begin{aligned} a_1A_2 + b_1B_2 + c_1C_2 = 0, \\ a_1A_3 + b_1B_3 + c_1C_3 = 0 \end{aligned}\right\} \tag{141-3}$$

and so on, as also

$$\left.\begin{aligned} a_1B_1 + a_2B_2 + a_3B_3 = 0, \\ a_1C_1 + a_2C_2 + a_3C_3 = 0 \end{aligned}\right\} \tag{141-4}$$

and so on (all in all we can write twelve such relations).

The left-hand members of all relations of (141-3) and (141-4) represent the expansions of the corresponding determinants of the third order containing two identical rows, and, hence, are equal to zero. For example,

$$a_1A_2 + b_1B_2 + c_1C_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(here the expansion should be carried out in the second row!).

III. *A common factor of all the elements of one row (or of one column) may be taken outside the sign of the determinant*, i.e.

$$\begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and so on.

This property follows directly from the expansion of a determinant in terms of elements of the corresponding row.

Corollary 1. *If all the elements of some row of a determinant are equal to zero, then the determinant is equal to zero.*

Corollary 2. *If the elements of some row of a determinant are proportional to the corresponding elements of its parallel row, then the determinant is equal to zero.* The same holds true for columns.

For instance, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

and so on.

IV. *If every element of some column (row) is the sum of two terms, then the determinant is equal to the sum of two determinants: one containing only the first term in place of each sum, the other only the second term* (the remaining elements of both determinants are the same as in the given determinant).

For instance, we have

$$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (a_1+\alpha_1) A_1 + (b_1+\beta_1) B_1 +$$

$$+ (c_1+\gamma_1) C_1 = (a_1A_1 + b_1B_1 + c_1C_1) + (\alpha_1A_1 + \beta_1B_1 + \gamma_1C_1) =$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and so on.

Corollary. *The numerical value of a determinant remains unchanged if to all the elements of some row we add (or subtract) terms proportional to the corresponding elements of a parallel row with one and the same proportionality factor* (so-called "elementary transformations of a determinant").

Indeed, let

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Let us consider, for instance, the determinants

$$\widetilde{D} = \begin{vmatrix} a_1 \pm ka_2 & b_1 \pm kb_2 & c_1 \pm kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Taking advantage of properties IV and III, we shall have

$$\widetilde{D}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \pm k \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = D \pm k \cdot 0 = D.$$

Elementary transformations yield a convenient method for computing determinants.

Example. Evaluate the symmetric determinant

$$D=\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix}.$$

Subtracting the doubled first row from the second row and the tripled first row from the third row, we get

$$D=\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & -4 & -8 \end{vmatrix} = 1 \cdot \begin{vmatrix} -3 & -4 \\ -4 & -8 \end{vmatrix} = 24-16=8.$$

Sec. 142. A System of Three Linear Equations

Consider a standard linear system of three equations

$$\left.\begin{aligned} a_1x+b_1y+c_1z&=d_1, \\ a_2x+b_2y+c_2z&=d_2, \\ a_3x+b_3y+c_3z&=d_3, \end{aligned}\right\} \tag{142-1}$$

whose constant terms represent the right-hand members of the equations. Any set of three numbers (x, y, z) satisfying this system is said to be its *solution*.

Let us introduce the *determinant of the system*

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \tag{142-2}$$

and also the *additional determinants*

$$D_x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad D_y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad D_z=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}. \tag{142-3}$$

Multiplying in succession the equations of system (142-1) by the cofactors A_1, A_2, A_3 of the corresponding elements a_1, a_2, a_3 of the first column of the determinant D, we obtain

$$(a_1A_1 + a_2A_2 + a_3A_3)\,x + (b_1A_1 + b_2A_2 + b_3A_3)\,y + \\ + (c_1A_1 + c_2A_2 + c_3A_3)\,z = d_1A_1 + d_2A_2 + d_3A_3. \quad (142\text{-}4)$$

Hence, using the expansion theorem (Sec. 140) and corollary 2 to property II, we shall have

$$Dx + 0\cdot y + 0\cdot z = D_x,$$

i.e.

$$Dx = D_x. \quad (142\text{-}5)$$

Using now the cofactors of the elements of the second and third columns of the determinant D, we find in a similar way

$$Dy = D_y, \quad Dz = D_z. \quad (142\text{-}5')$$

If the determinant of the system $D \neq 0$, then from equations (142-5) and (142-5′) we obtain the unique solution of system (142-1):

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}\,^{*}. \quad (142\text{-}6)$$

Thus, we have Cramer's rule: *The unknowns of a standard linear system* (142-1) *with a non-zero determinant represent fractions whose denominator is the determinant of the system, the numerators being equal to the corresponding additional determinants.*

Note. If the determinant of the system $D = 0$, then system (142-1) is either incompatible or has an infinite number of solutions.

Example. Solve the system

$$\left.\begin{aligned} x + 2y + 3z &= 1, \\ 2x + 3y + z &= 0, \\ 3x + y + 2z &= 0. \end{aligned}\right\}$$

We have

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}.$$

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* We have proved here that if a solution of system (142-1) exists, then it is expressed by formulas (142-6). We can get convinced by a direct substitution that for $D \neq 0$ formulas (142-6) yield the solution of system (142-1).

Subtracting the doubled first column from the second column and the tripled first column from the third column, we get

$$D=\begin{vmatrix}1 & 0 & 0\\ 2 & -1 & -5\\ 3 & -5 & -7\end{vmatrix}=1\cdot\begin{vmatrix}-1 & -5\\ -5 & -7\end{vmatrix}=7-25=-18\neq 0.$$

For the additional determinants we find the following values:

$$D_x=\begin{vmatrix}1 & 2 & 3\\ 0 & 3 & 1\\ 0 & 1 & 2\end{vmatrix}=1\cdot\begin{vmatrix}3 & 1\\ 1 & 2\end{vmatrix}=5,$$

$$D_y=\begin{vmatrix}1 & 1 & 3\\ 2 & 0 & 1\\ 3 & 0 & 2\end{vmatrix}=-1\cdot\begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix}=-1,$$

$$D_z=\begin{vmatrix}1 & 2 & 1\\ 2 & 3 & 0\\ 3 & 1 & 0\end{vmatrix}=1\cdot\begin{vmatrix}2 & 3\\ 3 & 1\end{vmatrix}=-7.$$

Using Cramer's rule, we obtain the solution of the system:

$$x=-\frac{5}{18},\quad y=\frac{1}{18},\quad z=\frac{7}{18}.$$

Sec. 143. A Homogeneous System of Three Linear Equations

Consider the linear system

$$\left.\begin{aligned}a_1x+b_1y+c_1z&=0,\\ a_2x+b_2y+c_2z&=0,\\ a_3x+b_3y+c_3z&=0,\end{aligned}\right\}\qquad(143\text{-}1)$$

whose constant terms are equal to zero. Such linear system is called *homogeneous.*

Homogeneous linear system (143-1), obviously, allows a zero solution: $x=0$, $y=0$, $z=0$ and, consequently, is always compatible.

It is of interest to find out the cases when a homogeneous system has non-zero solutions.

Theorem. *A linear homogeneous system of three linear equations in three unknowns has non-zero solutions if and only if its determinant is equal to zero,* i.e.

$$D=\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=0.\qquad(143\text{-}2)$$

Proof. Let us assume that system (143-1) has a non-zero solution (x_1, y_1, z_1). If its determinant $D \neq 0$, then, by Cramer's formulas, system (143-1) has *only* a zero solution which contradicts the assumption. Hence, $D = 0$.

Suppose $D = 0$. Then linear system (143-1) is either incompatible, or has an infinite number of solutions. But our system is compatible, since it has a zero solution. Consequently, system (143-1) allows an infinite number of solutions including non-zero ones.

Note. Let us indicate here the method for finding non-zero solutions of homogeneous system (143-1) in a typical case.

Suppose the determinant of the system $D = 0$, but not all of its second-order minors are equal to zero.

We shall assume that

$$C_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \tag{143-3}$$

(this can always be achieved by permuting the equations and changing the numeration of the unknowns).

Consider the subsystem consisting of the first two equations of system (143-1)

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0. \end{aligned}\right\} \tag{143-4}$$

According to Sec. 139, the solutions of this system have the form

$$x = A_3t, \quad y = B_3t, \quad z = C_3t \tag{143-5}$$

$(-\infty < t < +\infty)$, where A_3, B_3, C_3 are the corresponding signed minors. Substituting these numbers into the third (unused) equation of system (143-1) and taking into consideration that the determinant $D = 0$, we get

$$a_3x + b_3y + c_3z = (a_3A_3 + b_3B_3 + c_3C_3)\, t = Dt \equiv 0.$$

Consequently, formulas (143-5), where t is arbitrary, yield all solutions of *complete system* (143-1).

Geometrically, the equations of system (143-1) represent the equations of the three planes in the space $Oxyz$ (see Sec. 162, Chapter 19). If (1) the determinant $D \neq 0$, then all these planes intersect at the unique point O (0, 0, 0); but if (2) the determinant $D = 0$ but not all of its second-order minors are equal to zero, then in our case these planes intersect along a straight line (like the sheets of a book). We leave without consideration the case when the three planes *merge*.

Sec. 144. A System of Linear Equations in Many Unknowns. Gauss' Method

Consider a system of n linear equations in n unknowns:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= a_{1,\,n+1}, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= a_{2,\,n+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= a_{n,\,n+1}. \end{aligned}\right\} \tag{144-1}$$

Here for the coefficients a system of double subscripts has been introduced, namely, the first subscript of the coefficient a_{ij} denotes the number of the equation, and the second the number of the unknown. For the sake of convenience, the constant terms are denoted by $a_{i,\,n+1}$ $(i = 1, 2, \ldots, n)$.

The simplest method for solving system (144-1) is the *method of elimination*. We are going to set it forth in the form of *Gauss' scheme* (usually called *Gauss' method*).

Let, for definiteness $a_{11} \neq 0$ be the "leading coefficient". Dividing all the terms of the first equation by a_{11}, we shall have a *reduced* equation

$$x_1 + \alpha_{12}x_2 + \ldots + \alpha_{1n}x_n = \alpha_{1,\,n+1}, \tag{144-2}$$

where

$$\alpha_{1j} = \frac{a_{1j}}{a_{11}} \qquad (j = 1, 2, \ldots, n+1). \tag{144-3}$$

Consider the ith equation of system (144-1)

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = a_{i,\,n+1}. \tag{144-4}$$

To eliminate x_1 from this equation, we multiply reduced equation (144-2) by a_{i1} and subtract the obtained equation from equation (144-4). Then we shall have

$$a_{i2}^{(1)}x_2 + \ldots + a_{in}^{(1)}x_n = a_{i,\,n+1}^{(1)}, \tag{144-5}$$

where

$$a_{ij}^{(1)} = a_{ij} - a_{i1}\alpha_{1j} \qquad (j = 2, 3, \ldots, n+1). \tag{144-6}$$

Thus, we obtain a shortened system

$$\left.\begin{aligned} a_{22}^{(1)}x_2 + \ldots + a_{2n}^{(1)}x_n &= a_{2,\,n+1}^{(1)}, \\ \ldots\ldots\ldots\ldots\ldots & \\ a_{n2}^{(1)}x_2 + \ldots + a_{nn}^{(1)}x_n &= a_{n,\,n+1}^{(1)}, \end{aligned}\right\} \tag{144-7}$$

whose coefficients are determined by formulas (144-6).

If its *leading coefficient* is $a_{22}^{(1)} \neq 0$, then, using the above indicated method, we can eliminate the unknown x_2 from system (144-7), and the new coefficients will be computed by the formulas of type (144-6) and so on. This part of computations is called the *direct operation of Gauss' method.*

To determine the unknowns $x_1, x_2, \ldots, x_n$, let us consider the reduced equations

$$\left.\begin{aligned} x_1 + \alpha_{12}x_2 + \ldots + \alpha_{1n}x_n &= \alpha_{1,\,n+1}, \\ x_2 + \ldots + \alpha_{2n}^{(1)}x_n &= \alpha_{2,\,n+1}^{(1)}, \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ x_n &= \alpha_{n,\,n+1}^{(n-1)}. \end{aligned}\right\} \qquad (144\text{-}8)$$

Hence we find in succession the unknowns (*the reverse operation*)

$$\left.\begin{aligned} x_n &= \alpha_{n,\,n+1}^{(n-1)}, \\ x_{n-1} &= \alpha_{n-1,\,n+1}^{(n-2)} - \alpha_{n-1,\,n}^{(n-2)}x_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ x_1 &= \alpha_{1,\,n+1} - \alpha_{1n}x_n - \ldots - \alpha_{1,\,n-1}x_{n-1} - \ldots - \alpha_{12}x_2. \end{aligned}\right\} \qquad (144\text{-}9)$$

Note that operations (144-9) are carried out without division.

If the next leading coefficient turns out to be equal to zero, the equations of the system should be properly rearranged. Of course, it is possible that system (144-1) is incompatible. Then, naturally, Gauss' method allows no realization.

Example. Using Gauss' method, solve the system

$$\left.\begin{aligned} 2x_1 - 3x_2 + 4x_3 &= 20, \\ 3x_1 + 4x_2 - 2x_3 &= -11, \\ 4x_1 + 2x_2 + 3x_3 &= 9. \end{aligned}\right\} \qquad (144\text{-}10)$$

We compile a table of the coefficients of system (144-10), regarding its constant terms as coefficients of $x^0 = 1$:

x_1	x_2	x_3	x^0	Σ	
2	−3	4	20	23	
3	4	−2	−11	−6	I
4	2	3	9	18	
. . . . .					
1	−1.5	2	10	11.5	

Table (continued)

x_1	x_2	x_3	x^0	Σ	
	8.5	—8	—41	—40.5	
	8	—5	—31	—28	
					II
	1	—0.625	—3.875	—3.5	
		—2.6875	—8.0625	—10.75	
					III
		1	3	4	
		1	3	4	
	1		—2	—1	IV
1			1	2	

The last column Σ contains the sums of elements of the corresponding rows of the table; it serves to check the computations.

Regarding the marked coefficient 2 as leading and dividing by it all the elements of the first row of the table (including the one entering the column Σ), we obtain the coefficients of the first reduced equation (see the table). The current check of calculations is realized by that the element from the column Σ is equal to the sum of the remaining elements of this row. This is how section I of the table is filled in.

Then, using formula (144-6), we compute the coefficients of the shortened system not containing the unknown x_1. For the sake of obviousness, we shall call the row containing the coefficients of a reduced equation a *reduced row*, and the column containing the leading element of a section a *leading column*. Then, on the basis of formula (144-6), the following rule holds true: *The transformed coefficients of Gauss' scheme are equal to its initial coefficients minus the product of their "projections" on the corresponding reduced row and the leading column of the table.* Using this rule, we fill in section II of the table, the check column included. For the convenience of computations, we take element 8 as the leading coefficient of section II (see the table).

Section III is filled in in a similar way. And this is the end of the *direct operations* of Gauss' scheme.

The unknowns x_3, x_2, and x_1 are determined in succession from the reduced equations

$$x_3 = 3,$$

$$x_2 - 0.625x_3 = -3.875, \qquad x_1 - 1.5x_2 + 2x_3 = 10.$$

Hence

$$x_3 = 3,$$
$$x_2 = -3.875 + 0.625 \cdot 3 = -2,$$
$$x_1 = 10 - 2 \cdot 3 + 1.5 \cdot (-2) = 1$$

(*reverse operations*). The results of the reverse operations are tabulated in section IV.

Note that if the elements of the column Σ are taken as constant terms, then for the unknowns we obtain the values $\hat{x}_3 = 4$, $\hat{x}_2 = -1$, $\hat{x}_1 = 2$ which exceed by unity the value of the unknowns x_3, x_2, x_1. This ensures the final check of the computations carried out.

Exercises

1. Evaluate the following second-order determinants:

(a) $\begin{vmatrix} -7 & 6 \\ 8 & -7 \end{vmatrix}$; (b) $\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix}$.

2. With the aid of determinants solve the system

$$\left.\begin{aligned} x + 0.9y &= 1, \\ 1.1x + y &= -2. \end{aligned}\right\}$$

3. Find the solutions of the system

$$\left.\begin{aligned} x + y &= 1, \\ 2x + 2y &= \alpha \end{aligned}\right\}$$

(α parameter).

4. Find the point of intersection of the straight lines

$$\left.\begin{aligned} 5x + \alpha y &= 1, \\ x - 5y &= -3 \end{aligned}\right\}$$

(α parameter).

5. Solve the system

$$\left.\begin{aligned} x - 3y + 5z &= 0, \\ 7x - 9y - 11z &= 0. \end{aligned}\right\}$$

Indicate a certain integral non-zero solution of this system.

6. Evaluate the following third-order determinants:

(a) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{vmatrix}$; (b) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{vmatrix}$; (c) $\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$.

7. Solve the equation

$$\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0.$$

8. With the aid of determinants solve the system

$$\left.\begin{aligned} -x+3y+5z &= 1, \\ 3x+y+3z &= 2, \\ 5x+3y-z &= -3. \end{aligned}\right\}$$

9. Solve the system

$$\left.\begin{aligned} x+2y+3z &= 0, \\ 2x-3y+4z &= 0, \\ 3x-y+7z &= 0. \end{aligned}\right\}$$

10. Solve the system

$$\left.\begin{aligned} \alpha x &= 0, \\ x+y &= 0, \\ x+z &= 0 \end{aligned}\right\}$$

(α parameter).

11. Using Gauss' method solve the system

$$\left.\begin{aligned} x_1+2x_2+3x_3+4x_4 &= -10, \\ 2x_1+3x_2+4x_3-5x_4 &= -8, \\ 3x_1+4x_2-5x_3-6x_4 &= 4, \\ 4x_1-5x_2-6x_3-7x_4 &= 24. \end{aligned}\right\}$$

Chapter 18

Fundamentals of Vector Algebra

Sec. 145. Scalars and Vectors

A quantity completely characterized by its numerical value in a chosen system of units is called *scalar* or *the scalar*. The mass of a body, its volume and temperature, etc. are examples of scalars. The scalar is defined by a *number*, positive or negative, or equal to zero.

In addition to the numerical value, a quantity may also be specified by a certain direction. Such a quantity is called *vector* or *the vector*. Force, displacement, velocity, etc. belong to vectors. A vector is defined by a number and direction.

Vectors are usually denoted by boldface letters, for instance, **a**. Geometrically, a vector is represented by a directed line segment in space (Fig. 168). We denote such a directed segment by the symbol $\overrightarrow{AB}$ ($\mathbf{a} = \overrightarrow{AB}$), where the point A is the initial point and the point B the terminal point, and we usually draw it as an arrow.

The *length* of a vector **a**,

$$|\mathbf{a}| = a,$$

which is also called the absolute value (or *modulus*) of the vector, is understood as its numerical value without taking into consideration its direction. (Naturally, the symbol $|\overrightarrow{AB}|$, denotes the length (modulus) of the vector $\overrightarrow{AB}$.) Vector **0** whose modulus is equal to zero is called the *null-vector* (its direction being arbitrary).

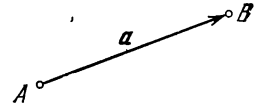


FIG. 168

Two (non-zero) vectors **a** and **b** are said to be *equal* if they lie on parallel or coinciding straight lines (parallelism in a broad sense), have the same length and are in the same direction. Let us agree here not to distinguish among equal vectors, and thus we have arrived to the concept of a *free vector*. In other words, a free vector may be transferred into any point in space provided its length and direction

are preserved. In particular, free vectors can be reduced to a common origin. Henceforward we shall discuss the *theory of free vectors* in a three-dimensional space.

Sec. 146. The Sum of Several Vectors

Definition. *The sum of several vectors*, for instance, **a**, **b**, **c**, **d**, (Fig. 169) *is a vector*

$$\mathbf{s} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d},$$

equal (by absolute value and direction) to the resultant $\overrightarrow{OM}$ *of a three-dimensional polygonal line constructed on the given vectors.*

In case of two vectors **a** and **b** (Fig. 170) their sum **s** is the diagonal of the parallelogram constructed on these vectors emanating from a common point of their application, or from the origin (*the parallelogram law*).

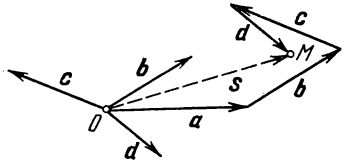


FIG. 169

Since in any triangle the length of one side does not exceed the sum of the lengths of two other sides, from Fig. 170 we have

$$|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|,$$

i.e. *the modulus of a sum of two vectors does not exceed the sum of the moduli of these vectors.*

The sum **s** of three vectors **a**, **b**, **c** (Fig. 171) is the diagonal $\overrightarrow{OM}$ of

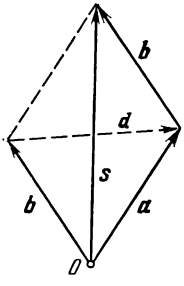


FIG. 170

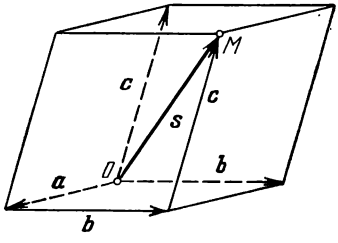


FIG. 171

a parallelepiped constructed on these vectors (*the parallelepiped law*).

It is easy to check that for the addition of vectors the following properties are valid:

(1) *commutative property*

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a},$$

i.e. *a sum of vectors does not depend on the order in which the vectors are added*;

(2) *associative property*

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + \mathbf{b} + \mathbf{c},$$

i.e. *the sum of three (or more) vectors is independent of the order in which the parentheses are arranged.*

For each vector $\mathbf{a} = \overrightarrow{OA}$ (Fig. 172) there exists an opposite vector

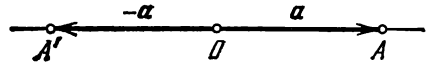


FIG. 172

$-\mathbf{a} = \overrightarrow{OA'}$ which has the same length but an opposite direction. By the parallelogram law, we obviously have

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0},$$

where $\mathbf{0}$ is a null-vector.

It is easy to check that

$$\mathbf{a} + \mathbf{0} = \mathbf{a}.$$

Sec. 147. The Difference of Vectors

The *difference of the vectors* $\mathbf{a}$ and $\mathbf{b}$ (Fig. 173) should be understood as the vector

$$\mathbf{d} = \mathbf{a} - \mathbf{b}$$

such that

$$\mathbf{b} + \mathbf{d} = \mathbf{a}. \qquad (147\text{-}1)$$

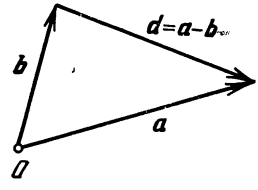


FIG. 173

Note that in the parallelogram constructed on the given vectors $\mathbf{a}$ and $\mathbf{b}$ (Fig. 170) their difference is the other diagonal of the parallelogram.

It is easy to check that the following rule of subtraction is valid:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

Sec. 148. Multiplication of a Vector by a Scalar

Definition. *The product of a vector* $\mathbf{a}$ *by a scalar* k (Fig. 174) *is a vector*

$$\mathbf{b} = k\mathbf{a} \equiv \mathbf{a}k,$$

whose length is $b = |k|a$ *and whose direction*: (1) *coincides with the direction of the vector* $\mathbf{a}$ *if* $k > 0$; (2) *is in the opposite sense if* $k < 0$; (3) *is arbitrary if* $k = 0$.

It is easy to get convinced that this vector operation possesses the following properties:

(1) $$(k + l)\,\mathbf{a} = k\mathbf{a} + l\mathbf{a},$$
$$k\,(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b};$$

(2) $$k\,(l\mathbf{a}) = (kl)\,\mathbf{a};$$

(3) $$1 \cdot \mathbf{a} = \mathbf{a}, \quad (-1)\,\mathbf{a} = -\mathbf{a}, \quad 0 \cdot \mathbf{a} = 0$$

(k, l scalars).

Example. $(\mathbf{a} + \mathbf{b}) + (\mathbf{a} - \mathbf{b}) = 2\mathbf{a}$.

If we divide a non-zero vector $\mathbf{a}$ by its length $a = |\mathbf{a}|$ (i.e. multiply by the scalar $\frac{1}{a}$), then we shall obtain a unit vector $\mathbf{e}$, which is in the same direction

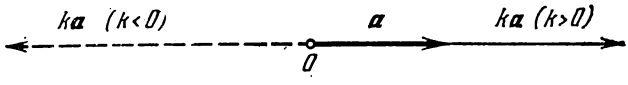


FIG. 174

$$\mathbf{e} = \frac{\mathbf{a}}{a}.$$

Hence, we have the standard formula of the vector

$$\mathbf{a} = a\mathbf{e}. \tag{148-1}$$

Formula (148-1) is formally true for the null-vector $\mathbf{a} = 0$ as well, where $\mathbf{a} = 0$ and $\mathbf{e}$ is an arbitrary unit vector.

Sec. 149. Collinear Vectors

Definition. *Two vectors* $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{O'B}$ (Fig. 175) *are called collinear if they are parallel in a broad sense* (i.e. *they are situated either on parallel lines, or on one and the same line*).

Since the direction of a null-vector is arbitrary, we may assume that *the null-vector is collinear to any vector.*

Theorem. *Two non-zero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are collinear if and only if they are proportional to each other*, i.e.

$$\mathbf{b} = k\mathbf{a} \tag{149-1}$$

(k scalar).

Proof. (1) Let the vectors **a** and **b** ($\mathbf{a} \neq 0$, $\mathbf{b} \neq 0$) be collinear and **e**, **e**′ be their unit vectors.

We have

$$\mathbf{a} = a\mathbf{e} \quad \text{and} \quad \mathbf{b} = b\mathbf{e}'. \tag{149-2}$$

Obviously,

$$\mathbf{e}' = \pm\mathbf{e}, \tag{149-3}$$

where the plus sign corresponds to the case when the vectors **a** and **b** are of the same direction, and the minus sign is taken when the vectors **a** and **b** have opposite directions.

From formulas (149-2) and (149-3) we obtain

$$\mathbf{b} = \pm b\mathbf{e} = \pm\frac{b}{a}\cdot(a\mathbf{e}) = \pm\frac{b}{a}\mathbf{a},$$

wherefrom formula (149-1) is derived, where $k = \pm b/a$.

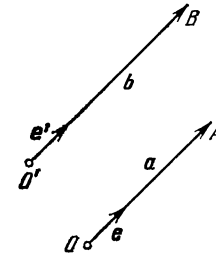


FIG. 175

(2) If equality (149-1) is fulfilled, then the collinearity of the vectors **a** and **b** follows directly from the sense of multiplication of vectors by a scalar (see Sec. 148).

Sec. 150. Coplanar Vectors

Definition. *Three vectors* **a**, **b**, *and* **c** *are called coplanar if they are parallel to a certain plane in a broad sense* (i.e. *either parallel to this plane, or lie in it*).

We may also state that the vectors **a**, **b**, and **c** are coplanar if and only if, when reduced to a common origin, they *lie in one and the same plane.*

According to the sense of the definition, three vectors among which there is at least one null-vector are coplanar.

Theorem. *Three non-zero vectors* **a**, **b**, *and* **c** *are coplanar if and only if one of them is a linear combination of the others*, i.e., for instance,

$$\mathbf{c} = k\mathbf{a} + l\mathbf{b} \tag{150-1}$$

(k, l scalars).

Proof. (1) Let the vectors **a**, **b**, and **c** be coplanar, lie in the plane P, and have a common origin O (Fig. 176).

Let us first assume that these vectors are not all collinear pairwise, for instance, the vectors **a** and **b** are not collinear. Then, decomposing the vector **c** into a sum of vectors $\mathbf{c}_a$ and $\mathbf{c}_b$ collinear to the vectors **a**

and **b**, respectively, by virtue of Sec. 149 we shall have

$$\mathbf{c} = \mathbf{c}_a + \mathbf{c}_b = k\mathbf{a} + l\mathbf{b}, \tag{150-2}$$

where k and l are the corresponding scalars.

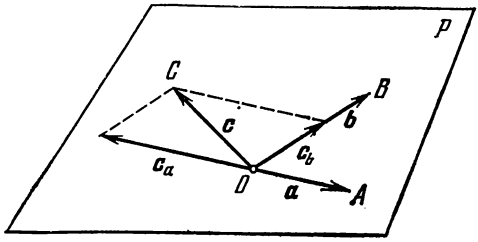


FIG. 176

If the vectors **a**, **b**, **c** are collinear pairwise, then we may write

$$\mathbf{c} = k\mathbf{a} = k\mathbf{a} + 0\mathbf{b}, \tag{150-3}$$

and thus condition (150-1) is fulfilled once again.

(2) Conversely, if for the vectors $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, and $\mathbf{c} = \overrightarrow{OC}$ (Fig. 176) condition (150-1) is fulfilled, then, by the sense of the corresponding vector operations, the vector **c** is situated in the plane containing the vectors **a** and **b**, i.e. these vectors are coplanar.

Example. The vectors **a**, **a** + **b**, **a** − **b** are coplanar, since

$$\mathbf{a} = \frac{1}{2}(\mathbf{a}+\mathbf{b}) + \frac{1}{2}(\mathbf{a}-\mathbf{b}).$$

Sec. 151. The Projection of a Vector on an Axis

A directed straight line is called the *axis*. Its direction is usually denoted by an arrow. We shall regard the specified direction of the axis as *positive* and the opposite direction as *negative*.

Definition 1. *The projection of the point A on the axis l is the foot A′ of a perpendicular AA′ dropped from the point A onto this axis.*

The perpendicular AA' is understood here as a straight line intersecting the axis l and making right angles with it*. Thus, the projection A' is the intersection of the plane passing through the point A and perpendicular to the axis l with this axis.

Definition 2. *The component (or geometric projection) of the vector* $\mathbf{a} = \overrightarrow{AB}$ *with respect to the axis l* (Fig. 177) *is the vector*

$$\mathbf{a}' = \overrightarrow{A'B'},$$

* Remember that all geometrical objects are considered here in a three-dimensional space.

whose origin A' is the projection of the origin A of the vector **a** *on the axis l, and the terminus of which B' is the projection of the terminus B of this vector on this axis.*

Definition 3. *The algebraic projection of the vector* **a** *on the axis l is the scalar*

$$a_l = \pm |\overrightarrow{A'B'}|,$$

equal to the length (modulus) of its component **a**′ *along the axis l taken with the plus sign if the component is in the same direction as the axis l, and with the minus sign if the component is in the opposite direction.*

If $\mathbf{a} = \mathbf{0}$, then we set $a_l = 0$.

It should be noted here that if **e** is the unit vector of the axis l, then the following equality holds true for the component **a**′

$$\mathbf{a}' = a_l \mathbf{e}. \qquad (151\text{-}1)$$

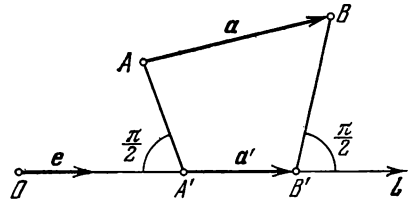


FIG. 177

Theorem 1. *The projection of a vector* **a** *on the axis l is equal to the product of the length a of the vector by the cosine of the angle between the direction of the vector and the direction of the axis*, i.e.

$$a_l = a \cos \varphi, \quad \varphi = \angle (\mathbf{a}, l). \qquad (151\text{-}2)$$

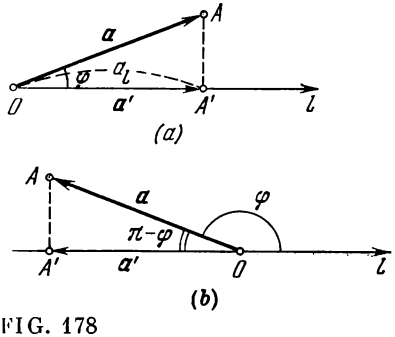


FIG. 178

Proof. Since $\mathbf{a} = \overrightarrow{OA}$ is a free vector (Fig. 178), we may assume that its origin O lies on the axis l.

(1) If the angle φ between the vector **a** and the axis l is acute $\left(0 \leqslant \varphi \leqslant \frac{\pi}{2}\right)$, the direction of the component $\mathbf{a}' = \overrightarrow{OA}'$ of the vector **a** coincides with the direction of the axis l (Fig. 178, *a*). In this case we have

$$a_l = \mathrm{pr}_l \mathbf{a} = + |\overrightarrow{OA}'| = OA \cos \varphi = a \cos \varphi.$$

(2) If the angle φ between the vector and the axis l is obtuse $\left(\frac{\pi}{2} < \varphi \leqslant \pi\right)$ (Fig. 178, *b*), the direction of the component, $\mathbf{a}' =$

$= \overrightarrow{OA'}$ of the vector **a** is opposite to that of the axis l. Then we obtain

$$a_l = \mathrm{pr}_l \mathbf{a} = -|\overrightarrow{OA'}| = -AO\cos(\pi - \varphi) = +a\cos\varphi.$$

Thus, formula (151-2) has been proved.

Corollary 1. *The projection of a vector on the axis is* (1) *positive if this vector forms an acute angle with the axis*; (2) *negative if this angle is obtuse, and* (3) *zero if this angle is a right one.*

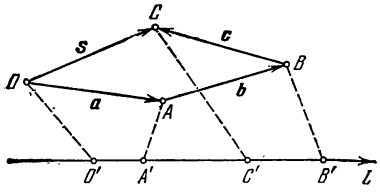


FIG. 179

Corollary 2. *The projections of equal vectors on one and the same axis are equal to each other.*

Theorem 2. *The projection of a sum of several vectors on a given axis is equal to the sum of their projections on this axis.*

Proof. Let, for instance,

$$\mathbf{s} = \mathbf{a} + \mathbf{b} + \mathbf{c},$$

where (Fig. 179) $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{c} = \overrightarrow{BC}$ and, consequently, $\mathbf{s} = \overrightarrow{OC}$.

Denoting the projections of the points O, A, B, C on the axis l by O', A', B', C' and taking into consideration the directions of the components (see Fig. 179), we have

$$\mathrm{pr}_l \mathbf{s} = +|\overrightarrow{O'C'}| = +|\overrightarrow{O'A'}| + |\overrightarrow{A'B'}| - |\overrightarrow{B'C'}| =$$

$$= \mathrm{pr}_l \mathbf{a} + \mathrm{pr}_l \mathbf{b} + \mathrm{pr}_l \mathbf{c}, \quad (151\text{-}3)$$

which was to be proved.

Corollary. *The projection of a closed vector line on any axis is equal to zero.*

Theorem 3. *When multiplying a vector by a scalar, its projection on a given axis is multiplied by this scalar*, i.e.

$$\mathrm{pr}_l (k\mathbf{a}) = k\,\mathrm{pr}_l \mathbf{a}. \quad (151\text{-}4)$$

Formula (151-4) follows from Theorem 1 as well as from the sense of multiplying a vector by a scalar.

Corollary. *The projection of a linear combination of vectors is equal to the linear combination of the projections of these vectors*, i.e.

$$\mathrm{pr}_l (k_1\mathbf{a} + k_2\mathbf{b}) = k_1 \mathrm{pr}_l \mathbf{a} + k_2 \mathrm{pr}_l \mathbf{b}.$$

Sec. 152. The Rectangular Cartesian Coordinates in Space

Let (Fig. 180) Ox, Oy, Oz be three mutually perpendicular axes in a three-dimensional space (*the coordinate axes*) emanating from a common point O (*the origin*) and forming a *right-handed triple* (*a right-handed coordinate system*), i.e. oriented according to the right-handed screw rule. In other words, for the observer viewing along the axis Oz the nearest rotation of the axis Ox towards the axis Oy is seen as occurring anticlockwise. Three mutually perpendicular planes Oyz, Ozx, and Oxy passing through the corresponding axes are called the *coordinate planes*. They separate the entire space into eight octants.

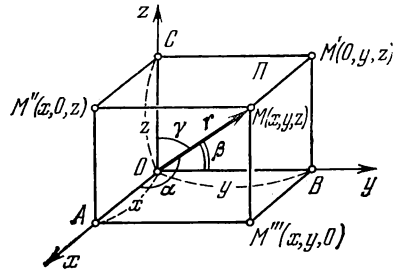


FIG. 180

For each point M in space (Fig. 180) there exists its *radius vector* $\mathbf{r} = \overrightarrow{OM}$ whose initial point is the origin O (of coordinates) and whose terminal point is the given point M.

Definition. *The rectangular Cartesian coordinates x, y, z of a point M are understood to be the projections of its radius vector* $\mathbf{r}$ *on the corresponding coordinate axes*, i.e.

$$x = r_x \quad y = r_y, \quad z = r_z. \tag{152-1}$$

Henceforward, for the sake of brevity, we shall call rectangular Cartesian coordinates simply the *rectangular coordinates*.

The point M with the coordinates x, y, z is denoted as M (x, y, z) where the first coordinate is called the *abscissa*, the second the *ordinate*, and the third the *z-coordinate* of the point M.

To find these coordinates, we pass through the point M three planes MA, MB, MC perpendicular to the axes Ox, Oy, Oz, respectively (Fig. 180). Then, on these axes we obtain the following directed segments

$$OA = x, \quad OB = y, \quad OC = z, \tag{152-2}$$

whose lengths are numerically equal to the coordinates of the point M.

The radius vector $\mathbf{r}$ is the diagonal of the parallelepiped Π with the dimensions $|x|$, $|y|$, $|z|$ formed by the planes MA, MB, MC and the coordinate planes. Therefore

$$r = \sqrt{x^2 + y^2 + z^2}. \tag{152-3}$$

Designating by α, β, γ ($0 \leqslant \alpha, \beta, \gamma \leqslant \pi$) the angles between the radius vector $\mathbf{r}$ and the coordinate axes, we shall have

$$x = r\cos\alpha, \quad y = r\cos\beta, \quad z = r\cos\gamma. \tag{152-4}$$

The cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the *direction cosines* of the radius vector $\mathbf{r}$. Taking into account (152-3), we obtain from (152-4)

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1, \tag{152-5}$$

i.e. *the sum of the squares of the direction cosines of the radius vector of a point in space is equal to unity.*

From (152-4) it follows that a coordinate of the point M is *positive* if the radius vector of this point forms an acute angle with the corresponding coordinate axis, and *negative* if the angle is obtuse. In particular, in the first octant of space whose edges constitute positive coordinate semi-axes all coordinates of the points contained are positive. In the remaining octants the coordinates corresponding to the negatively directed edges will be negative.

The dimensions $|x|$, $|y|$, $|z|$ of the parallelepiped Π are equal to the distances of the point M from the coordinate planes Oyz, Ozx, Oxy, respectively. Thus, *the rectangular coordinates of the point M in space represent the distances of this point from the coordinate planes taken with proper signs.*

In particular, if the point M (x, y, z) lies on the plane Oyz, then $x = 0$; if on the plane Ozx, then $y = 0$; and if on the plane Oxy, then $z = 0$; and conversely.

Sec. 153. The Length and Direction of a Vector

Let there be given a vector $\mathbf{a}$ in the space $Oxyz$. The projections of this vector on the coordinate axes

$$a_x = \mathrm{pr}_x\mathbf{a}, \quad a_y = \mathrm{pr}_y\mathbf{a}, \quad a_z = \mathrm{pr}_z\mathbf{a} \tag{153-1}$$

are called the *coordinates of the vector* $\mathbf{a}$, and the vector will be written in the following notation: $\mathbf{a} = \{a_x, a_y, a_z\}$.

Since $\mathbf{a}$ is a free vector, we may regard it as the radius vector of the point M (a_x, a_y, a_z). Hence we get the length of the vector

$$|\mathbf{a}| = a = \sqrt{a_x^2 + a_y^2 + a_z^2}, \tag{153-2}$$

i.e. *the modulus of a vector is equal to the square root of the sum of the squares of its coordinates.*

The direction cosines of the vector $\mathbf{a}$ are determined from the equations

$$a_x = a\cos\alpha, \quad a_y = a\cos\beta, \quad a_z = a\cos\gamma, \tag{153-3}$$

where

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1, \tag{153-4}$$

i.e. *the sum of the squares of the direction cosines of a vector is equal to unity*. The direction cosines of a non-zero vector determine its direction uniquely. Consequently, a vector is completely characterized by its coordinates.

Example. Find the length and direction of the vector

$\mathbf{a} = \{1, 2, -2\}$.

We have

$$a = \sqrt{1^2 + 2^2 + (-2)^2} = 3$$

and

$$\cos \alpha = \frac{a_x}{a} = \frac{1}{3}, \quad \cos \beta = \frac{a_y}{a} = \frac{2}{3}, \quad \cos \gamma = \frac{a_z}{a} = -\frac{2}{3}.$$

Hence

$$\alpha = \arccos \frac{1}{3} \approx 70°30',$$

$$\beta = \arccos \frac{2}{3} \approx 48°10',$$

$$\gamma = \arccos \left(-\frac{2}{3}\right) \approx 131°50'.$$

Thus, the vector $\mathbf{a}$ forms acute angles with the coordinate axes Ox and Oy and an obtuse angle with the coordinate axis Oz.

Sec. 154. The Distance Between Two Points in Space

Let $M_1 (x_1, y_1, z_1)$ be the initial point of the segment $\mathbf{l} = \overrightarrow{M_1 M_2}$ and $M_2 (x_2, y_2, z_2)$ its terminal point. The points M_1 and M_2 can be specified by their radius vectors $\mathbf{r}_1 = \{x_1, y_1, z_1\}$ and $\mathbf{r}_2 = \{x_2, y_2, z_2\}$ (Fig. 181). Considering the vector $\mathbf{l} = \overrightarrow{M_1 M_2}$, from the triangle OM_1M_2, we shall have

$$\mathbf{l} = \mathbf{r}_2 - \mathbf{r}_1. \tag{154-1}$$

Projecting this vector equality on the coordinate axes and taking into consideration the properties of projections, we shall obtain

$$l_x = x_2 - x_1, \quad l_y = y_2 - y_1, \quad l_z = z_2 - z_1. \tag{154-2}$$

Thus, *the projections of a directed segment on the coordinate axes are equal to the differences between the corresponding coordinates of the terminus and origin of the segment.*

From formulas (154-2) we obtain the length of the segment (or, in other words, the difference between two points M_1 and M_2)

$$l = \sqrt{l_x^2 + l_y^2 + l_z^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \qquad (154\text{-}3)$$

And so, *the distance between two points in space is equal to the square root of the sum of the squares of like coordinates of these points.*

Example. A rocket displaced rectilinearly from the point M_1 (10, —20, 0) to the point M_2 (—30, —50, 40) (the distances are

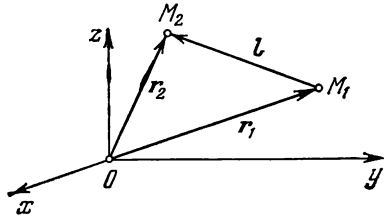


FIG. 181

given in kilometres). Find the distance l covered by the rocket.

By virtue of (154-3) we have

$$l = \sqrt{(-30-10)^2 + (-50+20)^2 + (40-0)^2} =$$

$$= \sqrt{1600 + 900 + 1600} = \sqrt{4100} \approx 64.4 \text{ km}.$$

We would like to note that on finding the direction cosines of the vector of displacement l, it is not difficult to determine the direction of motion of the rocket.

Sec. 155. Operations on Vectors Represented in the Coordinate Form

Let the vector $\mathbf{a} = \{a_x, a_y, a_z\}$ be given by its projections on the coordinate axes Ox, Oy, Oz.

We construct a parallelepiped (Fig. 182) whose diagonal is the vector $\mathbf{a}$ and whose edges are its components $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ along the corresponding coordinate axes. We have the following decomposition

$$\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3. \qquad (155\text{-}1)$$

If we introduce the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ directed along the coordinate axes, then on the basis of the relation between the components of the given vector and its projections set forth in Sec. 151 we shall

have

$$\mathbf{a}_1 = a_x\mathbf{i}, \quad \mathbf{a}_2 = a_y\mathbf{j}, \quad \mathbf{a}_3 = a_z\mathbf{k}. \tag{155-2}$$

Substituting these expressions into equality (155-1), we get the *coordinate form of the given vector*

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}. \tag{155-3}$$

Note that decomposition (155-3) for the vector $\mathbf{a}$ is *unique*. Indeed, let

$$\mathbf{a} = a'_x\mathbf{i} + a'_y\mathbf{j} + a'_z\mathbf{k}. \tag{155-3'}$$

Hence, subtracting (155-3') from (155-3) and taking advantage of the commutative and associative properties of the sum of vectors (Sec. 146) and also of the properties of the difference of vectors (Sec. 147), we shall have

$$\mathbf{0} = (a_x - a'_x)\,\mathbf{i} + (a_y - a'_y)\,\mathbf{j} + (a_z - a'_z)\,\mathbf{k}'$$

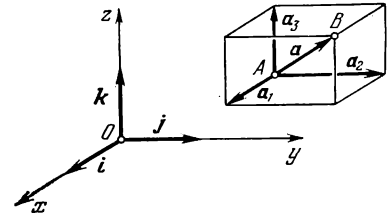


FIG. 182

If at least one of the coefficients of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ were non-zero, then the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ would be coplanar (Sec. 149), which is false. Therefore,

$$a'_x = a_x, \quad a'_y = a_y, \quad a'_z = a_z,$$

and the uniqueness of decomposition (153-3) has been proved.

If $\mathbf{b} = \{b_x, b_y, b_z\}$, then obviously, we also have

$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}. \tag{155-4}$$

Now we can write the above considered linear operations on vectors in the following form:

(1) $\lambda\mathbf{a} = \lambda a_x\mathbf{i} + \lambda a_y\mathbf{j} + \lambda a_z\mathbf{k}$,

or in a brief form: $\lambda\mathbf{a} = \{\lambda a_x, \lambda a_y, \lambda a_z\}$ (λ scalar). Thus, *when multiplying a vector by a scalar, the coordinates of the vector are multiplied by this scalar.*

(2) $\mathbf{a} \pm \mathbf{b} = (a_x \pm b_x)\,\mathbf{i} + (a_y \pm b_y)\,\mathbf{j} + (a_z \pm b_z)\,\mathbf{k}$,

or in a brief form: $\mathbf{a} \pm \mathbf{b} = \{a_x \pm b_x, a_y \pm b_y, a_z \pm b_z\}$.

Thus, *in addition (or subtraction) of vectors their like coordinates are added (or subtracted).*

Example. Find the magnitude and direction of the resultant $\mathbf{F}$ of two forces

$$\mathbf{F}_1 = \{10,\ 20,\ 30\} \quad \text{and} \quad \mathbf{F}_2 = \{30,\ 20,\ 10\}.$$

We have

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \{10 + 30,\ 20 + 20,\ 30 + 10\} = \{40,\ 40,\ 40\}.$$

Hence

$$F = |\mathbf{F}| = 40\sqrt{3}$$

and

$$\cos\alpha = \cos\beta = \cos\gamma = \frac{1}{\sqrt{3}},$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the resultant $\mathbf{F}$.

Sec. 156. Scalar Product of Two Vectors

Definition. *The scalar product of a vector* $\mathbf{a}$ *by a vector* $\mathbf{b}$ *is understood as a number equal to the product of their absolute values (lengths) by*

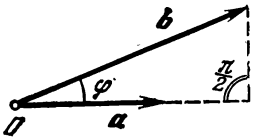


FIG. 183

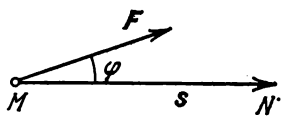


FIG. 184

the cosine of the angle between them, i.e., using the standard notation, we have

$$\mathbf{a}\cdot\mathbf{b} \equiv (\mathbf{a},\ \mathbf{b}) = ab\cos\varphi, \tag{156-1}$$

where $\varphi = \angle\,(\mathbf{a},\ \mathbf{b})$.

Note that in formula (156-1) the scalar product may also be written as $\mathbf{ab}$ omitting the point. Since (Fig. 183)

$$b\cos\varphi = \mathrm{pr}_{\mathbf{a}}\mathbf{b} \quad \text{and} \quad a\cos\varphi = \mathrm{pr}_{\mathbf{b}}\mathbf{a},$$

we may write

$$\mathbf{ab} = a\cdot\mathrm{pr}_{\mathbf{a}}\mathbf{b} = b\cdot\mathrm{pr}_{\mathbf{b}}\mathbf{a}. \tag{156-2}$$

In words, *the scalar product of two vectors is equal to the absolute value (the length) of one of them multiplied by the algebraic projection of the other vector on the direction of the first.*

The Physical Meaning of a Scalar Product. Let a constant force $\mathbf{F}$ ensure a rectilinear displacement $\mathbf{s} = \overrightarrow{MN}$ of a mass point. If the force $\mathbf{F}$ forms an angle φ with the displacement $\mathbf{s}$ (Fig. 184), then, as is known from physics, the work done by the force $\mathbf{F}$ during the displacement $\mathbf{s}$ is equal to

$$A = Fs \cos \varphi.$$

By (156-1), we have

$$A = \mathbf{F} \cdot \mathbf{s}. \tag{156-3}$$

In words, *the work of a constant force with the point of its application being displaced rectilinearly is equal to a scalar product of the force vector by the dispacement vector.*

A scalar product possesses the following basic properties:

(1) *The scalar product of two vectors does not depend on the order in which the factors follow* (commutative property):

$$\mathbf{ab} = \mathbf{ba}. \tag{156-4}$$

This formula follows directly from (156-1).

(2) For three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ the distributive property holds true:

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{ac} + \mathbf{bc}, \tag{156-5}$$

i.e. *in a scalar multiplication of a sum of vectors by a vector we may "remove the parentheses".*

Indeed, taking into account the properties of the projections of vectors by (156-2) (Sec. 151, theorem 2), we have

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathrm{pr}_c\, (\mathbf{a} + \mathbf{b}) \cdot c = (\mathrm{pr}_c\, \mathbf{a} + \mathrm{pr}_c\, \mathbf{b}) \cdot c =$$

$$= \mathrm{pr}_c\, \mathbf{a} \cdot c + \mathrm{pr}_c\, \mathbf{b} \cdot c = \mathbf{ac} + \mathbf{bc}.$$

(3) *The scalar square of a vector is equal to the square of its absolute value (modulus)*, i.e.

$$\mathbf{a}^2 = a^2.$$

Indeed,

$$\mathbf{a}^2 = \mathbf{a} \cdot \mathbf{a} = aa \cos \widehat{(\mathbf{a}, \mathbf{a})} = a^2.$$

Hence for the modulus we obtain the formula

$$|\mathbf{a}| = \sqrt{(\mathbf{a}, \mathbf{a})}. \tag{156-6}$$

(4) *A scalar factor may be taken outside the symbol of a scalar product*, i.e.

$$(\lambda \mathbf{a}, \mathbf{b}) = (\mathbf{a}, \lambda \mathbf{b}) = \lambda\, (\mathbf{a}, \mathbf{b}). \tag{156-7}$$

This property is also easily derived from (156-1).

(5) *A scalar product of a linear combination of vectors by an arbitrary vector is equal to the linear combination of the given vectors by this vector*, i.e.

$$(\lambda \mathbf{a} + \mu \mathbf{b},\ \mathbf{c}) = \lambda\,(\mathbf{a},\ \mathbf{c}) + \mu\,(\mathbf{b},\ \mathbf{c})$$

(λ and μ scalars).

This obviously follows from (2) and (4).

It follows from definition (156-1) that the cosine of the angle $\varphi = \angle\,(\mathbf{a}, \mathbf{b})$ between two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ is equal to

$$\cos\varphi = \frac{\mathbf{ab}}{ab}. \tag{156-8}$$

It follows from formula (156-8) that *two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are perpendicular*, i.e. $\varphi = \frac{\pi}{2}$, *if and only if*

$$\mathbf{ab} = 0. \tag{156-9}$$

This assertion is also valid for the case when at least one of the vectors $\mathbf{a}$ or $\mathbf{b}$ is a null vector.

Example. Find the projection of the vector $\mathbf{a}$ on the vector $\mathbf{b}$. Denoting by φ the angle between these vectors, we have

$$\mathrm{pr}_{\mathbf{b}}\,\mathbf{a} = a\cos\varphi = a \cdot \frac{\mathbf{ab}}{ab} = \mathbf{a} \cdot \frac{\mathbf{b}}{b} = \mathbf{ae},$$

where $\mathbf{e} = \frac{\mathbf{b}}{b}$ is the unit vector of the vector $\mathbf{b}$.

Sec. 157. Scalar Product of Vectors in the Coordinate Form

Let

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \tag{157-1}$$

and

$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}. \tag{157-2}$$

Multiplying these vectors as polynomials (which is valid by virtue of the properties considered in the preceding section) and taking into consideration the relations

$$\mathbf{ij} = \mathbf{jk} = \mathbf{ki} = 0$$

and

$$\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = 1,$$

we shall have

$$\mathbf{ab} = a_xb_x + a_yb_y + a_zb_z. \tag{157-3}$$

Thus, *a scalar product of vectors is equal to the sum of products of their like coordinates.* Hence, denoting by φ the angle between the vectors **a** and **b**, we get

$$\cos\varphi = \frac{\mathbf{ab}}{ab} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{b_x^2 + b_y^2 + b_z^2}}. \tag{157-4}$$

Example. Determine the angle φ between the vectors

$\mathbf{a} = \{1, +2, 3\}$ and $\mathbf{b} = \{-3, 2, -1\}$.

By (157-4), we have

$$\cos\varphi = \frac{1\cdot(-3)+2\cdot 2+3\cdot(-1)}{\sqrt{1^2+2^2+3^2}\cdot\sqrt{(-3)^2+2^2+(-1)^2}} = -\frac{2}{14} = -\frac{1}{7} = -0.143.$$

Hence $\varphi = \arccos\left(-\frac{1}{7}\right) \approx 98°10'$.

Let the vectors **a** and **b** be collinear (parallel). According to the collinearity condition (Sec. 149)

$$\mathbf{b} = k\mathbf{a}, \tag{157-5}$$

where k is a scalar, which is equivalent to

$$b_x = ka_x, \quad b_y = ka_y, \quad b_z = ka_z$$

or

$$\frac{b_x}{a_x} = \frac{b_y}{a_y} = \frac{b_z}{a_z}. \tag{157-6}$$

Thus, *two vectors are collinear if and only if their like coordinates are proportional.*

For perpendicular vectors **a** and **b** we have $\varphi = \frac{\pi}{2}$ and, consequently, $\cos\varphi = 0$, or, according to (157-4)

$$a_x b_x + a_y b_y + a_z b_z = 0.$$

Thus, *two vectors are mutually perpendicular if and only if the sum of products of their like coordinates is equal to zero.*

Sec. 158. Vector Product of Vectors

As we remember, a system of three non-coplanar vectors **a**, **b**, and **c** is called *right-handed* (Fig. 185, *a*) or *left-handed* (Fig. 185, *b*) if it is oriented according to the rule of a right-threaded or a left-threaded screw, respectively.

Note that if we interchange two vectors in a system of three non-coplanar vectors, it will change its orientation, i.e. a right-handed set will become left-handed and vice versa.

Henceforward we shall regard a right-handed system of three vectors as a standard one.

Definition. *A vector product of two vectors* **a** *and* **b** *is understood as the vector*

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} \equiv [\mathbf{a}, \mathbf{b}], \tag{158-1}$$

for which:

(1) *the modulus is equal to the area of the parallelogram constructed on the given vectors*, i.e.

$$c = |\mathbf{c}| = ab \sin \varphi, \tag{158-2}$$

where $\varphi = \angle(\mathbf{a}, \mathbf{b})$ $(0 \leqslant \varphi \leqslant \pi)$ (Fig. 186);

(2) *this vector is perpendicular to the vectors being multiplied* (in

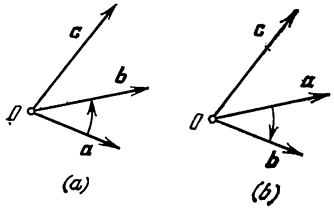


FIG. 185

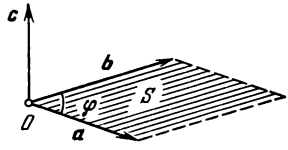


FIG. 186

other words, it is perpendicular to the plane of parallelogram constructed on them), i.e.

$$\mathbf{c} \perp \mathbf{a} \quad \text{and} \quad \mathbf{c} \perp \mathbf{b};$$

(3) *if the vectors are not collinear, then the vectors* **a**, **b**, **c** *form a right-handed system of vectors.*

Let us indicate the basic properties of a vector product.

(1) *If the factors are interchanged, the vector product reverses its sign* (or in other words, it is multiplied by —1):

$$\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b}). \tag{158-3}$$

Indeed, when interchanging the vectors **a** and **b**, the area of the parallelogram constructed on them remains unchanged, i.e. $|\mathbf{b} \times \mathbf{a}| = |\mathbf{a} \times \mathbf{b}|$. But the system of three vectors **b**, **a**, $\mathbf{a} \times \mathbf{b}$ is left-handed. Therefore, the direction of the vector $\mathbf{b} \times \mathbf{a}$ is opposite to the direction of the vector $\mathbf{a} \times \mathbf{b}$ (**a** and **b** are non-collinear). If **a** and **b** are collinear, then equality (158-3) is obvious.

Thus, *a vector product of two vectors does not possess the commutative property.*

(2) *The vector square is equal to the null-vector*, i.e.

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}.$$

This is an obvious corollary of property (1).

(3) *A scalar factor may be taken outside the symbol of a vector product*, i.e. if λ is a scalar, then

$$(\lambda \mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \lambda \mathbf{b}) = \lambda (\mathbf{a} \times \mathbf{b}).$$

This property directly follows from the sense of the product of a vector by a scalar and the definition of a vector product.

(4) *For any three vectors* **a**, **b**, **c** *the following equality holds true*

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}), \tag{158-4}$$

i.e. a vector product possesses the distributive property. We leave the proof of this property to the reader.

Example.

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = (\mathbf{a} \times \mathbf{a}) - (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{b}) - (\mathbf{b} \times \mathbf{b}) = \mathbf{0} + (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) + \mathbf{0} = 2 (\mathbf{a} \times \mathbf{b}).$$

Hence, in particular, we have

$$|(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b})| = 2 \, | \mathbf{a} \times \mathbf{b} \, |,$$

i.e. the area of the parallelogram constructed on the diagonals of the given parallelogram is equal to twice the area of this parallelogram.

With the aid of a vector product it is convenient to formulate the readily checked necessary and sufficient condition for collinearity of two vectors **a** and **b**:

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

Sec. 159. Vector Product in the Coordinate Form

Let

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \tag{159-1}$$

and

$$\mathbf{b} = b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}. \tag{159-2}$$

Multiplying these equalities as vectors and taking advantage of the properties of a vector product, we obtain the sum of nine terms

$$\begin{aligned}\mathbf{a}\times\mathbf{b} = {} & [a_xb_x(\mathbf{i}\times\mathbf{i}) + a_yb_x(\mathbf{j}\times\mathbf{i}) + a_zb_x(\mathbf{k}\times\mathbf{i})] + \\ & + [a_xb_y(\mathbf{i}\times\mathbf{j}) + a_yb_y(\mathbf{j}\times\mathbf{j}) + a_zb_y(\mathbf{k}\times\mathbf{j})] + \\ & + [a_xb_z(\mathbf{i}\times\mathbf{k}) + a_yb_z(\mathbf{j}\times\mathbf{k}) + a_zb_z(\mathbf{k}\times\mathbf{k})]. \quad (159\text{-}3)\end{aligned}$$

From the definition of a vector product it follows that for the unit

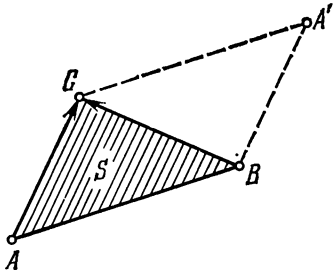


FIG. 187

vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the following "multiplication table" is valid:

$$\mathbf{i}\times\mathbf{i} = 0, \quad \mathbf{j}\times\mathbf{j} = 0, \quad \mathbf{k}\times\mathbf{k} = 0$$

and

$$\begin{aligned}\mathbf{i}\times\mathbf{j} &= -(\mathbf{j}\times\mathbf{i}) = \mathbf{k},\\ \mathbf{j}\times\mathbf{k} &= -(\mathbf{k}\times\mathbf{j}) = \mathbf{i},\\ \mathbf{k}\times\mathbf{i} &= -(\mathbf{i}\times\mathbf{k}) = \mathbf{j}.\end{aligned}$$

Therefore from formula (159-3) we get

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \mathbf{i}(a_yb_z - a_zb_y) + \mathbf{j}(a_zb_x - a_xb_z) + \mathbf{k}(a_xb_y - a_yb_x) = \\ &= \mathbf{i}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \quad (159\text{-}4)\end{aligned}$$

(with the succession of the letters x, y, z retained).

For a better memorizing formula (159-4) is written in the form of a third-order determinant (see Chapter 17)

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \quad (159\text{-}5)$$

It follows from formula (159-4) that

$$|\mathbf{a}\times\mathbf{b}|^2 = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}^2 + \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix}^2 + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}^2. \quad (159\text{-}6)$$

Geometrically, formula (159-6) yields the square of the area of the parallelogram constructed on the vectors **a** and **b**.

Example. Find the area of a triangle with the vertices A (1, 1, 0), B (1, 0, 1), and C (0, 1, 1).

The area S of the triangle ABC is equal to half the area of the parallelogram constructed on the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (Fig. 187). Using the formulas for the projections of directed segments (see Sec. 154), we have

$$\overrightarrow{AB}=\{0,\ -1,\ 1\} \quad \text{and} \quad \overrightarrow{AC}=\{-1,\ 0,\ 1\};$$

hence

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix}=$$

$$=\mathbf{i}\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix}-\mathbf{j}\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}+\mathbf{k}\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix}=-\mathbf{i}-\mathbf{j}-\mathbf{k}.$$

Consequently,

$$S=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{1}{2}\sqrt{3}.$$

Sec. 160. Triple Scalar Product

Definition. *A triple scalar product of three vectors* **a**, **b**, *and* **c**, *is understood as the number*

$$\mathbf{abc}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}. \tag{160-1}$$

Let us construct a parallelepiped Π (Fig. 188) whose edges are the vectors **a**, **b**, and **c** emanating from a common vertex O.

Then $|\mathbf{a}\times\mathbf{b}|=S$ represents the area of the parallelogram constructed on the vectors **a** and **b**, i.e. the area of the base of the parallelepiped. The altitude H of the parallelepiped is, obviously, equal to

$$H=\pm\mathrm{pr}_{\mathbf{S}}\mathbf{c}=\pm c\cos\varphi, \tag{160-2}$$

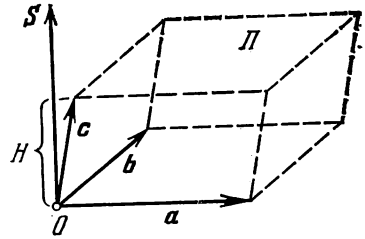


FIG. 188

where $\mathbf{S}=\mathbf{a}\times\mathbf{b}$ and the plus sign corresponds to the acute angle $\varphi=\angle(\mathbf{c},\ \mathbf{S})$ and the minus sign to an obtuse angle φ. In the first case the vectors **a**, **b**, **c** form a right-handed system, and in the second case a left-handed system.

By definition (Sec. 156), we have

$$(\mathbf{a} \times \mathbf{b})\,\mathbf{c} = \mathbf{Sc} = S \cdot \mathrm{pr}_{\mathbf{S}}\mathbf{c} = \pm SH = \pm V, \qquad (160\text{-}3)$$

where V is the volume of the parallelepiped constructed on the vectors **a**, **b**, **c**.

Hence

$$\mathbf{abc} = \pm V,$$

i.e. *a triple product of three vectors is equal to the volume V of a parallelepiped constructed on the vectors with the plus sign if the system* **a**, **b**, **c** *is right-handed and with the minus sign if the system is left-handed.*

A triple scalar product possesses the following basic properties:

(1) *A triple scalar product does not change in a circular permutation of the factors*, i.e.

$$\mathbf{abc} = \mathbf{bca} = \mathbf{cab}.$$

Indeed, in this case both the volume of the parallelepiped Π and the orientation of its edges remain unchanged.

(2) *An interchange of two neighbouring factors reverses the sign of a triple product*, i.e.

$$\mathbf{bac} = \mathbf{acb} = \mathbf{cba} = -\mathbf{abc}.$$

This follows from the fact that an interchange of neighbouring factors, while preserving the volume of the parallelepiped, changes the orientation of the system of three vectors, i.e. a right-handed system becomes left-handed, and vice versa.

A triple product provides a necessary and sufficient condition for the coplanarity of three vectors **a**, **b**, **c**:

$$\mathbf{abc} = 0$$

(the volume of the parallelepiped is equal to zero).

If

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k},$$
$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k},$$
$$\mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k},$$

then, using the expressions in coordinates for a vector (Sec. 159) and a scalar (Sec. 157) product, we get

$$\mathbf{abc} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c})\,\mathbf{a} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) =$$

$$= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} =$$

$$= a_x \begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - a_y \begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + a_z \begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix},$$

i.e.

$$\mathbf{abc} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

Exercises

1. Find the length and direction of the vector $\mathbf{a} = \{1, -1, \sqrt{2}\}$.

2. Find the magnitude and direction of the resultant $\mathbf{F}$ of three forces: $\mathbf{F}_1 = \{10, 20, 0\}$, $\mathbf{F}_2 = \{0, -10, 20\}$, $\mathbf{F}_3 = \{-10, 0, -20\}$.

3. Find the projection of the vector $\mathbf{a} = \{1, 2, -2\}$ on the vector $\mathbf{b} = \{1, 0, -1\}$.

4. Find the work done by the force $\mathbf{F} = \{10, 20, 30\}$ if the point of its application is in a rectilinear motion from the point M (0, 1, 2) to the point N (3, −4, 5).

5. Find the area S and the angle φ of a parallelogram constructed on the vectors $\mathbf{a} = \{1, -2, 3\}$ and $\mathbf{b} = \{3, 2, 1\}$.

6. Are the vectors $\mathbf{a} = \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{k} + \mathbf{i}$, $\mathbf{c} = \mathbf{i} + \mathbf{j}$ coplanar?

Chapter 19

Fundamentals of Solid Analytic Geometry

Sec. 161. The Equations of a Surface and a Line in Space

Definition 1. *An equation relating the coordinates x, y, z is called the equation of a surface in the space $Oxyz$ if the following two conditions hold*: (1) *the coordinates x, y, z of any point of the surface satisfy this equation*, (2) *the coordinates x, y, z of any point not lying on the surface do not satisfy this equation.*

That is, if

$$F(x, y, z) = 0 \qquad (161\text{-}1)$$

is the equation of the surface P (Fig. 189), then for $M(x, y, z) \in P$* we have $F(x, y, z) = 0$, and for $N(x, y, z) \notin P$** we have $F(x, y, z) \neq 0$.

Thus, equation (161-1) is fulfilled if and only if a point $M(x, y, z)$ belongs to a given surface. The coordinates of an arbitrary point of a surface are called the *current coordinates* of a point. Therefore, to form the equation of a surface means to find the relation between the current coordinates of its points.

Example 1. (The equations of the coordinate planes.)

Any point $M(x, y, z)$ lying on the coordinate plane Oyz has the abscissa $x = 0$; conversely, if for some point $M(x, y, z)$ $x = 0$, then this point is situated on the plane Oyz. Hence,

$$x = 0$$

is the equation of the coordinate plane Oyz.

Analogously,

$$y = 0$$

is the equation of the coordinate plane Oxz and

$$z = 0$$

is the equation of the coordinate plane Oxy

---

* The formula $M \in P$ denotes that the point M belongs to P

** The formula $N \notin P$ denotes that the point N does not belong to P.

In the general case

$$x = a, \quad y = b, \quad z = c \tag{161-2}$$

are the equations of three planes perpendicular to the corresponding coordinate axes Ox (the x-axis), Oy (the y-axis), Oz (the z-axis) and intercepting on them segments numerically equal to a, b, and c.

Theorem. *The equation of a cylindrical surface whose generatrix is parallel to a coordinate axis does not have the current coordinate denoted by the same letter as this coordinate axis, and conversely.*

Proof. Let, for instance, a cylindrical surface P be generated by displacing a straight line MN parallel to the z-axis (this line is called the *generatrix*) along a given line L lying in the plane Oxy (this line is called the *directrix*) (Fig. 190).

We denote by $M(x, y, z)$ the point of the surface P with the current coordinates x, y, z. The generatrix MN passing through the point M intersects, obviously, the directrix at point $N(x, y, 0)$.

Let

$$F(x \quad y) = 0 \tag{161-3}$$

be the equation of the directrix L in the coordinate plane Oxy. This equation is satisfied by the coordinates of the point N. Since the point M belonging to the surface P has the same abscissa x and the

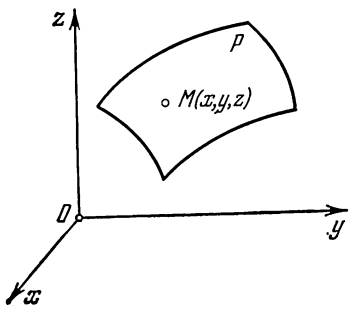


FIG. 189

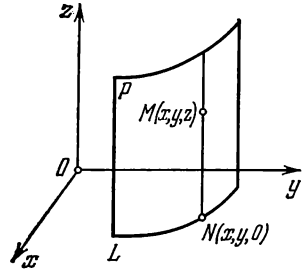


FIG. 190

same ordinate y as the point N, and the variable z does not enter equation (161-3), the coordinates of the point M also satisfy equation (161-3). Hence, the coordinates of any point $M(x, y, z)$ of the surface P satisfy equation (161-3). Conversely, if the coordinates of some point $M(x, y, z)$ satisfy equation (161-3), then this point is situated on a straight line MN (parallel to the z-axis) such that its trace on the plane Oxy, the point $N(x, y, 0)$, lies on the line L, and, hence, the point M belongs to the cylindrical surface P. Cons-

quently,

$$F(x, y) = 0$$

is the *equation of a cylindrical surface* in the space $Oxyz$, the z-coordinate being absent from this equation.

Example 2. (The equation of an elliptic cylinder.) By the preceding theorem, an *elliptic cylinder* whose base is an ellipse with semiaxes a and b and whose axis is the z-axis (Fig. 191) has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In particular, for $a = b$ we obtain the equation of a circular cylinder

$$x^2 + y^2 = a^2.$$

The line L can be specified in space as the intersection of two given surfaces P_1 and P_2 (Fig. 192). The point $M(x, y, z)$ on the line L

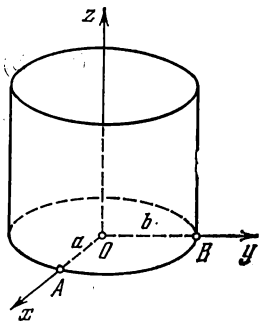


FIG. 191

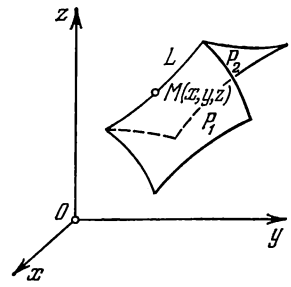


FIG. 192

belongs both to the surface P_1 and to the surface P_2, and, consequently, the coordinates of this point satisfy the equations of both these surfaces.

Therefore, the *equations of a line* in space are understood as a system of two equations:

$$\left.\begin{aligned} F_1(x, y, z) &= 0, \\ F_2(x, y, z) &= 0, \end{aligned}\right\} \qquad (161\text{-}4)$$

which are the equations of the surfaces defining the given line.

One should not think that system (161-4) must be "solved" in order to find the equations of the line. Generally speaking, this cannot be done, since the number of equations in system (161-4) is less than the number of the unknowns. This is the precise sense assigned to equations (161-4): *there belong to the line L only those points $M(x, y, z)$ whose coordinates satisfy both equations of system* (161-4).

Note that a given line can be specified as the intersection of surfaces in different ways. Therefore, an infinite number of equivalent systems of equations correspond to a line in space.

Definition 2. *Two equations (taken together) relating the coordinates x, y, z are called the equations of a line in the space $Oxyz$ if the following two conditions are fulfilled:* (1) *the coordinates of any point of the line satisfy both equations*; (2) *the coordinates of any point not lying on the line do not satisfy both equations at once* (although they may satisfy one of them).

Example 3. (The equations of the coordinate axes.)

The x-axis may be considered as the intersection of the coordinate planes Oxy and Oxz. Therefore,

$$\left.\begin{aligned} y&=0,\\ z&=0 \end{aligned}\right\}$$

are the *equations of the x-axis.*

Analogously,

$$\left.\begin{aligned} x&=0,\\ z&=0 \end{aligned}\right\}$$

are the *equations of the y-axis* and

$$\left.\begin{aligned} x&=0,\\ y&=0 \end{aligned}\right\}$$

are the *equations of the z-axis.*

Example 4. Write the equations of the circle Γ of radius $R = 1$

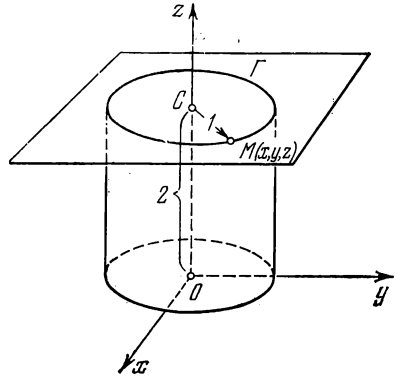


FIG. 193

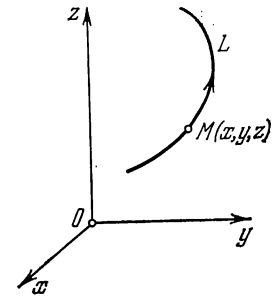


FIG. 194

whose centre is situated at point C (0, 0, 2) and whose plane is parallel to the coordinate plane Oxy (Fig. 193).

The circle Γ may be regarded as the intersection of a circular cylinder of radius 1 whose axis is Oz and a horizontal plane situated two units higher than the coordinate plane Oxy. Therefore, the equations of the given circle are

$$\left.\begin{aligned} x^2+y^2&=1,\\ z&=2. \end{aligned}\right\}$$

In mechanics the line L is frequently considered as the trace of a moving point (Fig. 194). Let x, y, z be the current coordinates of a point M belonging to the line L. Since the point M displaces and its coordinates change with time, they are functions of time t. Hence, we have

$$x=\varphi(t),\quad y=\psi(t),\quad z=\chi(t), \tag{161-5}$$

where φ, ψ, χ are definite functions. Generalizing these equations,

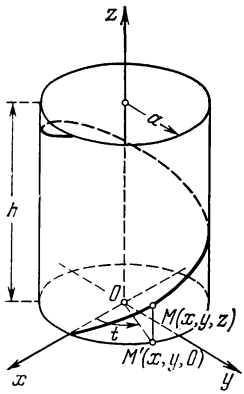


FIG. 195

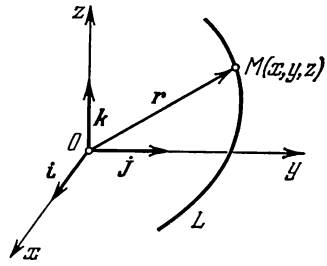


FIG. 196

we regard t as an auxiliary variable (*parameter*), not necessarily time; therefore, equations (161-5) are termed the *parametric equations of a line in space.*

Eliminating the parameter t from these equations, we shall obtain two relations for the current coordinates x, y, and z which represent the equations of certain surfaces passing through the given line.

Example 5. Write the equations of a helix of radius a and lead h (Fig. 195).

Let $M(x, y, z)$ be a moving point of the helix and $M'(x, y, 0)$ its projection on the plane Oxy.

Taking the angle $M'Ox$ for the parameter ($t=\angle M'Ox$) and taking into account that the z-coordinate of the helix increases propor-

tionally to the angle of rotation t, we shall have

$$\left.\begin{aligned} x &= a\cos t, \\ y &= a\sin t, \\ z &= bt. \end{aligned}\right\} \tag{161-6}$$

To determine the proportionality factor b, let us set $t = 2\pi$; then $z = h$. Hence,

$$h = 2\pi b \quad \text{and} \quad b = \frac{h}{2\pi}.$$

Eliminating the parameter t from the first and second equations, and also from the first and third equations of (161-6), we get

$$\left.\begin{aligned} x^2 + y^2 &= a^2, \\ x &= a\cos\frac{z}{b}. \end{aligned}\right\} \tag{161-6'}$$

Consequently, the helix represents the intersection of a circular cylinder with the generatrix parallel to the z-axis and a cylindrical surface whose generatrix is parallel to the y-axis and whose directrix is a cosine curve contained in the plane Oxz. It also follows from equations (161-6') that the projection of helix (161-6') on the coordinate plane Oxy is a circle, and its projection on the coordinate plane Oxz is a cosine curve.

A moving point $M(x, y, z)$ of the line L can be characterized by its adius vector (a "tracking" radius vector) (Fig. 196)

$$= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ unit vectors). Then from (161-5) we obtain the vector equation of a line

$$\mathbf{r} = \mathbf{f}(t), \tag{161-7}$$

where

$$\mathbf{f}(t) = \mathbf{i}\varphi(t) + \mathbf{j}\psi(t) + \mathbf{k}\chi(t)$$

which is the so-called *vector function of a scalar argument t.*

In mechanics time is usually taken as the parameter t. In this case line (161-7) is called the *path* of the point $M(x, y, z)$.

The set of all points $M(x, y, z)$ in space whose coordinates satisfy a given equation (or a system of equations) is called the *geometrical image* (*graph*) of the given equation (or the system of equations).

Example 6. What geometrical image corresponds to the equation

$$z^2 - 1 = 0? \tag{161-8}$$

From equation (161-8) we obtain: $z = 1$ or $z = -1$. Hence, the graph of this equation is a pair of planes parallel to the coordinate

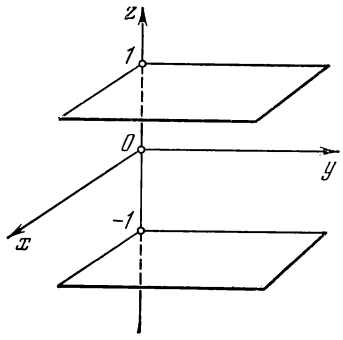


FIG. 197

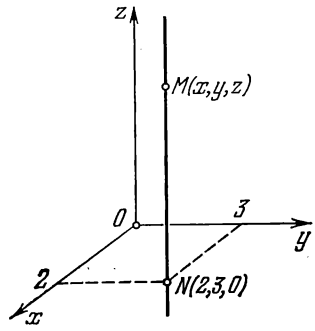


FIG 198

plane Oxy and passing from it at a distance of unity (Fig. 197).

Example 7. What geometrical image corresponds to the following pair of equations

$$x = 2, \quad y = 3?$$

The desired graph represents the intersection of the planes $x = 2$ and $y = 3$ and, consequently, is a straight line parallel to the z-axis and having a trace N (2, 3, 0) on the coordinate plane Oxy (Fig. 198).

Sec. 162. The General Equation of a Plane

A plane P in space can be specified by some of its points M_0 (x_0, y_0, z_0) and a non-zero vector $\mathbf{N}\{A, B, C\}$ $(A^2 + B^2 + C^2 \neq 0)$ perpendicular to this plane (a *normal* or *direction vector* of a plane). Let $\mathbf{r}_0 = \{x_0, y_0, z_0\}$ be the radius vector of the point M_0, and $\mathbf{r} = \{x, y, z\}$ the radius vector of an arbitrary point M of the plane (the current radius vector) (Fig. 199). Then the vector $\mathbf{r} - \mathbf{r}_0 = \{x - x_0, y - y_0, z - z_0\}$ is situated in the given plane and, consequently, is normal to the vector $\mathbf{N}$, i.e. $\mathbf{N} \perp (\mathbf{r} - \mathbf{r}_0)$. Hence, using the perpendicularity condition for two vectors (Sec. 156), we have

$$\mathbf{N} \cdot (\mathbf{r} - \mathbf{r}_0) = 0. \tag{162-1}$$

This is just the *equation of a plane in the vector form.* And this is the coordinate form of equation (162-1):

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0, \tag{162-2}$$

or

$$Ax + By + Cz + D = 0, \tag{162-3}$$

where

$$D = -Ax_0 - By_0 - Cz_0 \equiv -\mathbf{N}\mathbf{r}_0.$$

Equation (162-3) is called the *general equation of a plane* and represents an equation of the first degree with respect to the current coordinates x, y, z. Thus, the *plane is a surface of the first order.*

Conversely, let there be given a non-degenerate equation (162-3) ($A^2 + B^2 + C^2 \neq 0$). We choose a certain point $M_0(x_0, y_0, z_0)$ lying on surface (162-3) (for instance, if $A \neq 0$, then we may take the point $M_0\left(-\frac{D}{A}, 0, 0\right)$ and so on). We have

$$Ax_0 + By_0 + Cz_0 + D = 0. \tag{162-4}$$

Subtracting equation (162-4) from equation (162-3), we shall have

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{162-5}$$

Hence, introducing the vectors $\mathbf{N} = \{A, B, C\}$, $\mathbf{r}_0 = \{x_0, y_0, z_0\}$ and $\mathbf{r} = \{x, y, z\}$, we get

$$\mathbf{N}(\mathbf{r} - \mathbf{r}_0) = 0.$$

Hence, the surface specified by equation (162-3) is a plane passing through the point $M_0(x_0, y_0, z_0)$ perpendicular to the vector $\mathbf{N}$.

Example 1. Find the angle formed by the plane

$$x - 2y + 2z - 10 = 0$$

with the z-axis.

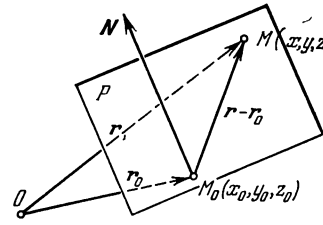


FIG. 199

The angle ψ between a straight line and a plane is understood to be the angle between the given line and its projection on this plane. This angle is a complementary one to the angle φ formed by the straight line and a perpendicular (normal) to the plane.

The normal vector of our plane is $\mathbf{N} = \{1, -2, 2\}$. Hence

$$\sin\psi = \cos\varphi = \frac{2}{\sqrt{1^2 + (-2)^2 + 2^2}} = \frac{2}{3}$$

and, consequently,

$$\psi = \arcsin\frac{2}{3} \approx 41°50'.$$

If in equation (162-1) we take the unit vector as the direction vector of the plane

$$\mathbf{n} = \frac{\mathbf{N}}{N} \qquad (N = \sqrt{A^2+B^2+C^2} \neq 0),$$

then we shall obtain the so-called *normalized equation of the plane*

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0, \tag{162-6}$$

or in the coordinate form

$$\frac{Ax+By+Cz+D}{\sqrt{A^2+B^2+C^2}} = 0, \tag{162-7}$$

where $D = -Ax_0 - By_0 - Cz_0$. Equation (162-7) is suitable in finding the distance of a point from a plane.

Problem. Find the distance h of the point $M_1(x_1, y_1, z_1)$ from the plane P specified by equation (162-6) (Fig. 200).

Let $h = M_1N_1$, where $M_1N_1 \perp P$, $N_1 \in P$.

Consider the vector $\overrightarrow{M_0M_1} = \mathbf{r}_1 - \mathbf{r}_0$, where $\mathbf{r}_0$ and $\mathbf{r}_1$ are the radius vectors of the points $M_0 \in P$ and M_1. From the triangle $M_0M_1N_1$, taking into account that M_1N_1 is parallel to $\mathbf{h}$, we find

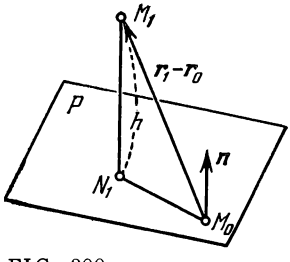


FIG. 200

$$h = |\mathrm{pr}_n(\mathbf{r}_1 - \mathbf{r}_0)| = |\mathbf{n} \cdot (\mathbf{r}_1 - \mathbf{r}_0)| = \frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}.$$

Consequently, we obtain the following rule: *To find the distance from a point to a plane, we have to substitute the coordinates of the given point into the left-hand side of a normalized equation of the plane and take the absolute value of the result thus obtained.*

In particular, setting $x_1 = 0$, $y_1 = 0$, $z_1 = 0$, we obtain the distance of a plane from the origin:

$$h_0 = \frac{|D|}{\sqrt{A^2+B^2+C^2}}.$$

Sec. 163. Angle Between Two Planes

Let there be given two planes

$$Ax + By + Cz + D = 0, \quad A'x + B'y + C'z + D' = 0 \tag{163-1}$$

with the direction vectors $\mathbf{N}=\{A, B, C\}$ and $\mathbf{N}'=\{A', B', C'\}$. Then φ (the dihedral angle between them) is equal to the angle formed

by the vectors N and N'. Thus, we have (see Sec. 156 of the preceding chapter)

$$\cos\varphi = \frac{\mathbf{NN'}}{NN'}, \tag{163-2}$$

where $\mathbf{NN'} = AA' + BB' + CC'$ and $N = \sqrt{A^2+B^2+C^2}$, $N' = \sqrt{A'^2+B'^2+C'^2}$.

Hence, we obtain: (1) *the condition of parallelism of planes* (in a broad sense)

$$\frac{A'}{A} = \frac{B'}{B} = \frac{C'}{C} \tag{163-3}$$

and (2) *the condition of their perpendicularity*

$$AA' + BB' + CC' = 0. \tag{163-4}$$

Note that if for the given planes condition (163-3) is not fulfilled, then these planes are neither parallel nor coincident, i.e. they intersect.

Example. Determine the angle φ between two bisector planes $x - z = 0$, $y - z = 0$.

Here

$$\mathbf{N} = \{1, 0, -1\}, \quad \mathbf{N'} = \{0, 1, -1\}.$$

We have

$$\cos\varphi = \frac{1\cdot 0 + 0\cdot 1 + (-1)\cdot(-1)}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2}$$

and, consequently $\varphi = 60°$.

Sec. 164. Equations of a Straight Line in Space

A straight line in space is uniquely defined by a point $M_0(x_0, y_0, z_0)$ and a certain direction (i.e. by some vector).

Let $\mathbf{r}_0 = \{x_0, y_0, z_0\}$ be the radius vector of the point M_0 and $\mathbf{s} = \{l, m, n\}$ be a non-zero direction vector of the straight line (its length is arbitrary). Denoting by $\mathbf{r} = \{x, y, z\}$ the radius vector of an arbitrary point M of the line (the current radius vector), from the vector triangle OM_0M (Fig. 201) we have

$$\mathbf{r} = \mathbf{r}_0 + \overrightarrow{M_0M}. \tag{164-1}$$

Since the vectors $\overrightarrow{M_0M}$ and s are collinear, we have

$$\overrightarrow{M_0M} = t\mathbf{s}, \tag{164-2}$$

where t is a certain scalar ($-\infty < t < +\infty$). Substituting this expression into equation (164-1), we shall obtain the *vector equation of a straight line in space*

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{s} \tag{164-3}$$

(t parameter).

Projecting equation (164-3) on the coordinate axes, we shall have the *parametric equations of a straight line in space*

$$\left.\begin{aligned} x &= x_0 + lt, \\ y &= y_0 + mt, \\ z &= z_0 + nt. \end{aligned}\right\} \tag{164-4}$$

Eliminating the parameter t from the above equations, we shall obtain the so-called *canonical equations of a straight line in space*

$$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n} \ *. \tag{164-5}$$

System (164-5) contains two equations, for instance, for $n \neq 0$ we may put

$$\frac{x-x_0}{l} = \frac{z-z_0}{n}, \quad \frac{y-y_0}{m} = \frac{z-z_0}{n}.$$

These equations represent the equations of two planes whose intersection is the given straight line. Note that the first equation has no coordinate y, and the second no coordinate x. Consequently, the first plane is parallel to the y-axis, and the second to the x-axis, i.e. these planes are the planes projecting our straight line on the coordinate planes Oxz and Oyz, respectively.

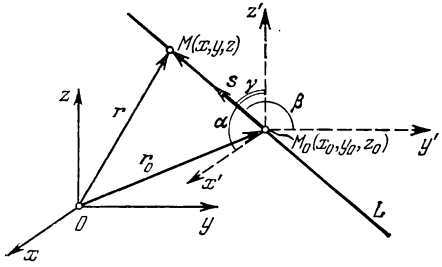


FIG. 201

The numbers l, m, n are called the *direction numbers of a straight line*. Denoting by α, β, γ the angles formed by the straight line with the coordinate axes (Fig. 201) and taking into consideration that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the vector s, we shall have

$$l = s\cos\alpha, \quad m = s\cos\beta, \quad n = s\cos\gamma, \tag{164-6}$$

* These equations have a sense of proportions, i.e. some (not more than two) of the numbers l, m, n may be zeros.

where

$$s = \sqrt{l^2 + m^2 + n^2} \neq 0 \tag{164-7}$$

is the length of the vector s. Hence we obtain

$$\cos\alpha = \frac{l}{s}, \quad \cos\beta = \frac{m}{s}, \quad \cos\gamma = \frac{n}{s}$$

$(\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1)$.

Thus, the direction numbers of a straight line are proportional to the corresponding direction cosines of this line.

The equations of straight line (164-5) can be written in the standard form

$$\frac{x - x_0}{\cos\alpha} = \frac{y - y_0}{\cos\beta} = \frac{z - z_0}{\cos\gamma}, \tag{164-5'}$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the straight line.

Example 1. The equations of motion of a rocket are

$$\left.\begin{aligned} x &= 2t, \\ y &= -4t, \\ z &= 4t, \end{aligned}\right\}$$

where the time t is given in seconds, and the coordinates of a moving point in kilometres.

What is the path of the rocket? At what distance will the rocket be from the starting point O (0, 0, 0) in 10 seconds?

Eliminating the time t from the given equations, we get the equation of the path

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{4},$$

or

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{2}.$$

Thus, the path of the rocket represents a straight line passing through the origin.

For $t = 10$ s we have: $x = 20$, $y = -40$, $z = 40$ and

$$r = OM = \sqrt{x^2 + y^2 + z^2} = \sqrt{400 + 1600 + 1600} = \sqrt{3600} = 60 \text{ km}.$$

Problem. Write the equation of a straight line passing through two non-coincident points: M_0 (x_0, y_0, z_0) and M_1 (x_1, y_1, z_1).

We take

$$\mathbf{s} = \{x_1 - x_0, y_1 - y_0, z_1 - z_0\} \neq 0$$

for the direction vector of the desired line.

Hence, by (164-5), we have

$$\frac{x-x_0}{x_1-x_0}=\frac{y-y_0}{y_1-y_0}=\frac{z-z_0}{z_1-z_0}. \tag{164-8}$$

Example 2. Write the equation of the straight line passing through the point M_0 (3, —4, 5) parallel to the z-axis.

Obviously, we have

$$\cos\alpha = 0, \quad \cos\beta = 0, \quad \cos\gamma = 1.$$

Thus, by (164-5′), we obtain the equations of the required straight line

$$\frac{x-3}{0}=\frac{y+4}{0}=\frac{z-5}{1}, \tag{164-9}$$

equivalent to the pair of equations

$$\frac{x-3}{0}=\frac{z-5}{1}, \quad \frac{y+4}{0}=\frac{z-5}{1}$$

or

$$x-3=0, \quad y+4=0$$

The direction vector of straight line (164-9) is $\{0, 0, 1\}$, i.e. this line is perpendicular to the x- and y-axes.

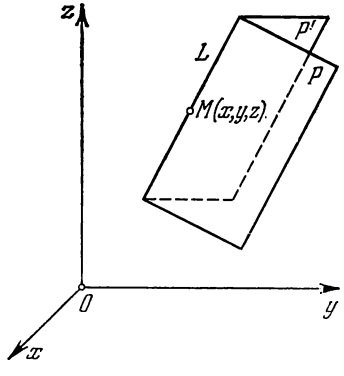


FIG. 202

The straight line L in space can be also specified as the line of intersection of two planes: P and P' (Fig. 202):

$$\left.\begin{aligned} Ax+By+Cz+D=0,\\ A'x+B'y+C'z+D'=0. \end{aligned}\right\} \tag{164-10}$$

It is assumed that the planes are neither parallel, nor coincident (see Sec. 159). The vectors $\mathbf{N}=\{A, B, C\}$ and $\mathbf{N}'=\{A', B', C'\}$ are the normal vectors of these planes. The direction vector $\mathbf{s}$ of the straight line, obviously, satisfies the conditions: $\mathbf{s}\perp\mathbf{N}$ and $\mathbf{s}\perp\mathbf{N}'$. We may put

$$\mathbf{s}=\mathbf{N}\times\mathbf{N}'$$

where $\times$ is the symbol of a vector product (see Sec. 159).

Example 3. Determine the direction cosines of the straight line

$$\left.\begin{aligned} x-2y+3z-4=0,\\ 3x-2y+z=0. \end{aligned}\right\}$$

We have

$$\mathbf{N}=\{1,\ -2,\ 3\},\quad \mathbf{N}'=\{3,\ -2,\ 1\}.$$

Hence

$$\mathbf{s}=\mathbf{N}\times\mathbf{N}'=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 3\\ 3 & -2 & 1 \end{vmatrix}=4\mathbf{i}+8\mathbf{j}+4\mathbf{k}=4(\mathbf{i}+2\mathbf{j}+\mathbf{k}).$$

We may take $s_0=\frac{1}{4}s=\{1,\ 2,\ 1\}$ for the direction vector of the line. Its length is $s_0=\sqrt{6}$. Hence

$$\cos\alpha=\frac{1}{\sqrt{6}},\quad \cos\beta=\frac{2}{\sqrt{6}},\quad \cos\gamma=\frac{1}{\sqrt{6}}.$$

Sec. 165. The Derivative of a Vector Function

Suppose we have a vector function

$$\mathbf{r}(t)=x(t)\,\mathbf{i}+y(t)\,\mathbf{j}+z(t)\,\mathbf{k} \qquad (165\text{-}1)$$

($\alpha<t<\beta$), where, for the sake of convenience, the functions of the parameter t representing the projections of the vector $\mathbf{r}(t)$ on the coordinate axes are denoted by the corresponding letters (cf. Sec. 161). If $\mathbf{r}(t)$ is treated as the radius vector of the point $M(x, y, t)$ of the space $Oxyz$, then the terminus of the variable vector $\mathbf{r}(t)$ will describe a certain curve K of the space $Oxyz$ whose parametric equation is vector equation (165-1). In mechanics this curve is called the *hodograph* of the variable vector $\mathbf{r}(t)$.

Naturally, we are going to determine the *limit of the vector function*, putting

$$\lim_{t\to t_0}\mathbf{r}(t)=\mathbf{i}\lim_{t\to t_0}x(t)+\mathbf{j}\lim_{t\to t_0}y(t)+\mathbf{k}\lim_{t\to t_0}z(t), \qquad (165\text{-}2)$$

if there exist the limits which stand in the right-hand side of equality (165-2).

Let us assign an increment Δt to the parameter t; then the point $M(x, y, z)$ of the curve K will displace to the point $M'(x+\Delta x, y+\Delta y, z+\Delta z)$, whose radius vector is

$$\mathbf{r}_1(t)=\mathbf{r}(t)+\Delta\mathbf{r}(t).$$

From the vector triangle OMM' (Fig. 203) we have

$$\Delta \mathbf{r}(t) = \overrightarrow{MM'} = \Delta x\mathbf{i} + \Delta y\mathbf{j} + \Delta z\mathbf{k}.$$

Hence assuming for definiteness that $\Delta t > 0$, we get

$$\frac{\Delta \mathbf{r}(t)}{\Delta t} = \frac{\Delta x}{\Delta t}\mathbf{i} + \frac{\Delta y}{\Delta t}\mathbf{j} + \frac{\Delta z}{\Delta t}\mathbf{k}, \tag{165-3}$$

i.e. the vector $\frac{\Delta \mathbf{r}(t)}{\Delta t}$ is directed along the secant $\overrightarrow{MM'}$.

In the general case for $\Delta t \neq 0$ the vector $\frac{\Delta \mathbf{r}(t)}{\Delta t}$ will be collinear with the vector $\overrightarrow{MM'}$.

Definition. *The derivative of a vector function* $\mathbf{r} = \mathbf{r}(t)$ *is the vector*

$$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}(t)}{\Delta t}. \tag{165-4}$$

If $x(t)$, $y(t)$, $z(t)$ are differentiable functions, then from formula (165-3), as $\Delta t \to 0$, we find

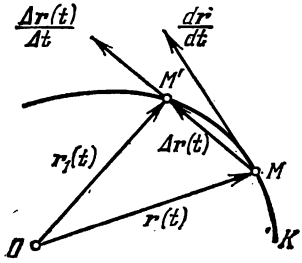


FIG. 203

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}. \tag{165-5}$$

Since the limit position of a secant is, by definition, a tangent, the vector $\frac{d\mathbf{r}}{dt}$ is directed along the *tangent to the curve* K at its point M (towards the increase of the parameter t).

From formula (165-5), as usually, we obtain

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}. \tag{165-6}$$

If t is time, then the vector $\frac{d\mathbf{r}}{dt} = \mathbf{v}$ represents the velocity of a moving point $M(x, y, z)$, which is understood as a vector.

Example. Write the equation of a tangent to a curve

$$x = t, \quad y = t^2, \quad z = t^3$$

at its point $M(1, 1, 1)$ $(t = 1)$.

Here

$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

and

$$\frac{d\mathbf{r}}{dt}=\mathbf{i}+2t\mathbf{j}+3t^2\mathbf{k}.$$

Hence, the direction of the tangent at the point M is determined by the vector

$$\left(\frac{d\mathbf{r}}{dt}\right)_M=\mathbf{i}+2\mathbf{j}+3\mathbf{k}.$$

Thus, the equation of the desired tangent is

$$\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}.$$

Sec. 166. The Equation of a Sphere

Definition. *A sphere of radius R is the set of points in space equidistant from a given point, called the centre. The distance of any point of the set to the centre is equal to R.*

Let us derive the equation of a sphere. Let C (x_0, y_0, z_0) be the centre of a sphere of radius R, and M (x, y, z) an arbitrary point lying on this sphere (Fig. 204). Then

$$CM = R.$$

By the formula for the distance between two points, we have

$$CM=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}.$$

Equating this expression to R, we get the equation of the sphere

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}=R$$

or, finally,

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2.$$

If the centre of a sphere coincides with the origin, then $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ and the equation attains the form

$$x^2+y^2+z^2=R^2.$$

Example 1. Determine the coordinates of the centre and the radius of the sphere

$$x^2+y^2+z^2-2y-3z=0.$$

Collecting the terms having like current coordinates and completing them to perfect squares, we shall have

$$x^2+(y-1)^2+\left(z+\frac{3}{2}\right)^2=\frac{13}{4}.$$

Consequently, the centre of the sphere is at the point $C\left(0,\ 1,\ -1\frac{1}{2}\right)$ and its radius is

$$R=\frac{1}{2}\sqrt{13}.$$

Note that the set

$$\left.\begin{aligned}(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2,\\ Ax+By+Cz+D=0\end{aligned}\right\}$$

of the equations of a sphere and a plane defines a circle in which

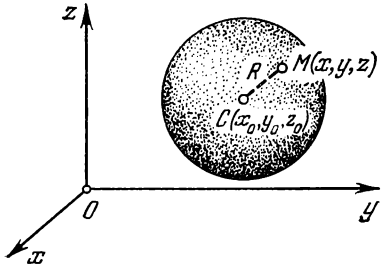


FIG. 204

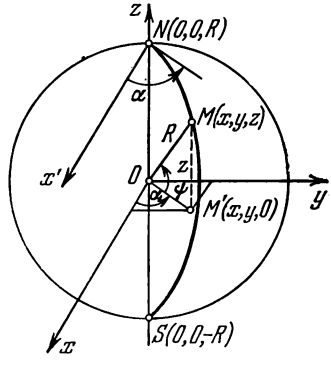


FIG. 205

they intersect (if this set is not empty). In particular, if $Ax_0 + By_0 + Cz_0 + D = 0$, then the system of these equations represents the circumference of the great circle.

The equation of a circle can also be written in the parametric form.

Example 2. Write the parametric equations of the meridian of the sphere

$$x^2 + y^2 + z^2 = R^2$$

passing through the poles N (0, 0, R) and S (0, 0, $-R$) if the plane containing this meridian forms an angle α with the coordinate plane Oxz (Fig. 205).

We take the angle $\psi = \angle MOM'$ (i.e. the latitude of a current point M (x, y, z) of the desired meridian) for the parameter of this point, where M' (x, y, 0) is the projection of the point M on the coordinate plane Oxy. Since $M'O = r = R\cos\psi$, we have (from

Fig. 205):

$$\left.\begin{aligned} x &= r\cos\alpha = R\cos\psi\cos\alpha, \\ y &= r\sin\alpha = R\cos\psi\sin\alpha, \\ z &= R\sin\psi, \end{aligned}\right\}$$

where

$$-\frac{\pi}{2}\leqslant\psi\leqslant\frac{\pi}{2}.$$

Sec. 167. The Equation of an Ellipsoid

Definition. *A three-axis ellipsoid is a surface obtained as the result of uniform deformation (expansion or compression) of a sphere along three mutually perpendicular directions* (cf. Sec. 24, Chapter 4).

Consider the sphere of radius R with centre at the origin:

$$X^2 + Y^2 + Z^2 = R^2, \tag{167-1}$$

where X, Y, Z are the current coordinates of a point on the sphere.

Let the given sphere be uniformly deformed along the directions of the coordinate axes Ox, Oy, and Oz with the coefficients of deformation equal to k_1, k_2, and k_3*.

As the result, the sphere becomes an ellipsoid, and the point M (X, Y, Z) of the sphere with the current coordinates X, Y, Z goes into the point M' (x, y, z) of the ellipsoid with the current coordinates x, y, z, and

$$\left.\begin{aligned} x &= k_1X, \\ y &= k_2Y, \\ z &= k_3Z \end{aligned}\right\}$$

(Fig. 206). Hence

$$X=\frac{x}{k_1},\quad Y=\frac{y}{k_2},\quad Z=\frac{z}{k_3}.$$

Substituting these formulas into equation (167-1), we shall have

$$\frac{x^2}{k_1^2}+\frac{y^2}{k_2^2}+\frac{z^2}{k_3^2}=R^2$$

or

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \tag{167-2}$$

* In other words, the linear dimensions of the sphere in the direction of the x-axis are reduced by $1/k_1$ times if $0 < k_1 \leqslant 1$, and increased by k_1 times if $k_1 > 1$, and so on.

where

$$a = k_1R, \quad b = k_2R, \quad c = k_3R.$$

Equation (167-2) relates the current coordinates of the point M' of the ellipsoid and, consequently, is the *equation of the three-axis ellipsoid.*

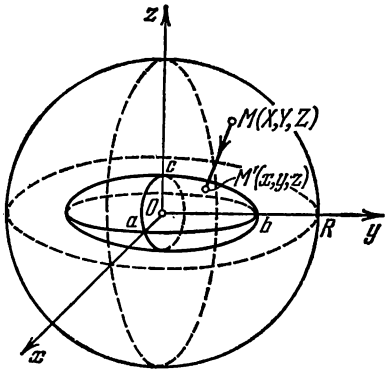


FIG. 206

The quantities a, b, c are called the *semi-axes* of the ellipsoid; the doubled quantities $2a$, $2b$, and $2c$ are called the *axes of the ellipsoid* and, obviously, represent its linear dimensions in the directions of deformation (in this case in the directions of the coordinate axes).

If two semi-axes of an ellipsoid are equal to each other, then such an ellipsoid is called the *ellipsoid of revolution*, since it can be obtained by revolving about one of its axes. For instance, in geodesy the surface of the Earth is regarded as an ellipsoid of revolution with the semi-axes

$$a = b = 6377 \text{ km} \quad \text{and} \quad c = 6356 \text{ km}.$$

If $a = b = c$, then ellipsoid becomes a sphere.

Sec. 168. The Equation of a Paraboloid of Revolution

Let the vertical parabola

$$X^2 = 2pZ \tag{168-1}$$

contained in the plane Oxz rotate about its axis Oz, thus generating a surface called the *paraboloid of revolution* (Fig. 207).

To derive the equation of the obtained surface, let us consider an arbitrary point M (x, y, z) of the paraboloid of revolution and let this point be obtained as the result of rotation of the point N $(X, 0, Z)$ of the given parabola about the point C $(0, 0, Z)$.

Since the points M and N are situated in one and the same horizontal plane and $CN = CM$ as radii of one and the same circle, we have

$$\left.\begin{aligned} X &= \sqrt{x^2+y^2}, \\ Z &= z. \end{aligned}\right\} \tag{168-2}$$

Substituting formulas (168-2) into equation (168-1), we obtain the *equation of the paraboloid of revolution*

$$x^2 + y^2 = 2pz.$$

Note that the surface of mercury contained in a vertical cylindri-

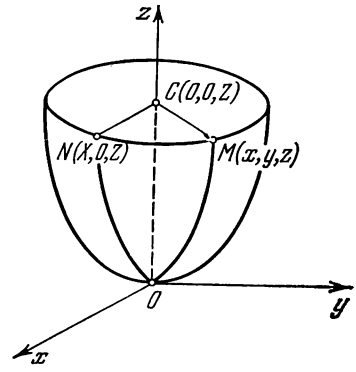


FIG. 207

cal vessel rapidly revolving about its axis has the shape of the paraboloid of revolution. This fact is of importance in manufacturing parabolic mirrors.

Exercises

1. What geometrical images in space correspond to the given equations:

(a) $xy = 0$;

(b) $xz = yz$;

(c) $y^2 + y - 2 = 0$;

(d) $z^2 = 2x$;

(e) $y = 1,\ z = -2$;

(f) $x^2 = 0$;

(g) $x^2 + y^2 = 0$;

(h) $x^2 + y^2 + z^2 = 0$.

2. Determine the length of a perpendicular dropped from the origin onto the plane

$$x - y + z\sqrt{2} - 8 = 0$$

and the angles formed by this perpendicular with the coordinate axes.

3. Write the equation of the plane parallel to the axis Oz and intercepting on the axes Ox and Oy segments of length 2 and 3, respectively.

4. Write the equation of the plane passing through the point M (1, 2, 3) perpendicular to the z-axis.

5. The function $z = f(x, y)$ is given in the tabular form:

y \ x	70	80
10	1.23	1.34
40	1.40	

Evaluate z for $x = 72$ and $y = 20$ regarding the function z as linear, i.e. of the first degree with respect to x and y.

6. What angles are formed by the straight line

$$\frac{x-1}{1} = \frac{y+2}{-1} = \frac{z}{-\sqrt{2}}$$

with the coordinate axes?

7. Find the traces (i.e. the points of intersection) of the straight line

$$\frac{x+1}{2} = \frac{y-1}{-3} = \frac{z+2}{4}$$

on the coordinate planes.

8. Write the equation of the sphere with centre at point C (2, —2,1) passing through the origin.

9. Write the parametric equations of a parallel of the sphere

$$x^2 + y^2 + z^2 = R^2$$

with the angular distance $\alpha \left(|\alpha| < \frac{\pi}{2} \right)$ from the equator $z = 0$, taking the longitude φ of the current point $M(x, y, z)$ for the parameter.

10. Determine the semi-axes of the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 4.$$

11. Determine the semi-axes of the ellipse formed by the intersection of the ellipsoid

$$\frac{x^2}{36} + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

and the plane $x = 3$.

12. Write the equation of the tangent to the helix

$$x = a \cos t, \quad y = a \sin t, \quad z = bt$$

at its point M_0 corresponding to the parameter $t = t_0$.

What angle is formed by this tangent with the z-axis?

Find the projections of the helix on the coordinate planes.

Chapter 20

Functions of Several Variables

Sec. 169. The Concept of a Function of Several Variables

In many problems of natural science we have to deal with functions of two, three, and more variables. Let us give some examples.

Example 1. The area of a triangle

$$U = \frac{1}{2} xy$$

with the base x and altitude y is a function of two variables x and y defined in the domain $x > 0$ and $y > 0$.

Example 2. Resolving the equation of a sphere

$$x^2 + y^2 + z^2 = R^2$$

with respect to z, for $z \geqslant 0$ we get

$$z = \sqrt{R^2 - x^2 - y^2}.$$

Here the z-coordinate of a point belonging to the upper hemisphere is a function of two variables x and y, i.e. of the abscissa and ordinate of this point. This function is defined in the circle $x^2 + y^2 \leqslant R^2$.

Example 3. The volume of a rectangular parallelepiped

$$V = xyz$$

with the dimensions x, y, and z is a function o three variables defined in the positive octant of the space $Oxyz$.

Example 4. According to Newton's law, the magnitude of the attraction force F of two mass points (m and m_1) occupying the positions M (x, y, z) and M_1 (x_1, y_1, z_1), respectively, is equal to

$$F = k \frac{mm_1}{(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2},$$

where k is a certain constant (the so-called "attraction constant"). Consequently, F is a function of six variables: x, y, z, x_1, y_1, z_1.

Let us draw reader's attention to one important thing: *any function of several variables becomes a function of less variables if some of them are fixed*, i.e. assigned constant values.

For instance, let us have a function

$$u = f(x, y, z)$$

of three variables x, y, and z.

If we assume that z retains a constant value (i.e. remains unchanged)

$$z = c,$$

then we obtain a function of two variables x and y:

$$u = f(x, y, c).$$

Furthermore, assuming that two variables y and z preserve unchanged values

$$y = b \quad \text{and} \quad z = c,$$

we get a function

$$u = f(x, b, c)$$

of one variable x.

Thus, in different situations, according to our desire, the function u may be regarded as a function of one, two or three variables.

Strictly speaking, almost any physical relation gives us an example of a function of rather a large number of variables. But in studying this relation, we ignore a number of unessential factors and thus limit the number of variables, reducing it to a minimum.

For instance, the distance s covered by a freely falling body during the time t depends on the following variables: the time of falling t, the area of the cross-section of the body Q, the latitude of the location φ, elevation above sea level h, air pressure P, ambient temperature T, coefficient of air viscosity η, and so on. Hence, we must write

$$s = f(t, Q, \varphi, h, P, T, \eta, \ldots).$$

In the first approximation, all the variables, except for the time t, are unessential. Ignoring them, we get

$$s = f(t)$$

and, thus, arrive at a known formula

$$s = \frac{gt^2}{2},$$

where g is acceleration due to gravity which is considered to be constant.

If only partly we take into consideration the role of other variables, then we shall have more and more precise formulas for s depending on the increasing number of the variables.

The *geometric representation* (*graph*) of a function of two variables

$$z = f(x, y) \tag{169-1}$$

is, generally speaking, a surface in the space $Oxyz$.

Indeed, let the given function be defined in a certain domain ω of the plane Oxy. Then to each pair of values x and y from the domain ω there corresponds, by (169-1), a certain value of z. In other words, each point $N(x, y, 0) \in \omega$ is associated with the point $M(x, y, z)$ belonging to the graph of the function and which is the end-point of the perpendicular NM to the plane Oxy ($MN \perp Oxy$).

If the point N occupies all possible positions exhausting the domain ω, then the point M associated with it will, in general case, describe in space some surface P* hanging over the domain ω. Obviously, we can imagine that P is a "roof" constructed over the area ω. The surface P is just the geometrical representation of function (169-1) (Fig. 208). Geometrical representations of functions of three and more variables have no simple geometrical sense.

In some cases we can obtain an obvious geometrical representation of the character of change of a function by considering its *level lines* (or *level surfaces*), i.e. lines (or surfaces) where a given function preserves a constant value.

Definition 1. *The level line of the function*

$$z = f(x, y)$$

is the set of all points of the xy-plane for which the given function has one and the same value (*isoline*).

Thus, the equation of a level line is

$$f(x, y) = C,$$

where C is a constant.

Example. Construct a family of level lines of the function

$$z = x^2 + y^2.$$

Assigning to z non-negative values $z = 0, 1, 2, \ldots$ (z, obviously, cannot be negative), we shall obtain the respective equations of the level lines of the given function: $x^2 + y^2 = 0$ is a point $O(0, 0)$; $x^2 + y^2 = 1$ a circle of radius $R = 1$ with centre $O(0, 0)$; $x^2 + y^2 = 2$, a circle of radius $R = \sqrt{2}$ with centre $O(0, 0)$, and so on.

Thus, the level lines of our function represent a family of concentric circles with centre O. Having constructed these lines, we get a

* Here we are speaking of simple elementary functions.

"surface map" for the given function with the altitudes (elevations) indicated (Fig. 209).

Figure 209 gives a distinct picture of how the function z increases along each radial direction. Therefore, in the xyz-space the geometrical image of the function represents a giant "pit" with steeply increasing edges. Theoretically, this is a paraboloid of revolution (see Sec. 168).

Definition 2. *The level surface of the function*

$$u = f(x, y, z)$$

is the set of all points of the Oxyz-space for which the given function has one and the same value (isosurfaces).

Level lines and level surfaces are frequently encountered in physical problems, as well as in other fields of science. For instance,

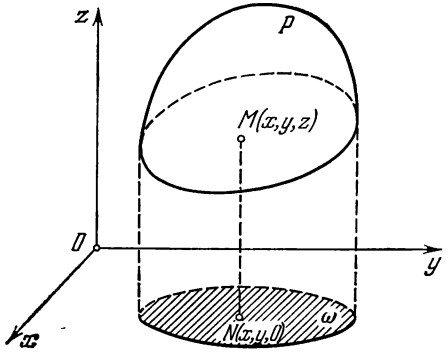


FIG. 208

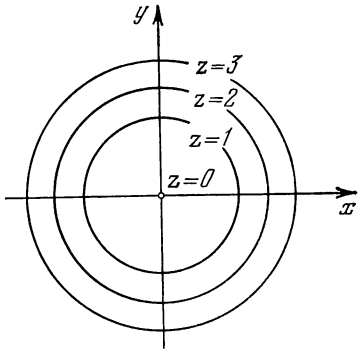


FIG. 209

joining on a map of Earth's surface the points with equal daily average pressure or with equal daily average temperature, we shall obtain isobars and isotherms, respectively, which are important initial data for weather forecast.

Sec. 170. Continuity

Let $z = f(x, y)$ be a function of two variables x and y, a pair of whose values (x, y) we shall call, for brevity, a point. Thus, z is a function of the "point".

Now we assign an increment Δx to the variable x, leaving the variable y unchanged. Then the difference

$$\Delta_x z = f(x + \Delta x, y) - f(x, y) \qquad (170\text{-}1)$$

is called the *partial increment of the function* $f(x, y)$ *with respect to the variable* x. Hence, we may write

$$\Delta_x f(x, y) = f(x + \Delta x, y) - f(x, y). \tag{170-2}$$

Analogously, if only the variable y receives an increment Δy, and the variable x remains unchanged, then the difference

$$\Delta_y f(x, y) = f(x, y + \Delta y) - f(x, y) \tag{170-2'}$$

is called the *partial increment of the function* $f(x, y)$ *with respect to the variable* y.

Finally, it may happen so that both variables x and y receive the increments Δx and Δy, respectively. Then the corresponding increment of the function

$$\Delta f(x, y) = f(x + \Delta x, y + \Delta y) - f(x, y) \tag{170-3}$$

is called the *total increment of the function* $f(x, y)$ (or simply the *increment of the function*).

Naturally, here we consider only such points

$$(x, y), \quad (x + \Delta x, y), \quad (x, y + \Delta y), \quad (x + \Delta x, y + \Delta y),$$

for which the function f has sense, i.e. defined.

Note that from formulas (170-2), (170-2'), and (170-3) it follows that *the total increment of a function is, generally speaking, not equal to the sum of partial increments of this function*:

$$\Delta f(x, y) \neq \Delta_x f(x, y) + \Delta_y f(x, y).$$

Example. Find the increment of the function

$$f(x, y) = x^2 + xy - 2y^2,$$

where x changes from 2 to 2.2 and y from 1 to 0.9.

Here

$$\Delta x = 0.2 \quad \text{and} \quad \Delta y = -0.1.$$

We have

$$f(2, 1) = 2^2 + 2 \cdot 1 - 2 \cdot 1^2 = 4$$

and

$$f(2.2, 0.9) = 2.2^2 + 2.2 \cdot 0.9 - 2 \cdot 0.9^2 = 5.20.$$

Hence,

$$\Delta f(2, 1) = 5.20 - 4 = 1.20.$$

The partial and total increments of a function of more than two variables are determined and written in a similar way.

Definition 1. *The function* $f(x, y)$ *is said to be continuous at point* (x_0, y_0) *if*: (1) *it is defined at this point and the latter is the limit point*

for the domain of existence of the function; (2) *to the infinitesimal increments*

$$\Delta x_0 = x - x_0 \quad \text{and} \quad \Delta y_0 = y - y_0$$

of the variables x *and* y *there corresponds the infinitesimal increment* $\Delta f(x_0, y_0)$ *of the function* $f(x, y)$, i.e. whatever the way by which the increments Δx_0 and Δy_0 tend to zero, for which $f(x_0 + \Delta x_0, y_0 + \Delta y_0)$ has sense, the following condition is fulfilled

$$\lim_{\substack{\Delta x_0 \to 0 \\ \Delta y_0 \to 0}} \Delta f(x_0, y_0) = \lim_{\substack{\Delta x_0 \to 0 \\ \Delta y_0 \to 0}} [f(x_0 + \Delta x_0, y_0 + \Delta y_0) - f(x_0, y_0)] = 0. \quad (170\text{-}4)$$

For obviousness, we may think that the function $f(x, y)$ continuous at the point (x_0, y_0), is defined both at the point itself and in some neighbourhood of this point, and for sufficiently small (by absolute value) Δx_0 and Δy_0 equality (170-4) holds true.

Definition 2. *The function* $f(x, y)$ *is said to be continuous in a given domain if this function is continuous at each point of the domain under consideration*, i.e. if for each point (x, y) of the domain we have

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \Delta f(x, y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} [f(x + \Delta x, y + \Delta y) - f(x, y)] = 0. \quad (170\text{-}5)$$

As usually, we assume here that the displaced point $(x + \Delta x, y + \Delta y)$ belongs to the given domain and $f(x + \Delta x, y + \Delta y)$ exists (the set of such points is non-empty in any neighbourhood of the point (x, y) by Definition 1). Thus, we may say that a *function is continuous if and only if to the infinitesimal increments of its arguments there corresponds an infinitesimal increment of the function.*

Example. The function

$$f(x, y) = \sqrt{x} + \sqrt{y} + \sqrt{1 - x - y}$$

is defined and continuous in the triangle: $\Delta = \{x \geqslant 0, y \geqslant 0; x + y \leqslant 1\}$. We should like to note that the points belonging to the boundary of the set Δ are not its interior points.

From formula (170-5) it follows that

$$f(x + \Delta x, y + \Delta y) = f(x, y) + \alpha, \quad (170\text{-}6)$$

where α is an infinitesimal as $\Delta x \to 0$ and $\Delta y \to 0$. Thus, *if a function* $f(x, y)$ *is continuous, then its values at two infinitely close points differ from each other by an infinitesimal function.*

Let us put $x + \Delta x = x_1$, $y + \Delta y = y_1$. Obviously, as $\Delta x \to 0$, $\Delta y \to 0$ we have $x_1 \to x$ and $y_1 \to y$, and conversely. Then from

(170-5) we get an equivalent definition for the continuity of a function

$$\lim_{\substack{x_1 \to x \\ y_1 \to y}} f(x_1, y_1) = f(x, y).$$

Sec. 171. Partial Derivatives of the First Order

Let there be given a function

$$z = f(x, y).$$

For simplicity, here and henceforward we shall assume that for each point (x, y) under consideration the function $f(x, y)$ is defined in some complete neighbourhood of this point.

Consider the ratio of the partial increment

$$\Delta_x z = f(x + \Delta x, y) - f(x, y),$$

of the function z with respect to the variable x to the increment Δx of this variable

$$\frac{\Delta_x z}{\Delta x} = \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}.$$

The limit of this ratio (if any), as $\Delta x \to 0$, is called the *partial derivative (of the first order) of the function* $z = f(x, y)$ *with respect to* x and is denoted as follows:

$$\frac{\partial z}{\partial x} = f'_x(x, y).$$

Hence, we have

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}.$$

The *partial derivative* $\frac{\partial z}{\partial y} = f'_y(x, y)$ *of the function* $x = f(x, y)$ *with respect to* y is determined in a similar manner:

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}.$$

Definition. *The partial derivative of a function of several variables with respect to one of these variables is the limit to which tends the ratio of the corresponding partial increment of the function to the increment of the independent variable under consideration provided the latter tends to zero.*

Note that if we differentiate the function $z = f(x, y)$ with respect to x $\left(\text{i.e. if we find } \frac{\partial z}{\partial x}\right)$, then y is regarded as constant, and if we find $\frac{\partial z}{\partial y}$, then x is considered to be constant.

Therefore, *the partial derivative of a function of several variables is equal to the derivative of the function of one variable which is obtained if all the independent variables of the given function, except for the corresponding one, are considered to be constant*, i.e.

$$\frac{\partial f}{\partial x} = \frac{d}{dx}[f(x, y)], \quad \text{where} \quad y = \text{const}, \text{ etc.}$$

Consequently, partial differentiation requires no new rules, and we may take advantage of the known differentiation formulas (see Sec. 82).

Example 1. Let

$$z = x^3 \sin y + y^4.$$

It is clearly seen that

$$\frac{\partial z}{\partial x} = 3x^2 \sin y, \quad \frac{\partial z}{\partial y} = x^3 \cos y + 4y^3.$$

Determined and evaluated in a similar way are the partial deriv-

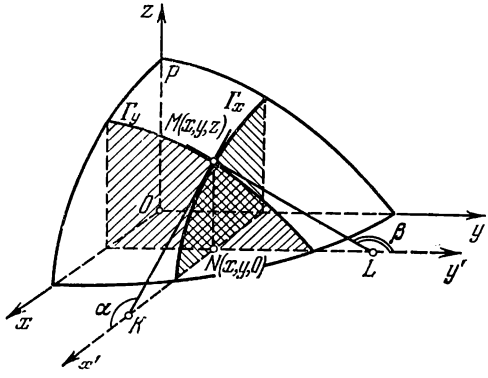


FIG. 210

atives of the function $u = f(x, y, z)$ of three variables x, y, z, and so on.

Example 2. Let $u = x^6 - y^4 + 3z^5$; then

$$\frac{\partial u}{\partial x} = 6x^5, \quad \frac{\partial u}{\partial y} = -4y^3, \quad \frac{\partial u}{\partial z} = 15z^4.$$

For the function

$$z = f(x, y)$$

it is not difficult to find out the geometrical meaning of its partial derivatives $\frac{\partial z}{\partial x} = f'_x(x, y)$ and $\frac{\partial z}{\partial y} = f'_y(x, y)$. The geometrical representation of the given function is a certain surface P (Fig. 210).

Putting $y = \text{const}$, we get a plane curve Γ_x representing the section of the surface P by a corresponding plane parallel to the coordinate plane Oxz. Let MK be the tangent to the curve Γ_x at point $M(x, y, z)$ and α the angle formed by this tangent with the positive direction of the x-axis. Since

$$\frac{\partial z}{\partial x} = \left[\frac{dz}{dx}\right]_{y=\text{const}},$$

by the geometrical meaning of an ordinary derivative, we have

$$\frac{\partial z}{\partial x} = \tan\alpha.$$

Analogously, if Γ_y is the section of the surface P by a plane $x = \text{const}$ and β is the angle formed with the y-axis by the tangent ML to the curve Γ_y at point $M(x, y, z)$, then

$$\frac{\partial z}{\partial y} = \tan\beta.$$

Sec. 172. The Total Differential of a Function

Let $z = f(x, y)$ be a function of two independent variables x and y. Then the increment of this function

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

represents the difference between the values of the given function at points $M(x, y)$ and $M'(x + \Delta x, y + \Delta y)$. Let us denote by ρ the distance between these points:

$$\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

If, as $\rho \to 0$, we can choose quantities A and B independent of Δx and Δy so that the expression

$$A\,\Delta x + B\,\Delta y$$

differs from the total increment Δz of the function by a quantity of a higher order of smallness as compared with ρ, then this expression is called the *principal linear part* of the total increment of the function. In this case we get

$$\Delta z = A\,\Delta x + B\,\Delta y + \gamma\rho, \tag{172-1}$$

where $\gamma \to 0$ as $\rho \to 0$ (or, which is the same $\gamma \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$).

Expression (172-1) can be written in another form. Since $\Delta x = \rho\cos\varphi$, $\Delta y = \rho\sin\varphi$ (Fig. 211), we have

$$\rho = \Delta x\cos\varphi + \Delta y\sin\varphi;$$

hence

$$\Delta z = A\ \Delta x + B\ \Delta y + \alpha\ \Delta x + \beta\ \Delta y \qquad (172\text{-}1')$$

where

$$\alpha = \gamma \cos \varphi \to 0 \text{ and } \beta = \gamma \sin \varphi \to 0$$

as $\rho \to 0$, i.e. as $\Delta x \to 0$ and $\Delta y \to 0$, and conversely.

Generalizing the definition of the differential of functions of one independent variable for the case of a function of two independent variables, we come to the following definitions.

Definition 1. *The differential of an independent variable is understood as the increment of this variable*, i.e.

$$dx = \Delta x \text{ and } dy = \Delta y.$$

Definition 2. *The total differential of a function (or simply, the differential of a function) $z = f(x, y)$ of two independent variables x*

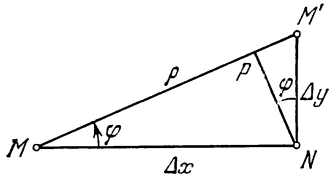


FIG. 211

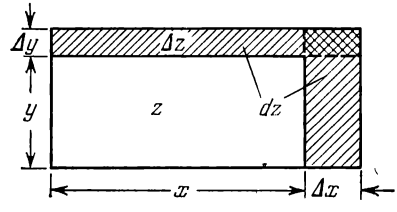


FIG. 212

and y is the principal linear part of the total increment of this function.

This definition is extended in a natural way to functions of any number of variables.

Denoting the differential of a function by d, we may write

$$dz = A\ \Delta x + B\ \Delta y, \qquad (172\text{-}2)$$

where A and B are independent of Δx and Δy, and, moreover,

$$\Delta z - dz = \alpha\ \Delta x + \beta\ \Delta y,$$

where α and β are infinitesimals as $\Delta x \to 0$ and $\Delta y \to 0$.

A function that has a differential in a given domain is called differentiable in this domain. If the function z is differentiable, then for the total increment Δz of the function formula (172-1) or (172-1') holds true.

Let us note that if a function $z = f(x, y)$ is differentiable, then this function is continuous. Indeed, passing to the limit in formula

(172-1′) as $\Delta x \to 0$ and $\Delta y \to 0$, we get

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \Delta z = 0,$$

i.e. the function z is continuous (see Sec. 170).

Example. Find the differential of the function

$$z = xy.$$

The function z may be regarded as the area of a rectangle with the sides x and y (Fig. 212)*. Assigning increments Δx and Δy to the sides x and y, we shall obtain an increment Δz of the area z representing the area of the "border":

$$\Delta z = (x + \Delta x)(y + \Delta y) - xy = y\,\Delta x + x\,\Delta y + \Delta x \cdot \Delta y.$$

The principal part of this increment as $\Delta x \to 0$ and $\Delta y \to 0$ consisting of two rectangles with the sides y, Δx and x, Δy is the differential dz of the area z; therefore

$$dz = y\,\Delta x + x\,\Delta y.$$

Theorem 1. *The differential of a function is equal to the sum of the products of derivatives of this function by the differentials of the corresponding independent variables.*

Proof. Let the function $z = f(x, y)$ be differentiable, i.e. let it have a differential

$$dz = A\,\Delta x + B\,\Delta y. \tag{172-3}$$

To determine the coefficients A and B, we write the total increment of the function

$$\Delta z = A\,\Delta x + B\,\Delta y + \alpha\,\Delta x + \beta\,\Delta y, \tag{172-4}$$

where α and β are infinitesimals as $\Delta x \to 0$ and $\Delta y \to 0$. Putting $\Delta y = 0$ in (172-4), we get the partial increment

$$\Delta_x z = A\,\Delta x + \alpha\,\Delta x.$$

Hence

$$\frac{\Delta_x z}{\Delta x} = A + \alpha$$

and, consequently, as $\Delta x \to 0$, we shall have

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \frac{\partial z}{\partial x} = A.$$

Analogously, setting $\Delta x = 0$ in formula (172-4), we find

$$\lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \frac{\partial z}{\partial y} = B.$$

* For the sake of obviousness, we consider x and y to be positive.

Thus,

$$A = \frac{\partial z}{\partial x}, \quad B = \frac{\partial z}{\partial y}.$$

Substituting these values into (172-3) and taking into account that $\Delta x = dx$ and $\Delta y = dy$, we finally have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy. \tag{172-5}$$

Corollary. *A given function has one differential.*

Indeed, from the proof of Theorem 1 it follows that the differential (if it exists) of the function $z = f(x, y)$ is necessarily expressed by formula (172-5).

Note. From formula (172-5) it follows that for the function $z = f(x, y)$ of two independent variables x and y its differential dz is a function of four independent variables x, y, dx, dy, which is linear (i.e. of the first degree) with respect to the second pair of the variables. The first pair of the variables, x and y, represents the coordinates of the point $M(x, y)$ at which the differential is taken. The second pair of the variables, dx and dy, are the coordinates of the displacement vector of the point $M(x, y)$ as the latter goes into an infinitely close point $M'(x + dx, y + dy)$, where dx and dy are the projections of the segment MM' on the corresponding coordinate axes Ox and Oy.

Theorem 2. (the sufficient condition of differentiability of a function). *If a function $z = f(x, y)$ possesses continuous partial derivatives $\frac{\partial z}{\partial x} = f'_x(x, y)$ and $\frac{\partial z}{\partial y} = f'_y(x, y)$ in a given domain, then this function is differentiable in this domain and its differential is expressed by formula* (172-5).

Proof. Consider the total increment of the function

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

Subtracting and adding the term $f(x, y + \Delta y)$, we shall have

$$\Delta z = [f(x + \Delta x, y + \Delta x) - f(x, y + \Delta y)] + [f(x, y + \Delta y) - f(x, y)]. \tag{172-6}$$

The expression in the first brackets of the above formula represents the increment of the function $f(x, y)$ with respect to the variable x with the fixed value $y + \Delta y$ of the second variable y, i.e. it may be regarded as an increment of a function of one variable x. Holding the quantity $y + \Delta y$ fixed and applying Lagrange's theo-

rem on finite increments of a function (see Sec. 85), we find

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = \Delta x f'_x(\bar{x}, y + \Delta y), \quad (172\text{-}7)$$

where $\bar{x}$ is some intermediate value between x and $x + \Delta x$.

Analogously, the expression in the second brackets of (172-6) is the increment of the function $f(x, y)$ with respect to the variable y with the value of the variable x unchanged. Therefore, by Lagrange's theorem, we have

$$f(x, y + \Delta y) - f(x, y) = \Delta y f'_y(x, \bar{y}), \quad (172\text{-}8)$$

where $\bar{y}$ is an intermediate value between y and $y + \Delta y$. From (172-6), (172-7), and (172-8) it follows

$$\Delta z = \Delta x f'_x(\bar{x}, y + \Delta y) + \Delta y f'_y(x, \bar{y}). \quad (172\text{-}9)$$

Let $\Delta x \to 0$ and $\Delta y \to 0$. Since the derivatives $f'_x(x, y)$ and $f'_y(x, y)$ are continuous, their values at infinitely close points $P(\bar{x}, y + \Delta y)$ and $M(x, y)$ and, correspondingly, $Q(x, \bar{y})$ and $M(x, y)$ (Fig. 213) differ from each other by infinitesimals (Sec. 170); therefore,

$$f'_x(\bar{x}, y + \Delta y) = f'_x(x, y) + \alpha$$

and $f'_y(x, \bar{y}) = f'_y(x, y) + \beta$,

where α and β are infinitesimals as $\Delta x \to 0$ and $\Delta y \to 0$. Hence, from (172-9) we have

$$\Delta z = [f'_x(x, y)\,\Delta x + f'_y(x, y)\,\Delta y] + (\alpha\,\Delta x + \beta\,\Delta y). \quad (172\text{-}10)$$

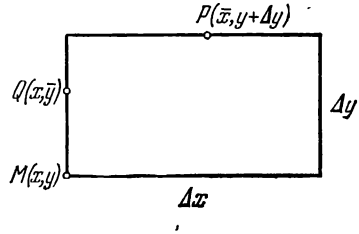


FIG. 213

By definition, the principal linear part of the total increment Δz of a function is the differential dz of this function. Consequently, from (172-10) we get

$$dz = f'_x(x, y)\,\Delta x + f'_y(x, y)\,\Delta y \equiv \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy,$$

which was to be proved.

Example 1. Find the differential of the function

$$z = x^y.$$

Here

$$\frac{\partial z}{\partial x} = yx^{y-1} \quad \text{and} \quad \frac{\partial z}{\partial y} = x^y \ln x.$$

Hence

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = yx^{y-1}\,dx + x^y \ln x\,dy.$$

Note. Analogously, if a function $u = f(x, y, z)$ has continuous partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$, then the differential of this function is expressed by the formula

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz,$$

where $dx = \Delta x$, $dy = \Delta y$ and $dz = \Delta z$.

Example 2. Find the differential of the function

$$u = \frac{x}{y}\,e^z.$$

We have

$$\frac{\partial u}{\partial x} = \frac{1}{y}\,e^z, \quad \frac{\partial u}{\partial y} = -\frac{x}{y^2}\,e^z, \quad \frac{\partial u}{\partial z} = \frac{x}{y}\,e^z.$$

Hence,

$$du = e^z\left(\frac{1}{y}\,dx - \frac{x}{y^2}\,dy + \frac{x}{y}\,dz\right).$$

If the increments Δx and Δy are small, then the increment of a differentiable function

$$\Delta f(x, y) = f(x + \Delta x, y + \Delta y) - f(x, y)$$

may be approximately substituted by the differential $df(x, y)$ of this function:

$$df(x, y) = f'_x(x, y)\,\Delta x + f'_y(x, y)\,\Delta y.$$

Hence, we have an approximate equality

$$f(x + \Delta x, y + \Delta y) - f(x, y) \approx f'_x(x, y)\,\Delta x + f'_y(x, y)\,\Delta y,$$

which will be relatively more precise with a decrease in $|\Delta x|$ and $|\Delta y|$.

Example 3. Given a rectangle with the sides $x = 6$ m and $y = 8$ m. What an increase in length will the diagonal of this rectangle receive if the side x is increased by 5 cm, and the side y is reduced by 10 cm?

Denoting the diagonal of the rectangle by u, we have

$$u = \sqrt{x^2 + y^2}.$$

Hence, replacing the increment Δx of the diagonal by the differential du of this diagonal, we approximately find

$$\Delta u \approx du = \frac{x}{\sqrt{x^2+y^2}}\Delta x + \frac{y}{\sqrt{x^2+y^2}}\Delta y = \frac{x\,\Delta x + y\,\Delta y}{\sqrt{x^2+y^2}}.$$

Setting

$x = 6$ m, $\Delta x = 0.05$ m;

$y = 8$ m, $\Delta y = -0.10$ m,

we get

$$\Delta u \approx \frac{6\cdot 0.05 + 8\cdot(-0.10)}{\sqrt{36+64}} = -0.05 \text{ m}.$$

Thus, the diagonal of the rectangle is reduced approximately by 5 cm. A precise computation yields $\Delta u = -0.045$ m.

Sec. 173. Application of the Differential of a Function to Approximate Computations

With the aid of the total differential of a function we can find out the effect of the errors of the arguments of the function on its value.

Problem. Determine the limiting absolute error Δ_z of the function

$$z = f(x, y),$$

knowing the limiting absolute errors Δ_x and Δ_y of the arguments x, y:

$$|\Delta x| \leqslant \Delta_x \text{ and } |\Delta_y| \leqslant \Delta_y.$$

We have

$$|\Delta z| = |f(x + \Delta x, y + \Delta y) - f(x, y)|.$$

Substituting the increment of the function by its differential, we obtain

$$|\Delta z| \approx |f'_x(x, y)\,\Delta x + f'_y(x, y)\,\Delta y|.$$

Hence we derive an approximate estimate:

$$|\Delta z| \leqslant \left|\frac{\partial z}{\partial x}\right| |\Delta x| + \left|\frac{\partial z}{\partial y}\right| |\Delta y|.$$

Consequently, we may take

$$\Delta_z = \left|\frac{\partial z}{\partial x}\right| \Delta_x + \left|\frac{\partial z}{\partial y}\right| \Delta_y \qquad (173\text{-}1)$$

for the limiting absolute error of the function z.

Example. The hypotenuse of a right-angled triangle $x = 120$ m $\pm$ $\pm$ 2 m, and the acute angle $y = 30° \pm 1°$. With what accuracy can we find the leg z of this triangle lying opposite the given angle?

We have

$$z = x \sin y. \tag{173-2}$$

Hence,

$$\frac{\partial z}{\partial x} = \sin y, \quad \frac{\partial z}{\partial y} = x \cos y.$$

Putting $x = 120$, $\Delta_x = 2$ and $y = \frac{\pi}{6}$, $\Delta_y = \frac{\pi}{180}$, by (172-2) and (172-1), we find

$$z = 120 \sin 30° = 60 \text{ m}$$

and

$$\Delta_z = \sin 30° \cdot 2 + 120 \cdot \cos 30° \cdot \frac{\pi}{180} = 1 + 1.8 = 2.8 \text{ m}.$$

Consequently,

$$z = 60 \text{ m} \pm 2.8 \text{ m}.$$

Using (172-1), we can also determine the limiting relative error of the function:

$$\delta_z = \frac{\Delta_z}{|z|}.$$

In particular, let us put

$$z = xy \quad (x \neq 0, \ y \neq 0).$$

Then

$$\Delta_z = |y| \Delta_x + |x| \Delta_y$$

and, hence,

$$\delta_z = \frac{\Delta_x}{|x|} + \frac{\Delta_y}{|y|}$$

or

$$\delta_z = \delta_x + \delta_y,$$

i.e. *the limiting relative error of a product is equal to the sum of limiting relative errors of its factors.*

Sec. 174. Directional Derivatives

Let $u = f(x, y)$ be a function defined in the domain ω. We are going to consider some point $M(x, y) \in \omega$ and a certain direction l defined by the direction cosines $\cos \alpha$ and $\cos \beta = \sin \alpha$ (i.e. $\cos \alpha$

and cos β are cosines of the angles formed by the ray l with the coordinate axes Ox and Oy). As the point $M(x, y)$ displaces along the given direction l to the point $M'(x + \Delta x, y + \Delta y) \in \omega$, the function $u = f(x, y)$ receives the increment

$$\Delta_l u = f(x + \Delta x, y + \Delta y) - f(x, y), \tag{174-1}$$

which is called the *increment of a function in a given direction* l (Fig. 214). If $MM' = \Delta l$ is the magnitude of displacement of the point M, then from the right-angled triangle MPM' we obtain

$$\left.\begin{aligned} \Delta x &= \Delta l \cdot \cos\alpha, \\ \Delta y &= \Delta l \cdot \cos\beta, \end{aligned}\right\} \tag{174-2}$$

hence,

$$\Delta_l u = f(x + \Delta l \cos\alpha, y + \Delta l \cos\beta) - f(x, y).$$

Definition. *The (directional) derivative* $\frac{\partial u}{\partial l}$ *of the function* u *in the given direction* l *is the limit of the ratio of the increment of the function in this direction to the magnitude of displacement, provided the latter tends to zero,* i.e.

$$\frac{\partial u}{\partial l} = \lim_{\Delta l \to 0} \frac{\Delta_l u}{\Delta l}. \tag{174-3}$$

From this point of view, the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ may be considered as the derivatives of the function u in the positive directions of the x- and y-axes.

The derivative $\frac{\partial u}{\partial l}$ gives the *rate of change of the function in the direction* l.

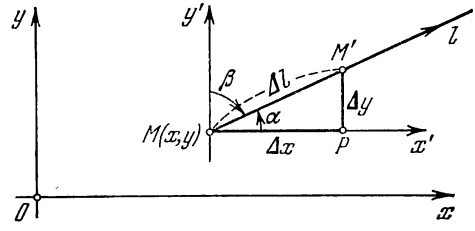


FIG. 214

Let us derive the formula for the derivative $\frac{\partial u}{\partial l}$, assuming that the function $u = f(x, y)$ is differentiable. It follows from the definition of the differential that the increment of the function differs from its differential by a magnitude of a higher order of smallness as compared with the increments of the independent variables. Therefore, using the formula for the total differential (172-5), we shall have

$$\Delta_l u = \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. Hence, by relations (174-2), we get

$$\Delta_l u = \left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta\right)\Delta l + (\varepsilon_1\cos\alpha + \varepsilon_2\cos\beta)\,\Delta l.$$

Consequently,

$$\frac{\Delta_l u}{\Delta l} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \varepsilon_1\cos\alpha + \varepsilon_2\cos\beta.$$

Passing to the limit in the above formula, as $\Delta l \to 0$, i.e. as $\Delta x \to 0$ and $\Delta y \to 0$, and applying (174-3), we shall obtain the sought-for formula of the derivative of the function in the given direction:

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta, \qquad (174\text{-}4)$$

where $\cos\beta = \sin\alpha$.

Example 1. Find the increment of the function

$$u = x^2 + 2xy - y^2$$

as the point M (1, 2) displaces in the direction l which makes an angle $\alpha = \arctan\frac{3}{4}$ with the positive direction of the x-axis over a distance $\Delta l = 0.1$. Evaluate the derivative $\frac{\partial u}{\partial l}$ at the point M.

We have $\tan\alpha = \frac{3}{4}$, and $0 < \alpha < \frac{\pi}{2}$. Hence

$$\sec\alpha = \sqrt{1+\tan^2\alpha} = \sqrt{1+\frac{9}{16}} = \frac{5}{4},$$

hence,

$$\cos\alpha = \frac{1}{\sec\alpha} = \frac{4}{5}$$

and

$$\sin\alpha = \tan\alpha\cdot\cos\alpha = \frac{3}{4}\cdot\frac{4}{5} = \frac{3}{5}.$$

Using the obtained direction cosines $\cos\alpha = \frac{4}{5}$ and $\cos\beta = \frac{3}{5}$ of the direction l, we find the increments of coordinates for the point M

$$\Delta x = \Delta l\cdot\cos\alpha = 0.1\cdot\frac{4}{5} = 0.08$$

and

$$\Delta y = \Delta l\cdot\cos\beta = 0.1\cdot\frac{3}{5} = 0.06.$$

Thus, the displaced point M' has the following coordinates

$$x_1 = x + \Delta x = 1 + 0.08 = 1.08$$

and

$$y_1 = y + \Delta y = 2 + 0.06 = 2.06.$$

Hence, the required increment of the function u is equal to

$$\Delta_l u = (1.08^2 + 2 \cdot 1.08 \cdot 2.06 - 2.06^2) - (1^2 + 2 \cdot 1 \cdot 2 - 2^2) = \\ = 1.3724 - 1 = 0.3724.$$

Note that

$$\frac{\Delta_l u}{\Delta l} \approx 3.7.$$

Further, we have

$$\frac{\partial u}{\partial x} = 2x + 2y, \quad \frac{\partial u}{\partial y} = 2x - 2y;$$

therefore,

$$\left(\frac{\partial u}{\partial x}\right)_M = 6, \quad \left(\frac{\partial u}{\partial y}\right)_M = -2$$

and, consequently,

$$\left(\frac{\partial u}{\partial l}\right)_M = \left(\frac{\partial u}{\partial x}\right)_M \cos\alpha + \left(\frac{\partial u}{\partial y}\right)_M \cos\beta = 6 \cdot \frac{4}{5} + (-2) \cdot \frac{3}{5} = 3.6.$$

Note. For the function $u = f(x, y, z)$ its derivative in the direction $\mathbf{l} = \{\cos\alpha, \cos\beta, \cos\gamma\}$ is equal to

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x} \cos\alpha + \frac{\partial u}{\partial y} \cos\beta + \frac{\partial u}{\partial z} \cos\gamma.$$

Sec. 175. The Gradient

Definition 1. *A scalar field is said to be defined in a given domain* ω* *if for each point* $M \in \omega$ *a certain scalar* (i.e. a number)

$$u = f(M) \tag{175-1}$$

is specified.

Thus, u is a numerical function of a point.

A temperature field (i.e. the distribution of temperature in a heated body), the distribution of concentration of substance in a solution, etc. are examples of scalar fields.

If the domain ω is situated on the xy-plane, then any of its points, say M, is defined by two coordinates (x, y), and a plane scalar field

* Traditionally, the word "domain" is used here as synonym for the word "set". For the precise definition of the notion "domain" see Sec. 179.

(175-1) can be written in the form

$$u = f(x, y) \quad ((x, y) \in \omega). \tag{175-2}$$

Analogously, for the domain ω, contained in the xyz-space, we shall have

$$u = f(x, y, z) \quad ((x, y, z) \in \omega).$$

Thus, the notion of a scalar field represents a physical interpretation of a function of several variables.

Definition 2. *A vector field is said to be defined in a given domain if for any point* $M \in \omega$ *a certain vector*

$$\mathbf{a} = \mathbf{F}(M). \tag{175-3}$$

is specified.

A velocity field of points of a fluid flux in a given moment of time, a force field generated by an attractive centre, etc. are examples of vector fields.

For the case of a plane vector field (175-3) ($\omega \in Oxy$) we shall have a vector function

$$\mathbf{a} = \mathbf{F}(x, y) \quad ((x, y) \in \omega). \tag{175-4}$$

Hence, passing over to the coordinates of the vector $\mathbf{a}$, we obtain

$$a_x = F_1(x, y), \quad a_y = F_2(x, y). \tag{175-5}$$

Thus, specifying a plane vector field (175-4) is equivalent to specifying two scalar fields (175-5).

Analogously, for the case of a space vector field ($\omega \in Oxyz$) we get

$$\mathbf{a} = \mathbf{F}(x, y, z), \tag{175-6}$$

or in coordinates

$$a_x = F_1(x, y, z), \quad a_y = F_2(x, y, z), \quad a_z = F_3(x, y, z). \tag{175-7}$$

And so, vector field (175-6) is equivalent to three scalar fields (175-7). This explains the convenience of the vector language: it allows us to write several scalar relations in one vector formula.

The set of all points M for which scalar field (175-1) preserves a constant value

$$f(M) = \text{const},$$

is called the *level surface* (or *level line*) of a scalar field (*isosurfaces*) (see Sec. 169).

Definition 3. *Let*

$$u = f(x, y) \tag{175-8}$$

be a differentiable plane scalar field. Then the vector*

$$\operatorname{grad} u = \left\{ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\} \tag{175-9}$$

is called the gradient of the field; or

$$\operatorname{grad} u = \mathbf{i} \frac{\partial u}{\partial x} + \mathbf{j} \frac{\partial u}{\partial y},$$

where $\mathbf{i}$, $\mathbf{j}$ are unit vectors directed along the x- and y-axes (coordinate unit vectors).

Analogously, for a space scalar field

$$u = f(x, y, z) \tag{175-8'}$$

its gradient is the vector

$$\operatorname{grad} u = \left\{ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\}. \tag{175-9'}$$

Hence, a scalar field generates a vector field, i.e. a *gradient field*

The derivative of a scalar field (175-8') in a given direction $\mathbf{l}$ is understood to be the expression (see Sec. 174)

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta + \frac{\partial u}{\partial z} \cos \gamma, \tag{175-10}$$

where $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the direction cosines of the vector $\mathbf{l}$. The derivative $\frac{\partial u}{\partial l}$ represents the rate of change of the field in the given direction.

Theorem 1. *The derivative of a scalar field in a given direction is equal to the projection of the field gradient on the given direction (at an appropriate point).*

Proof. Let us denote by $\mathbf{l}_0 = \{\cos \alpha, \cos \beta, \cos \gamma\}$ the *unit vector of the direction* $\mathbf{l}$. Then, taking into consideration formula (175-9') and recalling the definition of a scalar product (Sec. 156), expression (175-10) can be written in the following form:

$$\frac{\partial u}{\partial l} = \operatorname{grad} u \cdot \mathbf{l}_0 = |\operatorname{grad} u| \, |\mathbf{l}_0| \cos \varphi = |\operatorname{grad} u| \cos \varphi, \tag{175-11}$$

where $\varphi = \angle(\operatorname{grad} u, \mathbf{l})$ (Fig. 215).

Hence

$$\frac{\partial u}{\partial l} = \operatorname{pr}_l \operatorname{grad} u. \tag{175-12}$$

Corollary. *The gradient of a scalar field at a given point is equal (by magnitude and direction) to the maximum rate of change of the field at this point.*

* That is, $f(x, y)$ is a differentiable function of two variables.

Indeed, from formula (175-11) we obtain that

$$\max_{l} \frac{\partial u}{\partial l} = \frac{\partial u}{\partial l^*} = |\operatorname{grad} u|,$$

and $\cos \varphi = 1$. Hence, we find that $\varphi = 0$ and, consequently, the direction of the vector $\mathbf{l} = \mathbf{l}^*$ must coincide with the direction grad u, i.e. $\mathbf{l}^* = k \operatorname{grad} u$, where $k > 0$. Besides, for this direction we have

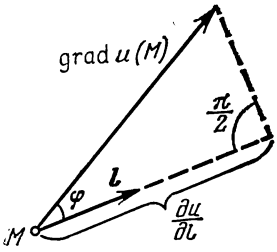


FIG. 215

$$\frac{\partial u}{\partial l^*} = |\operatorname{grad} u| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2}. \tag{175-13}$$

Note. It follows from the corollary that a field gradient is independent of the choice of the coordinate system $Oxyz$.

Example. Find the magnitude and direction of the gradient of the field

$$u = \frac{x}{y} + z^2$$

at the point $M_0\,(2, 1, 0)$.

We have

$$\left(\frac{\partial u}{\partial x}\right)_{M_0} = \left(\frac{1}{y}\right)_{M_0} = 1,$$

$$\left(\frac{\partial u}{\partial y}\right)_{M_0} = \left(-\frac{x}{y^2}\right)_{M_0} = -2,$$

$$\left(\frac{\partial u}{\partial z}\right)_{M_0} = (2z)_{M_0} = 0.$$

Consequently,

$$\operatorname{grad} u\,(M_0) = \mathbf{i} - 2\mathbf{j}.$$

Hence

$$|\operatorname{grad} u\,(M_0)| = \sqrt{5}$$

and

$$\cos\alpha = \frac{1}{\sqrt{5}}, \quad \cos\beta = -\frac{2}{\sqrt{5}}, \quad \cos\gamma = 0.$$

The point M_0 at which grad $u\,(M_0) = 0$ is called *singular* for this scalar field; otherwise it is termed *non-singular* (or *regular*).

Theorem 2. *At any non-singular point of a plane scalar field the field gradient is directed along the normal to the level line passing through this point to the side of the field growth.* The proof of this theorem is left to the reader.

Sec. 176. Partial Derivatives of Higher Orders

Suppose we have a function $z = f(x, y)$ of two variables x and y. Its partial derivatives

$$\frac{\partial z}{\partial x} = f'_x(x, y) \quad \text{and} \quad \frac{\partial z}{\partial y} = f'_y(x, y)$$

are functions of the variables x and y. In some cases for these functions there exist again partial derivatives called *partial derivatives of the second order* (or simply, *second partial derivatives*):

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f''_{xx}(x, y), \qquad \frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f''_{xy}(x, y),$$

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f''_{yx}(x, y), \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f''_{yy}(x, y).$$

Proceeding in this way, we can determine *partial derivatives of the third order* (*third partial derivatives*), and so on.

Partial derivatives of higher orders of a function of three and more variables are determined and written in a similar manner.

We can prove the following theorem: *if all partial derivatives participating in the process of computation which are considered as the functions of their independent variables are continuous, then the result of partial differentiation is independent of the order of differentiation.*

In particular, for instance, if the derivatives $\frac{\partial^2 z}{\partial x\, \partial y}$ and $\frac{\partial^2 z}{\partial y\, \partial x}$ are continuous, then the following equality holds

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}.$$

Without giving the proof in the general form, let us verify the validity of the last assertion by considering the following example.

Example. Let

$$z = x^y \quad (x > 0).$$

We have

$$\frac{\partial z}{\partial x} = yx^{y-1}, \qquad \frac{\partial^2 z}{\partial x\, \partial y} = yx^{y-1} \ln x$$

and

$$\frac{\partial z}{\partial y} = x^y \ln x, \quad \frac{\partial^2 z}{\partial y\, \partial x} = yx^{y-1} \ln x.$$

We see that for the given function z the equality

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$$

is observed, as it was to be expected.

Sec. 177. Test for the Total Differential

If the function $u = f(x, y)$ is differentiable, then its total differential has the form (172-5)

$$du = P(x, y)\, dx + Q(x, y)\, dy, \tag{177-1}$$

where

$$P(x, y) = \frac{\partial u}{\partial x} \quad \text{and} \quad Q(x, y) = \frac{\partial u}{\partial y}. \tag{177-2}$$

There arises a converse problem: under what conditions is the differential expression

$$P(x, y)\, dx + Q(x, y)\, dy, \tag{177-3}$$

where the functions $P(x, y)$ and $Q(x, y)$ are continuous together with their derivatives of the first order, the total differential of some function u?

A necessary condition for a total differential is given by the following theorem.

Theorem. *For differential expression* (177-3) *to be in a domain G the total differential of some function $u = F(x, y)$ it is necessary that the following condition be fulfilled identically in this domain*

$$\frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y} \qquad ((x, y) \in G) \tag{177-α}$$

(the condition for a total differential).

Proof. Let (177-3) be the total differential of the function $u = F(x, y)$. We have

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy. \tag{177-4}$$

Hence, by virtue of uniqueness of the differential (Sec. 172, Corollary from Theorem 1), we get

$$P(x, y) = \frac{\partial u}{\partial x}, \quad Q(x, y) = \frac{\partial u}{\partial y}. \tag{177-5}$$

Differentiating the first equality with respect to y and the second with respect to x, we shall have

$$\frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}. \tag{177-6}$$

Since for continuous mixed derivatives the result of differentiation is independent of the order of differentiation, from (177-6) we obtain

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

i.e. the condition (177-α) is fulfilled.

Corollary. *If the condition* (177-α) *is not fulfilled, then the expression* $P(x, y)\,dx + Q(x, y)\,dy$ *is not a total differential of some function in the domain* G.

Note. We can prove that for a finite or infinite rectangular domain

$$G = \{a < x < b; \quad A < y < B\}$$

fulfilment of the condition (177-α) is just sufficient for the existence of a function u such that

$$P(x, y)\,dx + Q(x, y)\,dy = du.$$

Example. Are the expressions

$$y\,dx - x\,dy \quad \text{and} \quad y\,dx + x\,dy$$

total differentials of some functions?

For the first expression we have $P = y$, $Q = -x$. Hence

$$\frac{\partial P}{\partial y} = 1, \quad \frac{\partial Q}{\partial x} = -1$$

and, consequently, the condition of a total differential is not fulfilled, i.e. there is no function whose total differential is equal to $y\,dx - x\,dy$.

For the second expression we get

$$P = y, \quad Q = x$$

and, hence,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 1.$$

The condition for a total differential is fulfilled. Since the plane may be regarded as an infinite rectangular domain, then $y\,dx + x\,dy$ is a total differential of some function. Indeed,

$$y\,dx + x\,dy = d(xy).$$

Sec. 178. The Extremum (Maximum or Minimum) of a Function of Several Variables

Let us recall (see Sec. 46) that a *neighbourhood of a point* in the plane is understood to be the interior of any rectangle surrounding this point, excluding the point itself.

Analogously, a neighbourhood of a point in space is understood to be the interior of an arbitrary parallelepiped containing this point less the point itself.

Definition. *A (strict) maximum of a function $f(x, y)$ is a value $f(x_1, y_1)$ of this function such that it is the greatest among all the values $f(x, y)$ attained by the given function at the points of some neighbourhood of the point (x_1, y_1)**. (This neighbourhood may be very small by its linear dimensions.)

Analogously, a (strict) minimum of a function $f(x, y)$ is a value $f(x_2, y_2)$ of this function such that it is the least among all the values $f(x, y)$ attained by the given function at the points of some neighbourhood of the point (x_2, y_2).

A maximum or a minimum of a function $f(x, y)$ is called the *extremum* of this function and a point at which an extremum is obtained is called the *point of extremum* (i.e. the *point of maximum* or the *point of minimum* of the function).

An extremum of a function $f(x, y, z)$, etc. is determined in a similar fashion.

Now we are going to indicate a necessary condition for an extremum of a function of several variables.

Theorem. *At a point of extremum of a function of several variables each of its partial derivatives of the first order is either equal to zero or does not exist.*

Proof. For the sake of simplicity, let us consider a function of two variables $u = f(x, y)$, and let $f(x_0, y_0)$ be its maximum (the reasoning for a minimum is analogous).

Fixing one of the variables, say y, and putting $y = y_0$, we obtain a function of one variable

$$u_1 = f(x, y_0),$$

which, obviously, has a maximum at $x = x_0$. Hence, on the basis of the theory of a function of one variable (Sec. 91) we obtain that

$$\left(\frac{du_1}{dx}\right)_{x=x_0} = f'_x(x_0, y_0) = 0$$

or $f'_x(x_0, y_0)$ does not exist.

We prove just in the same way that $f'_y(x_0, y_0) = 0$ or it does not exist.

* By the meaning of the definition, the function $f(x, y)$ must make sense on a certain set of points of this neighbourhood.

Corollary. *At a point of extremum* $M_0(x_0, y_0)$ *of a differentiable function* $f(x, y)$ *the following equalities are fulfilled*

$$f'_x(x_0, y_0) = 0, \quad f'_y(x_0, y_0) = 0.$$

Analogously, if a differentiable function $f(x, y, z)$ has an extremum at point $M_0(x_0, y_0, z_0)$, then

$$f'_x(x_0, y_0, z_0) = 0, \quad f'_y(x_0, y_0, z_0) = 0, \quad f'_z(x_0, y_0, z_0) = 0.$$

Note 1. Points at which partial derivatives of the first order of some function are either equal to zero or do not exist will be called *critical* for this function.

Then the theorem is equivalent to the following statement: ***extrema of a function of several variables can exist only at its critical points.***

Note 2. The above derived conditions for an extremum of a function are, generally speaking, not sufficient, i.e. if, for instance, at a certain point all partial derivatives of the first order of a function are equal to zero, then the function not necessarily has an extremum at this point.

Example 1. For the function

$$f(x, y) = xy$$

we have

$$f'_x(x, y) = y, \quad f'_y(x, y) = x.$$

Hence,

$$f'_x(0, 0) = f'_y(0, 0) = 0.$$

But the point $O(0, 0)$ is not a point of extremum of the function, since in any neighbourhood of the point O there are points $A(\varepsilon, \varepsilon)$ and $B(-\varepsilon, \varepsilon)$ ($\varepsilon > 0$ is arbitrary) such that

$$f(A) = \varepsilon^2 > 0 = f(0) \quad \text{and} \quad f(B) = -\varepsilon^2 < f(0).$$

Example 2. Of all rectangular parallelepipeds having the sum of the three dimensions equal to a given positive quantity a find the one whose volume is the greatest.

Let us denote by x, y, and z ($x > 0$, $y > 0$, $z > 0$) the dimensions of the rectangular parallelepiped under consideration. Then, its volume V will be expressed as follows:

$$V = xyz.$$

Besides, according to the condition of the problem, we have

$$x + y + z = a.$$

Expressing z in terms of x and y from the last equation and substituting it into the expression for V, we get

$$V = xy\,(a - x - y),$$

where x and y are independent variables.

We now take partial derivatives of V with respect to x and y:

$$\frac{\partial V}{\partial x} = ay - 2xy - y^2, \quad \frac{\partial V}{\partial y} = ax - x^2 - 2xy.$$

Equating these partial derivatives to zero, we shall have

$$\left.\begin{aligned} ay - 2xy - y^2 &= 0, \\ ax - x^2 - 2xy &= 0. \end{aligned}\right\}$$

Since for the desired parallelepiped the quantities x and y are automatically not equal to zero, we may reduce our equations by them. After simple transformations we obtain the system

$$\left.\begin{aligned} 2x + y &= a, \\ x + 2y &= a. \end{aligned}\right\}$$

Solving this system in a usual way, we find

$$x = \frac{a}{3} \quad \text{and} \quad y = \frac{a}{3}.$$

Hence, also

$$z = \frac{a}{3}.$$

Thus, the sought-for parallelepiped is a cube whose edge is equal to $\frac{a}{3}$ (we can prove strictly that its volume under the given conditions is the greatest).

Sec. 179. An Absolute Extremum of a Function

Let us consider a certain set G of points in the plane (or in space).

A point M is called *interior* for the set G if it belongs to this set together with some neighbourhood (Fig. 216).

A point N is termed *boundary* for the set G if in any of its complete neighbourhoods there are points both belonging to G, and not belonging to it (Fig. 216). The point N itself not necessarily belongs to the set G.

The set of all boundary points of the set G is called its *boundary* Γ.

Definition 1. *A set G will be called the domain if all of its points are interior*.*

The set $G = G \cup \Gamma$ with the boundary Γ attached is called a *closed domain.*

* Such a set is usually termed *open.*

A domain is said to be *bounded* if it is entirely contained inside a circle (or a sphere) of a sufficiently large radius.

Example 1. The interior K of the circle (Fig. 217)

$$x^2 + y^2 < 1$$

is a domain. Its boundary is the circumference of the circle $x^2 + y^2 = 1$. The circle (or rather, the disk) with the boundary attached, i.e. the set of points for which $x^2 + y^2 \leqslant 1$, is a closed domain.

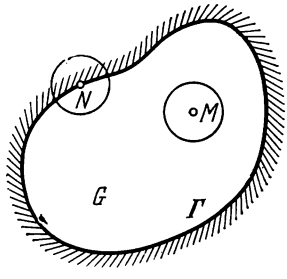


FIG. 216

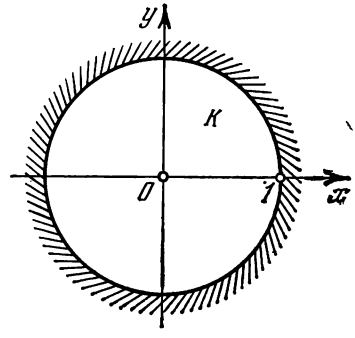


FIG. 217

Definition 2. *The least or the greatest value of a function in a given domain is called the absolute extremum* (*absolute minimum* or *absolute maximum*, respectively) *of the function in this domain.*

Weierstrass' theorem states: *A function continuous in a bounded and closed domain reaches its least and greatest values in this domain.*

Theorem. *The absolute extremum of a function in a given domain is reached either at a critical point of the function belonging to this domain or at a boundary point of the domain.*

Example 2. Find the absolute extremum of the function $z = xy$ in a triangular domain S with the vertices O (0, 0), A (1, 0), and B (0, 2) (Fig. 218).

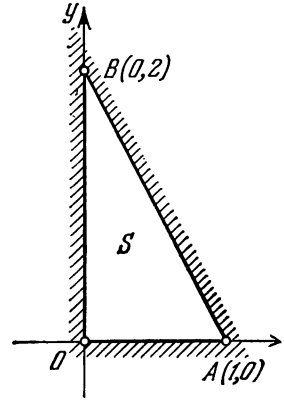


FIG. 218

We have

$$\frac{\partial z}{\partial x} = y, \quad \frac{\partial z}{\partial y} = x.$$

Hence, we find the critical point O (0, 0) with the coordinates $x = 0$, $y = 0$ belonging to the domain S.

Let us investigate the behaviour of the function z on the boundary $\Gamma = OABO$ of the domain S.

On the segment OA we have $y = 0$ $(0 \leqslant x \leqslant 1)$. Therefore, $z = 0$.

Analogously, on the segment OB: $x = 0$ $(0 \leqslant y \leqslant 2)$ and we get $z = 0$.

Finally, the segment AB has the equation $\frac{x}{1} + \frac{y}{2} = 1$, or $y = 2 - 2x$ $(0 \leqslant x \leqslant 1)$.

Hence,

$$z = xy = 2x - 2x^2.$$

We have

$$\frac{dz}{dx} = 2 - 4x = 0$$

for $x = \frac{1}{2}$, whence $y = 1$.

Since

$$\frac{d^2z}{dx^2} = -4 < 0,$$

at the point $D\left(\frac{1}{2}, 1\right)$ the function z reaches its maximum $M = \frac{1}{2} \cdot 1 = \frac{1}{2}$ on the interval AB.

And so, the least value of the function z in the domain S is $m = 0$ and it is realized at the points of the segments OA and OB constituting a part of the boundary Γ of the domain S, its greatest value $M = \frac{1}{2}$ is reached at the point $D\left(\frac{1}{2}, 1\right)$ belonging to the segment AB of the boundary Γ.

Sec. 180. Constructing Empirical Formulas by the Method of Least Squares

In natural science, in physics and biology in particular, we have to use empirical formulas prompted by experience and observations. One of the best methods of obtaining such formulas is the *method of least squares*. We are going to set forth the main idea of this method, confining ourselves to the case of linear relation of two quantities.

Suppose we want to establish a relationship between two quantities x and y (for instance, between temperature and elongation of a rectilinear metal rod). To this end, we carry out appropriate measure-

ments (say, n measurements) and tabulate the results thus obtained:

x	x_1	x_2	x_3	...	x_n
y	y_1	y_2	y_3	...	y_n

We shall consider x and y as rectangular coordinates of some points in the plane. Suppose the points with the corresponding coordinates taken from the above table lie almost on a straight line; for instance, they are arranged in the way shown in Fig. 219. Naturally, in this case we regard that between x and y there exists an approximate *linear relation*, i.e. y is a linear function of x expressed by the formula

$$y = ax + b, \tag{180-1}$$

where a and b are certain constant coefficients to be determined. Formula (180-1) can be represented in such a form:

$$ax + b - y = 0. \tag{180-2}$$

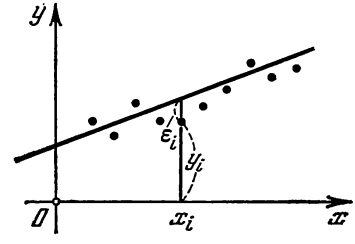


FIG. 219

Since the points (x, y) only approximately lie on our straight line, formulas (180-1) and (180-2) turn out to be approximate. Consequently, substituting in (180-2) x and y by their values x_1, y_1; x_2, y_2; ...; x_n, y_n, taken from the above table, we obtain the following equalities:

$$\left.\begin{aligned} ax_1 + b - y_1 &= \varepsilon_1, \\ ax_2 + b - y_2 &= \varepsilon_2, \\ &\cdots\cdots\cdots \\ ax_n + b - y_n &= \varepsilon_n, \end{aligned}\right\} \tag{180-3}$$

where

$$\varepsilon_1, \ \varepsilon_2, \ \ldots, \ \varepsilon_n \tag{180-4}$$

are certain numbers, generally speaking, not equal to zero which will be called *errors*.

It is required to choose the coefficients a and b so as to obtain the errors as small by their absolute value as possible*. The *method of the least squares* consists in the following: we have to choose the coefficients a and b so that the sum of the squared errors is as small as

* This problem is a simplest particular case of the general problem on best approximation of functions set forth and solved for quite broad conditions by the well-known Russian mathematician P. L. Chebyshev.

possible, i.e. we require that the sum

$$U = \varepsilon_1^2 + \varepsilon_2^2 + \ldots + \varepsilon_n^2 \qquad (180\text{-}5)$$

be the least possible. If this minimal sum of the squares turns out to be small, then the errors themselves will be small by absolute value.

Note. We could try to take the sum of the errors instead of the sum of their squares and to find the coefficients a and b so that this sum is as small as possible by absolute value. But this, obviously, will not ensure the smallness of the errors, since the latter can have different signs. This cannot happen if the problem is solved by the method of the least squares.

Substituting in (180-5) numbers (180-4) by their values from equalities (180-3), we shall obtain such a quantity:

$$U = (ax_1 + b - y_1)^2 + (ax_2 + b - y_2)^2 + \ldots$$
$$\ldots + (ax_n + b - y_n)^2. \qquad (180\text{-}6)$$

In formula (180-6) the numbers x_1, y_1, x_2, y_2, . . ., x_n, y_n are yielded by measurements and, therefore, they are considered as given, whereas the coefficients a and b are unknown quantities to be determined.

Thus, U may be regarded as a function of two variables a and b. Let us choose the coefficients a and b so that the function U attains the least possible value. According to the preceding section, we must observe the following conditions:

$$\frac{\partial U}{\partial a} = 0, \quad \frac{\partial U}{\partial b} = 0.$$

Taking these partial derivatives and supplying them with the coefficient $\frac{1}{2}$ (for the convenience of further computations), we shall have

$$\frac{1}{2}\frac{\partial U}{\partial a} = (ax_1 + b - y_1)\,x_1 + (ax_2 + b - y_2)\,x_2 + \ldots$$
$$\ldots + (ax_n + b - y_n)\,x_n,$$

$$\frac{1}{2}\frac{\partial U}{\partial b} = (ax_1 + b - y_1) + (ax_2 + b - y_2) + \ldots + (ax_n + b - y_n).$$

Hence, equating these partial derivatives to zero, we shall obtain a linear system of two equations in two unknowns a and b:

$$\left.\begin{aligned}(ax_1 + b - y_1)\,x_1 + (ax_2 + b - y_2)\,x_2 + \ldots + (ax_n + b - y_n)\,x_n = 0,\\ (ax_1 + b - y_1) + (ax_2 + b - y_2) + \ldots + (ax_n + b - y_n) = 0.\end{aligned}\right\}$$

Carrying out ordinary algebraic transformations, we bring this system to a simpler form:

$$a(x_1^2 + x_2^2 + \ldots + x_n^2) + b(x_1 + x_2 + \ldots + x_n) = \\ = x_1y_1 + x_2y_2 + \ldots + x_ny_n,$$

$$a(x_1 + x_2 + \ldots + x_n) + b \cdot n = y_1 + y_2 + \ldots + y_n,$$

or, introducing brief notations, we have

$$\left.\begin{aligned} a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} x_iy_i, \\ a\sum_{i=1}^{n} x_i + bn &= \sum_{i=1}^{n} y_i. \end{aligned}\right\} \qquad (180\text{-}7)$$

This is the final form of the so-called *normal system* of the method of the least squares. From this system we find a and b and then substitute them into our empirical formula

$$y = ax + b.$$

Example. Let the results of measurements of the quantities x and y and their processing be tabulated as follows:

i	x_i	y_i	x_i^2	x_iy_i	ε_i
1	−2	0.5	4	−1.0	−0.175
2	0	1	0	0	0.175
3	1	1.5	1	1.5	0.100
4	2	2	4	4	0.025
5	4	3	16	12	−0.125
Σ	5	8.0	25	16.5	0

Let us put

$$y = ax + b.$$

Normal system (180-7) has the form

$$\left.\begin{aligned} 25a + 5b &= 16.5, \\ 5a + 5b &= 8. \end{aligned}\right\}$$

Solving these equations, we get

$$a = 0.425, \quad b = 1.175.$$

Hence,

$$y = 0.425\,x + 1.175.$$

The last column of the table gives the corresponding errors.

Exercises

1. Determine the domains of existence of the given functions:

(a) $u = x\sqrt{1-y^2}$; (b) $u = \sqrt{x^2+y^2-1}$; (c) $u = \ln(x+y)$.

2. Construct the level lines of the following functions:

(a) $z = x - y$; (b) $z = \frac{y}{x}$; (c) $z = x^2 - y^2$; (d) $z = y - x^2$.

3. Construct the level surfaces of the given functions:

(a) $u = x + y + z$; (b) $u = y^2 + z^2$.

4. Find the first-order partial derivatives and the total differentials of the following functions:

(a) $u = x^2 - 2xy - y^2$; (b) $u = \frac{x}{\sqrt{x^2+y^2}}$; (c) $u = \sqrt{\frac{x}{y}}$.

5. Find the derivative $\frac{\partial u}{\partial l}$ of the function

$$u = \frac{2x}{\sqrt{y}}$$

at the point $M_0(1, 1)$ in the direction l making an angle α with the x-axis if $\alpha = 0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$, $\frac{7\pi}{4}$, 2π.

Evaluate $|\operatorname{grad} u\,(M_0)|$.

6. The isotherms of temperature u have the form

$$x^2 + y^2 = C.$$

Knowing that for the isotherm passing through the point A (3, 4) the value of the function is $u = 30°$, and for the isotherm passing through the point B (5, 1) the value of the function is $u = 35°$, find approximately $|\operatorname{grad} u\,(A)|$ if the linear distances are given in kilometres.

7. Find the second-order derivatives of the following functions:

(a) $u = x\ln y + \sqrt{\sin x}$; (b) $u = \frac{xy}{z^2}$; (c) $u = \ln(x+y^2)$;

(d) $u = \sqrt[3]{x} + \arctan y$; (e) $u = \left(\frac{x}{y}\right)^z$.

8. Check that

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x},$$

if

$$u = x\arcsin\sqrt{\frac{x}{y}}.$$

9. The weight of a body in air is $p = 5.2\ \text{N} \pm 0.2\ \text{N}$ and its weight in water is $q = 3.1\ \text{N} \pm 0.1\ \text{N}$. With what accuracy is it possible to determine the specific weight of the body?

10. Find out which of the given expressions are total differentials:

(a) $dx + y\,dy$; (b) $\frac{dx}{y^2} - \frac{2x\,dy}{y^3}$ $(y > 0)$; (c) $\frac{y\,dx - x\,dy}{(x+y)^2}$ $(x > 0,\ y > 0)$.

11. Under what condition will the sum of three positive terms x, y, and z be the least if the product of these terms is a constant equal to a?

12. In a hemisphere of radius a inscribe a rectangular parallelepiped of the greatest volume.

13. In a right circular cone the radius of whose base is r and the altitude is h inscribe a rectangular parallelepiped of the greatest volume.

14. The results of measurement of the quantities x and y are given by the following table:

x	10	20	30	40	50	60
y	150	100	40	0	−60	−100

Assuming that between x and y there exists a linear relation

$$y = ax + b,$$

determine the coefficients a and b, using the method of the least squares.

15. Substitute approximately the parabola $y = x^2$ on the region $2 \leqslant x \leqslant 4$ by a straight line $Y = kx + b$ so that the mean square error $\omega = \int_2^4 (y - Y)^2\,dx$ is the least.

16. Find the absolute extremum of the function

$$u = 3x + 4y + 5$$

in the domain $x^2 + y^2 \leqslant 1$.

Hint. Use the parametric equations of the boundary of the given domain.

Chapter 21

Series

Sec. 181. Examples of Infinite Series

In this chapter we shall study the properties of infinite series, as well as expansion of functions into power series and trigonometric series.

An example of an infinite series which is considered in elementary algebra is an infinitely decreasing geometric progression*

$$a + aq + aq^2 + \ldots + aq^{n-1} + \ldots, \tag{181-1}$$

where $|q| < 1$. Here each subsequent term is formed from the preceding one according to a definite law, namely: each subsequent term is obtained from the preceding one by multiplying it by the ratio of the progression q. Consequently, the nth term, the so-called *general term* of the progression is expressed by the formula

$$u_n = aq^{n-1} \qquad (n = 1, 2, 3, \ldots).$$

Another example of an infinite series is a *harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n} + \ldots, \tag{181-2}$$

whose nth term is $u_n = \frac{1}{n}$.

There also exist series composed of functions, for instance,

$$\frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots + \frac{x^n}{1 \cdot 2 \ldots n} + \ldots, \tag{181-3}$$

where the nth term is

$$u_n = \frac{x^n}{1 \cdot 2 \ldots n} = \frac{x^{n**}}{n!}.$$

---

* Precisely speaking, a series formed from the terms of an infinitely decreasing geometric progression. Henceforward, for the sake of brevity, we shall call series (181-1) simply geometric progression.

** The brief notation $n!$ (read: "n factorial") denotes the product of all natural numbers not exceeding the number n.

The law of formation of the terms of a series is given by its nth term which is called the *general term of a series*. Having the formula of the general term of a series, we can find any term of this series.

There arises a problem: *to investigate the properties of an infinite series, assuming that its nth term is known.*

We should like to note here that the theory of series is frequently applied for practical purposes, since it provides the possibility to represent under broad conditions a given function in the form of an infinite series of simpler functions, for instance, of a series of polynomials which enables us to easily find the approximate values of a function for a given value of the argument.

Sec. 182. Convergence of a Series

We are going to give the general notion of an infinite series. Suppose we have a certain infinite sequence of numbers or functions formed according to a definite law and formally connected to one another with the plus sign:

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots = \sum_{n=1}^{\infty} u_n. \tag{182-1}$$

Such an expression is called an *infinite series* or simply *series* and the summands $u_1, u_2, u_3, \ldots, u_n, \ldots$ are called the *terms* of this series. If the terms of a series are numbers, then the series is said to be *numerical*; and if its terms are functions, then the series is called *functional*.

Note that the study of functional series is reduced to studying numerical series. Indeed, if

$$u_n = f_n(x) \quad (n = 1, 2, \ldots),$$

then for each fixed value of the argument x we obtain the corresponding numerical series (182-1) whose properties are to be investigated.

The term u_n of series (182-1) occupying the nth place as counted from the beginning is called the *general term* of this series. Series (182-1) is considered to be specified if its general term expressed as the function of the number n is known. This is exemplified by series (181-1), (181-2), and (181-3) from the preceding section, where we have respectively:

$$u_n = aq^{n-1}, \quad u_n = \frac{1}{n}, \quad u_n = \frac{x^n}{n!}.$$

Considering that series (182-1) is given, we can form *partial sums* of this series, i.e.

$$S_1 = u_1,$$

$$S_2 = u_1 + u_2,$$

$$S_3 = u_1 + u_2 + u_3,$$

. . . . . .

$$S_n = u_1 + u_2 + u_3 + \ldots + u_{n-1} + u_n.$$

We first assume that (182-1) is a numerical series. Two cases are possible here.

I. Let, as the number n increases without bound, the sum of the first n terms S_n of series (182-1) tend to a finite limit S:

$$\lim_{n \to \infty} S_n = S.$$

Then series (182-1) is said *to be convergent* and the number S is called the *sum* of this series.

II. Let, as the number n increases without bound, the sum of the first n terms S_n of series (182-1) either increase infinitely, or in general not tend to any limit. Then we say that series (182-1) *diverges* and has no sum.

Definition. *A numerical series is called convergent if the sequence of its partial sums has a finite limit. This limit is termed the sum of the convergent series. If a sequence of partial sums has no finite limit, then the series is called divergent.*

If (182-1) is a functional series, i.e.

$$u_n = f_n(x) \qquad (n = 1, 2, \ldots),$$

then for each fixed value x_0 of the argument x the corresponding numerical series

$$f_1(x_0) + f_2(x_0) + \ldots + f_n(x_0) + \ldots \qquad (182\text{-}1')$$

either converges or diverges. Accordingly, x_0 is called either the *point of convergence* or the *point of divergence* of the given functional series, and the set of all points of convergence of a functional series is termed the *domain of its convergence.*

If $u_n = f_n(x)$ $(n = 1, 2, \ldots)$ and functional series (182-1) converges at each point x of a certain set, then it is said *to be convergent on this set*, and the function $S = S(x)$ determined for each value of x under consideration by the formula

$$S = \lim_{n \to \infty} S_n(x)$$

is called the *sum of this series* on the given set.

If series (182-1) converges, then the difference between the sum S and its partial sum S_n

$$R_n = S - S_n$$

is called the *nth remainder of the series*. The remainder R_n of the series represents the error committed by substituting the partial sum S_n of the first n terms of this series for the sum S of the series.

Since S is the limit of the sequence S_n, we, obviously, have

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} (S - S_n) = 0.$$

Therefore, *taking a sufficiently large number of terms of a convergent series, we can compute the sum of this series to any extent of accuracy.*

Hence, it is clear that the primary task of the theory of series is investigation of convergence of a series. The problem of finding the sum of a convergent series is of minor importance, since, after the convergence of a series is established, its sum can be easily found in most practically important cases.

Let us illustrate the notions of convergence and divergence by a couple of examples.

Example 1. Consider an infinite geometric progression

$$a + aq + aq^2 + \ldots + aq^{n-1} + \ldots, \qquad (182\text{-}2)$$

where $a \neq 0$.

It is known that the sum S_n of the first n terms of a geometric progression is expressed by the formula

$$S_n = \frac{a(1-q^n)}{1-q} = \frac{a}{1-q} - \frac{a}{1-q} \cdot q^n.$$

Here, we have to consider separately four cases.

(1) Let $|q| < 1$. Then q^n tends to zero, as $n \to \infty$, and, consequently,

$$\lim_{n\to\infty} S_n = \frac{a}{1-q}.$$

In this case series (182-2) converges and its sum is equal to

$$S = \frac{a}{1-q}.$$

(2) Let $|q| > 1$. Then, as n increases without bound, q^n increases infinitely by absolute value and, consequently, the sum of the first n terms S_n also increases unboundedly. Therefore, in this case series (182-2) diverges and has no sum.

(3) Let $q = 1$. Then series (182-2) has the following form

$$a + a + \ldots + a + \ldots \qquad (a \neq 0).$$

It is easy to see that $S_n = na$ and, consequently, as n increases without bound, the sum S_n increases infinitely. Therefore, in this case series (182-2) diverges.

(4) Let $q = -1$. In this case series (182-2) attains the form

$$a - a + a - a + \ldots + (-1)^{n-1} a + \ldots$$

The quantity S_n will be equal either to zero or to a depending on whether n is even or odd. It is clear that S_n for $a \neq 0$ does not tend to any limit as n increases infinitely. In this case series (182-2) diverges.

Consequently, *infinite geometric progression* (182-2) *converges if and only if the absolute value of its ratio is less than unity*, i.e. $|q| < 1$.

Example 2. Suppose we have a series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \ldots + \frac{1}{n(n+1)} + \ldots \quad (182\text{-}3)$$

We are going to show that this series converges. Let us take the sum of its first n terms:

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \ldots + \frac{1}{n(n+1)}.$$

As is easily seen, separate terms may be represented as follows

$$\frac{1}{1\cdot 2} = 1 - \frac{1}{2}, \quad \frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}, \quad \frac{1}{3\cdot 4} = \frac{1}{3} - \frac{1}{4},$$

$$\frac{1}{4\cdot 5} = \frac{1}{4} - \frac{1}{5}, \ \ldots, \quad \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore,

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots$$

$$\ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Hence,

$$S_n = 1 - \frac{1}{n+1}$$

and, consequently,

$$\lim_{n\to\infty} S_n = 1.$$

Thus, series (182-3) converges, and its sum is equal to unity.

The further properties of series belong to *numerical series* if otherwise is not stated explicitly.

We are going now to indicate some elementary properties of series

Theorem 1. *The convergence of the series*

$$\sum_{n=1}^{\infty} u_n$$

will not be violated if all of its terms are multiplied by one and the same non-zero number k, and for the sums of these series the following equality is fulfilled:

$$\sum_{n=1}^{\infty} ku_n = k \sum_{n=1}^{\infty} u_n.$$

The proof of this theorem follows directly from the passage to the limit, as $N \to \infty$, in the equality

$$\sum_{n=1}^{N} ku_n = k \sum_{n=1}^{N} u_n.$$

The *sum* (*difference*) of two series

$$\sum_{n=1}^{\infty} u_n \quad \text{and} \quad \sum_{n=1}^{\infty} v_n$$

is understood as a series of the form (respectively)

$$\sum_{n=1}^{\infty} (u_n \pm v_n).$$

Theorem 2. *The sum* (*difference*) *of two convergent series is a convergent series, where*

$$\sum_{n=1}^{\infty} (u_n \pm v_n) = \sum_{n=1}^{\infty} u_n \pm \sum_{n=1}^{\infty} v_n. \tag{182-4}$$

Indeed, since

$$\sum_{n=1}^{N} (u_n \pm v_n) = \sum_{n=1}^{N} u_n \pm \sum_{n=1}^{N} v_n$$

for any finite N in the limit, as $N \to \infty$, we shall obtain equality (182-4).

Sec. 183. A Necessary Condition for Convergence of a Series

Theorem. *If the series*

$$u_1 + u_2 + \ldots + u_{n-1} + u_n + \ldots$$

converges, then, as the number n increases without bound, its nth term u_n *tends to zero.*

Proof. We have

$$S_{n-1} = u_1 + u_2 + \ldots + u_{n-1}$$

and

$$S_n = u_1 + u_2 + \ldots + u_{n-1} + u_n.$$

Hence

$$u_n = S_n - S_{n-1}.$$

Since the given series converges, we have

$$\lim_{n\to\infty} S_n = S \quad \text{and} \quad \lim_{n\to\infty} S_{n-1} = S.$$

Hence

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0,$$

which was to be proved.

Corollary. *If the nth term of a series, as its number n increases infinitely, does not tend to zero, then this series diverges.*

The above proved necessary condition for convergence of a series is, generally speaking, not sufficient. We can give examples of series whose general term u_n tends to zero as $n \to \infty$, but the series, nevertheless, diverges.

Example. Consider the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots + \frac{1}{n} + \ldots \qquad (183\text{-}1)$$

The general term of this series

$$u_n = \frac{1}{n}$$

tends to zero as n increases infinitely. Nevertheless, we shall show that series (183-1) diverges. To this effect, let us take the sum of 2^m first terms of series (183-1) and group these terms in the following way:

$$S_{2^m} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{2 \text{ terms}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{2^2 \text{ terms}} +$$

$$+ \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right)}_{2^3 \text{ terms}} + \ldots$$

$$\ldots + \underbrace{\left(\frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \ldots + \frac{1}{2^m}\right)}_{2^{m-1} \text{ terms}}.$$

It is easily seen that

$$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2},$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2},$$

$$\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}>$$

$$>\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{8}{16}=\frac{1}{2},$$

. . . . . .

$$\frac{1}{2^{m-1}+1}+\frac{1}{2^{m-1}+2}+\ldots+\frac{1}{2^m}>$$

$$>\underbrace{\frac{1}{2^m}+\frac{1}{2^m}+\ldots+\frac{1}{2^m}}_{2^{m-1}\text{ terms}}=\frac{2^{m-1}}{2^m}=\frac{1}{2}.$$

Thus, the sum of the terms in each parentheses is more than $\frac{1}{2}$. Since the total number of the parentheses (without counting the first two terms) is, obviously, equal to $m-1$, we have

$$S_{2^m}>1+\frac{m}{2}.$$

If the number of terms $n=2^m$ in the sum S_{2^m} increases infinitely, then the exponent m also increases unboundedly. Therefore S_{2^m} tends to infinity and, consequently, harmonic series (183-1) diverges.

Hence, the above considered necessary condition (or test) for convergence, generally speaking, gives no possibility to judge whether a given series converges or not. We shall pass over now to establishing such tests which will enable us in a number of cases to give an exact answer whether a given series converges or diverges.

Sec. 184. Comparison Tests

To prove the further theorems, we shall need the following lemma:

Lemma. *If in the series*

$$u_1+u_2+\ldots+u_p+u_{p+1}+u_{p+2}+\ldots \tag{184-1}$$

we reject a finite number of first initial terms, for instance, p terms, then we shall get the series

$$u_{p+1}+u_{p+2}+u_{p+3}+\ldots, \tag{184-2}$$

which converges (diverges) simultaneously with the given series.

Proof. We denote the sum of the rejected terms by Q:

$$Q=u_1+u_2+\ldots+u_p.$$

Let S_n be the sum of the first n terms of series (184-1) and S'_n the sum of the first n terms of series (184-2). Then, obviously, we have the equality

$$S_{n+p} = Q + S'_n.$$

Hence

$$S'_n = S_{n+p} - Q.$$

Let us assume that series (184-1) converges, and let

$$\lim_{n\to\infty} S_n = S,$$

and, consequently, also

$$\lim_{n\to\infty} S_{n+p} = S.$$

In this case

$$\lim_{n\to\infty} S'_n = S - Q,$$

and, hence, series (184-2) also converges.

Let us now assume that series (184-2) converges, and let

$$\lim_{n\to\infty} S'_n = S';$$

then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n+p} = S' + Q.$$

Therefore, series (184-1) also converges.

Thus, we have proved that from convergence of one of our series there follows convergence of the other, and vice versa.

The lemma is proved completely.

Corollary 1. *When investigating a series for convergence, a finite number of its terms may be ignored.*

Corollary 2. *If series* (184-1) *converges and* S *is its sum, then the nth remainder of this series*

$$R_n = S - S_n$$

represents the sum of the series

$$u_{n+1} + u_{n+2} + u_{n+3} + \ldots,$$

i.e.

$$R_n = u_{n+1} + u_{n+2} + u_{n+3} + \ldots$$

Let us now prove the following theorem.

Comparison Test. *If the terms of the series*

$$u_1 + u_2 + \ldots + u_n + \ldots \qquad (184\text{-}3)$$

are positive (or, which is more precise, non-negative) and do not exceed the corresponding terms of the convergent series

$$v_1 + v_2 + \ldots + v_n + \ldots, \tag{184-4}$$

then the given series (184-3) *also converges.*

Proof. We introduce the following notations

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n$$

and

$$S'_n = v_1 + v_2 + v_3 + \ldots + v_n.$$

Since (184-4) converges, we have

$$\lim_{n\to\infty} S'_n = S',$$

where S' is the sum of series (184-4). According to the condition of the theorem, the inequalities

$$0 \leqslant u_1 \leqslant v_1, \quad 0 \leqslant u_2 \leqslant v_2, \ \ldots, \quad 0 \leqslant u_n \leqslant v_n, \ \ldots$$

are fulfilled.

Hence it follows that

$$S_n \leqslant S'_n \leqslant S'.$$

Since the terms of series (184-3) are positive, with an increase in n the sum S_n increases monotonically but never exceeds S'. As is known (Sec. 53), *any monotonically increasing bounded sequence has a limit.* Therefore, as n increases unboundedly, S_n tends to a definite limit and, consequently, series (184-3) converges.

Corollary. *If the terms of a certain series are not less than the corresponding terms of a positive series* and this latter series diverges, then the given series also diverges.*

Indeed, if the first series converged, then by virtue of the theorem, the second series would also converge which contradicts our condition.

Note. By the lemma, the test for comparing series (184-3) and (184-4), as well as the corresponding corollary, hold true if the corresponding inequalities relating their terms are fulfilled beginning with a certain number ($n \geqslant N$).

Let us apply this test to prove convergence of certain series by comparing them with the series whose convergence is already known.

Example 1. Consider the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} + \ldots \tag{184-5}$$

* That is a series with positive (more precisely, non-negative) terms.

Rejecting the first term, we compare it with the convergent series (181-3):

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\ldots+\frac{1}{n(n+1)}+\ldots$$

Obviously,

$$\frac{1}{2^2}<\frac{1}{1\cdot 2},\quad \frac{1}{3^2}<\frac{1}{2\cdot 3},\ \ldots,\quad \frac{1}{(n+1)^2}<\frac{1}{n(n+1)},\ \ldots$$

Hence, by the lemma and comparison test, series (184-5) converges. Then, from the comparison with (184-5) it follows that the series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots$$

converges if $p>2$. We can prove that this latter series converges for $p>1$ and diverges for $p\leqslant 1$.

Example 2. Consider the series

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}+\ldots \qquad (184\text{-}6)$$

Since $\frac{1}{\sqrt{n}}>\frac{1}{n}$ $(n=2,\ 3,\ \ldots)$, it follows from the comparison with the harmonic series that (184-6) diverges.

Sec. 185. D'Alembert's Test for Convergence

There are many tests for convergence of series that enable us to judge, by the behaviour of its coefficients, whether a given series is convergent or divergent. Let us consider one of them.

D'Alembert Test. *Let all terms of the series*

$$u_1+u_2+\ldots+u_n+\ldots$$

be positive and let the ratio of the $(n+1)$*th term to the nth term have a limit equal to a certain number* l*, as* $n\to\infty$, i.e.

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=l.$$

Then, three cases are possible:

1°. $l<1$. *The given series converges.*

2°. $l>1$. *The given series diverges.*

3°. $l=1$. *The series can either converge or diverge, i.e. the test gives no definite answer.*

Proof. Suppose we have a series

$$u_1+u_2+\ldots+u_n+u_{n+1}+\ldots, \qquad (185\text{-}1)$$

composed of positive numbers, and let

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=l.$$

Then for a sufficiently large n, i.e. for n not less than a certain number N, we have

$$\left|\frac{u_{n+1}}{u_n}-l\right|<\varepsilon,$$

where ε is a preassigned arbitrary small positive number. Hence,

$$-\varepsilon<\frac{u_{n+1}}{u_n}-l<+\varepsilon$$

or

$$l-\varepsilon<\frac{u_{n+1}}{u_n}<l+\varepsilon, \tag{185-2}$$

only if $n\geqslant N$.

Let us consider each case separately.

1°. Let $l<1$. We can take the number ε so small that $l+\varepsilon$ will also be less than 1; then, setting $l+\varepsilon=q$, we get

$$0<q<1.$$

By (185-2), we have

$$\frac{u_{n+1}}{u_n}<q$$

or

$$u_{n+1}<u_nq,$$

and this last inequality will be fulfilled if

$$n=N,\quad N+1,\quad N+2,\ \ldots$$

Assigning these values to the number n, we shall obtain a set of inequalities:

$$u_{N+1}<u_Nq,$$

$$u_{N+2}<u_{N+1}q<u_Nq^2,$$

$$u_{N+3}<u_{N+2}q<u_Nq^3,$$

$$\ldots\ldots\ldots\ldots\ldots$$

Thus, the terms of the series

$$u_{N+1}+u_{N+2}+u_{N+3}+\ldots \tag{185-3}$$

are less than the corresponding terms of the geometric progression

$$u_Nq+u_Nq^2+u_Nq^3+\ldots \tag{185-4}$$

Since the ratio q of progression (185-4) is less than unity, (185-4) converges (see Sec. 182, Example 1). But then, by the comparison

test and the corresponding note, both series (185-3) and (185-1) converge.

2°. Let now $l > 1$. We can take $\varepsilon > 0$ so small that the number $l - \varepsilon$ will also be more than unity. Then for a sufficiently large n, by (185-2), we shall have

$$\frac{u_{n+1}}{u_n} > 1,$$

or

$$u_{n+1} > u_n$$

for

$$n = N, \quad N + 1, \quad N + 2, \ldots$$

Hence

$$u_N < u_{N+1} < u_{N+2} < \ldots$$

Thus, the terms of series (185-1), beginning with a certain number N increase with an increase in their number, all the time being positive. Consequently, u_n does not tend to zero as $n \to \infty$. Therefore, by the corollary from the necessary condition (test) for convergence, (Sec. 183), series (185-1) diverges, and its general term does not tend to zero.

3°. If $l = 1$, D'Alembert's test is inapplicable: there exist both convergent and divergent series for which l is equal to 1. In this case we must resort either to the comparison theorem, or to other tests.

Note 1. If (185-1) is a functional series, i.e.

$$u_n = f_n(x) > 0$$

and $l = l(x)$ is the corresponding limit, then our scheme 1°, 2°, and 3° holds for each x.

Note 2. From the proof of D'Alembert's test for case 2° it follows that *if for a certain series*

$$u_1 + u_2 + \ldots + u_n + \ldots$$

the inequality

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} > 1,$$

is fulfilled, then the nth term u_n *of this series does not tend to zero as* $n \to \infty$.

Example 1. Consider the series

$$\frac{a}{1} + \frac{a^2}{2} + \frac{a^3}{3} + \ldots + \frac{a^n}{n} + \frac{a^{n+1}}{n+1} + \ldots \tag{185-5}$$

where a is a positive number.

We have

$$\frac{u_{n+1}}{u_n}=\frac{a^{n+1}}{n+1}:\frac{a^n}{n}=a\cdot\frac{n}{n+1}$$

and, consequently,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=a\lim_{n\to\infty}\frac{n}{n+1}=a\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=a.$$

On the basis of D'Alembert's test series (185-5) converges for $0<a<1$ and diverges for $a>1$.

For $a=1$ D'Alembert's test gives no answer, but in this case (185-5) takes the form

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots$$

This is a harmonic series which, as it was shown in Sec. 183, diverges.

Example 2. Consider the series

$$\frac{1000}{1!}+\frac{1000^2}{2!}+\frac{1000^3}{3!}+\ldots$$

with the general term

$$u_n=\frac{1000^n}{n!}.$$

We have

$$u_{n+1}=\frac{1000^{n+1}}{(n+1)!}.$$

Hence

$$\frac{u_{n+1}}{u_n}=\frac{1000^{n+1}\cdot n!}{(n+1)!\cdot 1000^n}=\frac{1000}{n+1}$$

and, consequently,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{1000}{n+1}=0.$$

Therefore, the series converges. Note that the terms of the given series first keep increasing (up to 1000th term!), and then begin to rapidly decrease. Such a series is not suitable for practical computations.

Example 3. According to D'Alembert's test, for the series

$$\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}+\ldots$$

the corresponding limit $l=1$. As is known, (Sec. 184, Example 1), this series converges.

Sec. 186. Absolute Convergence

The above considered sufficient conditions (tests) for convergence referred to positive series (i.e. to series with positive terms). Series with negative terms possess similar properties.

Let us now consider series consisting of positive, negative, and zero terms. Such series are called *alternating*.

Theorem. *If for the alternating series*

$$u_1+u_2+\ldots+u_n+\ldots=\sum_{n=1}^{\infty}u_n \tag{A}$$

the series made up from the absolute values of its terms:

$$|u_1|+|u_2|+\ldots+|u_n|+\ldots=\sum_{n=1}^{\infty}|u_n|, \tag{B}$$

converges, then the given series also converges.

Proof. Consider the auxiliary series

$$(u_1+|u_1|)+(u_2+|u_2|)+\ldots+(u_n+|u_n|)+\ldots=$$

$$=\sum_{n=1}^{\infty}(u_n+|u_n|). \tag{C}$$

Since

$0 \leqslant u_n + |u_n| \leqslant 2|u_n|$

($n = 1, 2, \ldots$) and the series

$$\sum_{n=1}^{\infty}(2|u_n|) \tag{B'}$$

converges, by virtue of convergence of the series (B), then, on the basis of the comparison test (Sec. 184), the series (C) also converges. But the series (A) represents the difference of two convergent series

$$\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}(u_n+|u_n|)-\sum_{n=1}^{\infty}|u_n|$$

and, hence, is a convergent series (see Sec. 182).

The theorem is proved.

Note. The converse is false. Namely, *if a given series converges, then the series formed from the absolute values of its terms not necessarily converges*; *this series can also diverge.*

Thus, all convergent series fall into two classes.

Convergent series for which the absolute values of their terms form convergent series belong to the first class. Such series are said to be *absolutely convergent.*

Convergent series for which the absolute values of their terms form divergent series belong to the second class. Such convergent series are said to be *not absolutely convergent* or *conditionally convergent.*

Definition. *A series for which the absolute values of its terms form a convergent series is called absolutely convergent.*

If a given series is convergent whereas the series composed of the absolute values of its terms is divergent the given series is said to be conditionally convergent.

For instance, the convergent series

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\ldots$$

is absolutely convergent, since the series composed of the absolute values of its terms

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots,$$

also converges. (Both series are geometric progressions with the ratios equal to $-\frac{1}{2}$ and $\frac{1}{2}$, respectively).

On the contrary, the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots,$$

as we shall see below, is convergent (see the next section), but not absolutely convergent, since the series composed of its terms

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots,$$

diverges (a harmonic series).

The test for absolute convergence of a series. *Let for a certain series*

$$u_1+u_2+\ldots+u_n+\ldots \qquad \text{(A)}$$

the condition

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=l$$

be fulfilled.

In this case: (1) if $l<1$, *then the given series* (A) *converges absolutely*; (2) *if* $l>1$, *then the series* (A) *diverges.*

Indeed, our condition is just D'Alembert's test applied to the series of absolute values

$$|u_1|+|u_2|+\ldots+|u_n|+\ldots \qquad \text{(B)}$$

Hence, it follows that if $l<1$, then both series (A) and (B) converge and, consequently, the given series (A) is absolutely convergent.

And if $l > 1$, then, by the note to D'Alembert's test, $|u_n|$ does not tend to zero as $n \to \infty$. In this case both series (A) and (B) diverge.

Sec. 187. Alternating Series. Leibniz' Test

An *alternating series* is defined as the series of the form

$$v_1 - v_2 + v_3 - v_4 + v_5 - v_6 + \ldots + (-1)^{n-1}v_n + \ldots *, \quad (187\text{-}1)$$

where $v_n \geqslant 0$ for $n = 1, 2, 3, \ldots$, i.e. as the series in which any of its neighbouring terms have opposite signs.

Leibniz theorem (Leibniz' test). *If the absolute values of the terms of an alternating series* (187-1) *form a monotone decreasing sequence tending to zero*

$$v_1 \geqslant v_2 \geqslant v_3 \geqslant v_4 \geqslant \ldots \quad (187\text{-}2)$$

and the nth term of the series tends to zero as $n \to \infty$, *i.e.*

$$\lim_{n\to\infty} v_n = 0, \quad (187\text{-}3)$$

then the series is convergent (generally speaking, not absolutely).

Proof. Let us take the sum S_{2m} of the first $2m$ terms of series (187-1) and write it in the following way:

$$S_{2m} = (v_1 - v_2) + (v_3 - v_4) + \ldots + (v_{2m-1} - v_{2m}). \quad (187\text{-}4)$$

Since, by condition (187-2), the differences standing in parentheses in sum (187-4) are positive or equal to zero, we have

$$S_{2m} \geqslant 0.$$

If $2m$ increases, then S_{2m} does not decrease, since positive or zero terms are added each time.

On the other hand, this sum can be represented as follows:

$$S_{2m} = v_1 - (v_2 - v_3) - (v_4 - v_5) - \ldots - (v_{2m-2} - v_{2m-1}) - v_{2m}. \quad (187\text{-}5)$$

Hence,

$$S_{2m} \leqslant v_1.$$

Consequently, S_{2m}, being monotonically increasing (more precisely, not decreasing) and bounded sequence, tends to a certain limit S as $m \to \infty$ (see Sec. 53), i.e.

$$\lim_{m\to\infty} S_{2m} = S.$$

* More precisely, series (187-1) must be written in the following manner:

$$v_1 + (-v_2) + v_3 + (-v_4) + \ldots$$

But it is obvious that

$$S_{2m+1} = S_{2m} + v_{2m+1},$$

and, by (187-3), we have

$$\lim_{m\to\infty} v_{2m+1} = 0.$$

Taking this into consideration, we get

$$\lim_{m\to\infty} S_{2m+1} = \lim_{m\to\infty} S_{2m} + \lim_{m\to\infty} v_{2m+1} = S + 0 = S.$$

Thus, S_n, as $n \to \infty$, tends to one and the same limit S irrespective of whether n is even or odd. Therefore, series (187-1) converges.

Note. In approximate computation of the sum of a convergent alternating series satisfying the conditions of Leibniz' theorem, the error does not exceed the absolute value of the first rejected term.

Indeed, rejecting in a convergent alternating series all the terms after the term $(-1)^{n-1} v_n$ and denoting by ρ_n the error thus obtained, we have

$$\rho_n = (-1)^n v_{n+1} + (-1)^{n+1} v_{n+2} + \ldots =$$
$$= (-1)^n [(v_{n+1} - v_{n+2}) + (v_{n+3} - v_{n+4}) + \ldots];$$

hence

$$|\rho_n| = (v_{n+1} - v_{n+2}) + (v_{n+3} - v_{n+4}) + \ldots$$

or

$$|\rho_n| = v_{n+1} - (v_{n+2} - v_{n+3}) - (v_{n+4} - v_{n+5}) - \ldots$$

Consequently,

$$|\rho_n| \leqslant v_{n+1}.$$

Example. The series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots,$$

considered at the end of Sec. 186 is convergent, since for this series all conditions of Leibniz' theorem are fulfilled.

Sec. 188. Power Series

A series of the form

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots, \qquad (188\text{-}1)$$

arranged by increasing integral non-zero powers of the variable x and having the coefficients $a_0, a_1, a_2, \ldots, a_n, \ldots$, independent

of x is called a *power series*. Sometimes they consider the power series of a more general form:

$$a_0 + a_1 (x - a) + a_2 (x - a)^2 + \ldots + a_n (x - a)^n + \ldots, \quad (188\text{-}2)$$

where a is a certain constant number. Series (188-2) is readily reduced to the form (188-1) if we put

$$x - a = x'.$$

Therefore, henceforth we shall deal almost exclusively with power series of the form (188-1).

Let us find out the question about convergence of power series (188-1). Assigning a fixed value to the variable x, we shall obtain a numerical series which either converges or diverges depending on x.

We can prove that for any power series (188-1) there exists a finite or an infinite non-zero number R (called the *radius of convergence of a series*) such that if $R > 0$, then for $|x| < R$ the series converges, and for $|x| > R$ it diverges. For $|x| = R$, i.e. for $x = R$ and for $x = -R$, the series either converges, or diverges. The interval $(-R, R)$ is called the *interval of convergence* of a power series. If $R = +\infty$, then the interval of convergence represents the entire number line. If $R = 0$, then power series (188-1) converges only at the point $x = 0$, and, strictly speaking, there is no interval of convergence.

In simple cases the radius of convergence of power series (188-1) can be determined with the aid of D'Alembert's test. To this effect, let us consider the series composed of the absolute values of the terms of series (188-1):

$$|a_0| + |a_1||x| + |a_2||x|^2 + \ldots + |a_n||x|^n + \ldots \quad (188\text{-}3)$$

As is known from the previous sections, if series (188-3) converges, then series (188-1) will be absolutely convergent. To solve the problem on convergence of series (188-3), let us take advantage of D'Alembert's test for convergence. We denote the $(n + 1)$th term of series (188-3) by v_n, i.e.

$$v_n = |a_n||x|^n;$$

hence,

$$v_{n+1} = |a_{n+1}||x|^{n+1}.$$

Let us form the ratio

$$\frac{v_{n+1}}{v_n} = \left|\frac{a_{n+1}}{a_n}\right| |x|.$$

Suppose there exists the limit of the ratio $\left|\frac{a_{n+1}}{a_n}\right|$ as $n \to \infty$. We denote this limit by l:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l. \quad (188\text{-}4)$$

Then

$$\lim_{n\to\infty}\frac{v_{n+1}}{v_n}=l|x|. \tag{188-5}$$

Obviously, if $|x|<\frac{1}{l}$, then $l|x|<1$ and series (188-3) converges. Consequently, series (188-1) converges, and it converges absolutely.

And if $|x|>l$, then $l|x|>1$. By Note 2 from Sec. 185, both series (188-3) and (188-1) diverge.

Thus, $R=\frac{1}{l}\geqslant 0$ is the radius of convergence of power series (188-1) and on the basis of relation (188-4), we have the formula

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|. \tag{188-6}$$

The following question remains unanswered: will series (188-1) converge for $R>0$ at the end-points of the interval of convergence $(-R, R)$, i.e. when $x=R$ or $x=-R$? In each particular case this problem is solved separately.

Example. Consider the series

$$\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}+\ldots \tag{188-7}$$

Here

$$a_n=\frac{1}{n} \quad \text{and} \quad a_{n+1}=\frac{1}{n+1}.$$

According to (188-6), for the radius of convergence R of series (188-7) we have

$$R=\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}}=\lim_{n\to\infty}\frac{n+1}{n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1.$$

Consequently, series (188-7) converges in the interval $(-1, 1)$.

To solve the problem on convergence of series (188-7) at the end-points of the interval, we first set $x=1$. We obtain the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots,$$

which, as we saw, diverges.

Now we set $x=-1$. Then series (188-7) will attain the form

$$-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\ldots+\frac{(-1)^n}{n}+\ldots$$

This series converges conditionally by virtue of Leibniz' theorem.

Hence, the domain of convergence of series (188-7) is the interval $[-1, 1)$.

Sec. 189. Differentiation and Integration of Power Series

The sum of the power series

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \qquad (189\text{-}1)$$

represents the function defined in the interval of convergence $(-R, R)$ of this series where it is assumed that $R > 0$.

We can prove that the function $f(x)$ is differentiable and its derivative $f'(x)$ can be found by termwise differentiation of series (189-1), i.e.

$$f'(x) = a_1 + 2a_2x + \ldots + na_nx^{n-1} + \ldots$$

for $-R < x < R$. The same is true with respect to derivatives of higher orders.

Analogously, the indefinite integral of the function $f(x)$ for all values of x belonging to the interval of convergence can be obtained by termwise integration of series (189-1), i.e.

$$\int f(x)\,dx = C + a_0x + \frac{a_1x^2}{2} + \frac{a_2x^3}{3} + \ldots + \frac{a_nx^{n+1}}{n+1} + \ldots,$$

if $-R < x < R$.

Thus, a power series in its interval of convergence behaves with respect to the operations of differentiation and integration in the same manner as a polynomial with a finite number of terms.

Sec. 190. Expanding a Given Function into a Power Series

For applications it is important to know how to expand a given function $f(x)$ into a power series, i.e. to know how to represent the function $f(x)$ in the form of the sum of a power series, since, by the same token, we get the possibility to evaluate this function to any extent of accuracy.

Prior to considering this problem in the general form, we are going to discuss some particular cases.

Consider the power series

$$1 + x + x^2 + \ldots + x^n + \ldots$$

This series represents a geometric progression with the ratio x and, as we saw, converges for $|x| < 1$, its sum being equal to $\frac{1}{1-x}$. Hence, we may write

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots \qquad (190\text{-}1)$$

This equality may be regarded as the expansion of the function $1/1-x$ into a power series arranged by increasing powers of the variable x. From expansion (190-1) it is easy to obtain other expansions which are of great interest.

Expanding the function ln (1 + x). Substituting $-z$ for x in expansion (190-1), we shall have

$$\frac{1}{1+z}=1-z+z^2-\ldots+(-1)^n z^n+\ldots \qquad (190\text{-}2)$$

If

$$0\leqslant|z|\leqslant|x|<1,$$

then equality (190-2), as it was said in the preceding section, can be integrated termwise with respect to z between the limits 0 and x. Therefore, multiplying equality (190-2) by dz and integrating termwise between the limits 0 and x, we get

$$\int_0^x \frac{dz}{1+z}=\int_0^x dz-\int_0^x z\,dz+\int_0^x z^2\,dz-\ldots+(-1)^n\int_0^x z^n\,dz+\ldots$$

Hence,

$$\ln(1+z)\Big|_0^x=\frac{z}{1}\Big|_0^x-\frac{z^2}{2}\Big|_0^x+\frac{z^3}{3}\Big|_0^x-\ldots+(-1)^n\frac{z^{n+1}}{n+1}\Big|_0^x+\ldots,$$

or

$$\ln(1+x)=\frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^n\frac{x^{n+1}}{n+1}+\ldots,$$

if $|x|<1$. We can show that this expansion holds true for $x=1$ as well and, consequently,

$$\ln 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

Expanding the function arctan x. Let us set in expansion (190-1) $x=-z^2$:

$$\frac{1}{1+z^2}=1-z^2+z^4-\ldots+(-1)^n z^{2n}+\ldots$$

Multiplying this equality by dz and integrating termwise between the limits 0 and x where $|x|<1$, we get

$$\int_0^x \frac{dz}{1+z^2}=\int_0^x dz-\int_0^x z^2\,dz+\int_0^x z^4\,dz-\ldots+(-1)^n\int_0^x z^{2n}\,dz+\ldots,$$

or

$$\arctan z\Big|_0^x=z\Big|_0^x-\frac{z^3}{3}\Big|_0^x+\frac{z^5}{5}\Big|_0^x-\ldots+(-1)^n\frac{z^{2n+1}}{2n+1}\Big|_0^x+\ldots$$

Since arctan $0 = 0$, we finally have

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots,$$

if $|x| < 1$. We can prove that this expansion holds true for $x = 1$ and for $x = -1$ as well.

In particular, for $x = 1$ we derive

$$\arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

We see that many functions, as, for instance, $\ln(1 + x)$, arctan x, etc., allow to be expanded into a power series with respect to the argument x. It is only natural to discuss the general problem on expanding a given function $f(x)$ in a series of increasing integral non-negative powers of the variable x. This problem will be delt with in the next section.

Sec. 191. Maclaurin's Series

Suppose a given function $f(x)$ can be expanded into a power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots, \qquad (191\text{-}1)$$

where a_0, a_1, a_2, ... are indefinite coefficients and the interval of convergence $|x| < R$ of this series is not reduced to a point, i.e. $R > 0$.

As it was indicated above, power series (191-1) can be differentiated termwise in its interval of convergence any number of times, understanding by this that all series thus obtained will converge and their sums are equal to the corresponding derivatives.

Differentiating series (191-1) termwise in succession infinite number of times, we shall have

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots,$$

$$f''(x) = 2a_2 + 2 \cdot 3a_3 x + 3 \cdot 4a_4 x^2 + 4 \cdot 5a_5 x^3 + \ldots,$$

$$f'''(x) = 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4 x + 3 \cdot 4 \cdot 5a_5 x^2 + \ldots,$$

$$f^{IV}(x) = 2 \cdot 3 \cdot 4a_4 + 2 \cdot 3 \cdot 4 \cdot 5a_5 x + \ldots,$$

. . . . . .

Putting in these equalities, as also in (191-1), $x = 0$, we get

$$f(0) = a_0, \quad f'(0) = a_1, \quad f''(0) = 2a_2,$$

$$f'''(0) = 2 \cdot 3a_3, \quad f^{IV}(0) = 2 \cdot 3 \cdot 4a_4, \ldots$$

Hence

$$a_0 = f(0), \quad a_1 = \frac{f'(0)}{1}, \quad a_2 = \frac{f''(0)}{1 \cdot 2},$$

$$a_3 = \frac{f'''(0)}{1 \cdot 2 \cdot 3}, \quad a_4 = \frac{f^{IV}(0)}{1 \cdot 2 \cdot 3 \cdot 4}, \ \ldots$$

Substituting then the values of the coefficients a_0, a_1, a_2, a_3, ... into series (191-1), we obtain *Maclaurin's series**

$$f(x) = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots + \frac{f^{(n)}(0)}{n!} x^n + \ldots \qquad (191\text{-}2)$$

(cf. Sec. 90, formula (90-6)).

ec. 192. Applying Maclaurin's Series to Expanding Some Functions into Power Series

(1) **Expanding the function e^x.** Let

$$f(x) = e^x.$$

We have

$$f'(x) = e^x, \quad f''(x) = e^x, \quad f'''(x) = e^x, \quad f^{IV}(x) = e^x, \ \ldots$$

Setting here $x = 0$, we obtain

$$f(0) = e^0 = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad f'''(0) = 1, \quad f^{IV}(0) = 1, \ \ldots$$

Substituting these values into Maclaurin's series (191-2), we shall finally have

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!} + \ldots \qquad (192\text{-}1)$$

The general term of series (192-1) is

$$u_n = \frac{x^n}{n!}.$$

Applying D'Alembert's test to the series of absolute magnitudes, we get

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} : \frac{x^n}{n!} \right| = \lim_{n \to \infty} \frac{|x|}{n+1} = 0;$$

consequently, power series (192-1) *converges for any x*, i.e. its interval of convergence is equal to $(-\infty, +\infty)$. In detailed courses it is

* In the general case, Maclaurin's series formally composed for a function $f(x)$ not necessarily converges to this function.

proved that the sum of this series for any value of x is equal to the function e^x.

(2) **Expanding the function sin x.** Let

$$f(x) = \sin x;$$

hence

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x,$$

$$f^{IV}(x) = \sin x, \ldots$$

Putting $x = 0$, we have

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad f^{IV}(0) = 0, \ldots$$

Substituting these functions into (191-2), we get

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \ldots, \qquad (192\text{-}2)$$

where x is measured in radians. We can easily get convinced that this series converges for any x. It is possible to prove that its sum is equal to $\sin x$.

(3) **Expanding the function cos x.** If

$$f(x) = \cos x,$$

then we have

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{IV}(x) =$$

$$= \cos x, \ldots$$

Setting $x = 0$, we obtain

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0,$$

$$f^{IV}(0) = 1, \ldots$$

Substituting these values into Maclaurin's formula (191-2), we find

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \ldots, \qquad (192\text{-}3)$$

where x is measured in radians. This series, the same as (192-2), converges for any x. It is proved that the sum of this series is equal to $\cos x$.

Expansion (192-3) could be obtained from expansion (192-2) by termwise differentiation.

(4) **Expanding the binomial formula $(1 + x)^m$.** Let

$$f(x) = (1 + x)^m,$$

where m is a number, integral or fractional, positive or negative. Then we have

$$f'(x) = m(1+x)^{m-1},$$
$$f''(x) = m(m-1)(1+x)^{m-2},$$
$$f'''(x) = m(m-1)(m-2)(1+x)^{m-2},$$
$$\dots\dots\dots\dots\dots\dots$$
$$f^{(n)}(x) = m(m-1)(m-2)\dots(m-n+1)(1+x)^{m-n},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

Setting $x = 0$ in all these formulas, we obtain

$$f(0) = 1, \quad f'(0) = m, \quad f''(0) = m(m-1),$$
$$f'''(0) = m(m-1)(m-2), \dots,$$
$$f^{(n)}(0) = m(m-1)(m-2)\dots(m-n+1), \dots$$

Substituting the expressions for $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, ..., $f^{(n)}(0)$, ... into Maclaurin's series (191-2), we shall have

$$(1+x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \dots$$
$$\dots + \frac{m(m-1)(m-2)\dots(m-n+1)}{1\cdot 2\cdot 3\dots n}x^n + \dots \qquad (192\text{-}4)$$

Formally, the binomial formula for a non-integral or negative exponent looks the same as for an integral positive exponent. If m is an integral positive number, then for $n = m + 1$ the factor $m - n + 1$ is equal to zero. Consequently, series (192-4) will terminate, and instead of an infinite expansion a finite sum is obtained (cf. Sec. 89).

Using the formula

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|,$$

we shall find the interval of convergence $(-R, R)$ of series (192-4). We have

$$a_n = \frac{m(m-1)(m-2)\dots(m-n+1)}{1\cdot 2\cdot 3\dots n};$$
$$a_{n+1} = \frac{m(m-1)(m-2)\dots(m-n+1)(m-n)}{1\cdot 2\cdot 3\dots n(n+1)}.$$

Hence,

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{n+1}{m-n}\right|$$

and consequently,

$$R=\lim_{n\to\infty}\left|\frac{n+1}{m-n}\right|=\lim_{n\to\infty}\left|\frac{1+\frac{1}{n}}{\frac{m}{n}-1}\right|=\left|\frac{1}{-1}\right|=1.$$

Thus, the binomial series converges inside the interval

$$-1<x<+1$$

and diverges outside it. Whether this series converges for $x=1$ and $x=-1$ should be investigated for each case separately.

It is considerably more difficult to prove that for $|x|<1$ the sum of series (192-4) is equal to $(1+x)^m$.

Sec. 193. Applying Power Series to Approximate Calculations

The obtained expansions make it possible to compute particular values of a function, to approximately evaluate some "inexpressible" definite integrals, etc. Let us consider a number of examples.

(1) **Computing sin 1.** Setting $x=1$ in the expansion for $\sin x$, we have

$$\sin 1=1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\ldots$$

If we reject all the terms, beginning with the 4th, then the absolute value of the error will be less than $\frac{1}{7!}=\frac{1}{5040}$ (since the series for $\sin 1$ is a series satisfying the conditions of Leibniz' theorem). Hence,

$$\sin 1\approx\sin 57°18'\approx 1-\frac{1}{3!}+\frac{1}{5!}\approx 0.8417$$

accurate to 0.0002.

(2) **Computing roots.** Let it be required to compute

$$\sqrt[3]{9}.$$

Writing this expression in the form

$$\sqrt[3]{9}=\sqrt[3]{8+1}=2\left(1+\frac{1}{8}\right)^{\frac{1}{3}}$$

and putting in the binomial formula $m = \frac{1}{3}$ and $x = \frac{1}{8}$, we shall have

$$\sqrt[3]{9} = 2\left[1 + \frac{1}{3}\cdot\frac{1}{8} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1\cdot 2}\cdot\left(\frac{1}{8}\right)^2 + \right.$$

$$\left. + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1\cdot 2\cdot 3}\cdot\left(\frac{1}{8}\right)^3 + \ldots\right] =$$

$$= 2\left(1 + \frac{1}{24} - \frac{1}{576} + \frac{5}{41\,472} - \ldots\right) \approx$$

$$\approx 2\,(1 + 0.0417 - 0.0017 + 0.0001) = 2.0802.$$

Taken from the table,

$$\sqrt[3]{9} = 2.0801.$$

(3) **Computing natural logarithms.** In Sec. 190 we derived the following expansion:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \ldots \qquad (193\text{-}1)$$

This series is not suitable for computing natural logarithms of the numbers exceeding 2, since it diverges for $x > 1$. But on its basis we can obtain another series suitable for our purpose. To this end, let us replace x by $-x$ in formula (193-1). We then obtain

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} - \ldots \qquad (193\text{-}2)$$

Both series (193-1) and (193-2) have a common interval of convergence: $|x| < 1$. As is known (see Sec. 181), converging series can be added and subtracted termwise. Therefore, assuming that $|x| < 1$ and subtracting (193-2) from (193-1), we shall have

$$\ln\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right). \qquad (193\text{-}3)$$

Putting

$$\frac{1+x}{1-x} = \frac{N+1}{N} \qquad (N > 0),$$

we find

$$x = \frac{1}{2N+1}.$$

Substituting these values into series (193-3), we get

$$\ln (N+1) - \ln N = 2\left[\frac{1}{2N+1} + \frac{1}{3}\cdot\frac{1}{(2N+1)^3} + \frac{1}{5}\cdot\frac{1}{(2N+1)^5} + \frac{1}{7}\cdot\frac{1}{(2N+1)^7} + \ldots\right]. \quad (193\text{-}4)$$

Applying D'Alembert's test, it is easy to see that series (193-4) converges for any positive number N. Consequently, using this series, we can determine step by step the natural logarithms of all integral positive numbers.

For large N series (193-4) converges very rapidly. Let us estimate for $N > 0$ the error ρ_n to be obtained if we reject in (193-4) all the terms contained in brackets after the nth term. We have

$$\rho_n = 2\left[\frac{1}{2n+1}\cdot\frac{1}{(2N+1)^{2n+1}} + \frac{1}{2n+3}\cdot\frac{1}{(2N+1)^{2n+3}} + \frac{1}{2n+5}\cdot\frac{1}{(2N+1)^{2n+5}} + \ldots\right].$$

It is obvious that

$$\rho_n < \frac{2}{2n+1}\cdot\frac{1}{(2N+1)^{2n+1}}\left[1 + \frac{1}{(2N+1)^2} + \frac{1}{(2N+1)^4} + \ldots\right]$$

or, computing the sum of the infinitely decreasing geometric progression standing in the brackets, we get finally

$$\rho_n < \frac{1}{2(2n+1)}\cdot\frac{1}{(2N+1)^{2n-1}}\cdot\frac{1}{N(N+1)} \quad (193\text{-}5)$$

Let us put in expansion (193-4) for instance $N = 1$ and $n = 3$ We have

$$\ln 2 \approx 2\left(\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5}\right) = 0.6932;$$

where, by (193-5), the error

$$\rho_3 < \frac{1}{2\cdot 7\cdot 3^5\cdot 1\cdot 2} < \frac{1}{5000},$$

i.e. we have three correct decimal places.

Further, setting $N = 2$ and confining ourselves to two terms ($n = 2$), we obtain

$$\ln 3 \approx \ln 2 + 2\left(\frac{1}{5} + \frac{1}{3}\cdot\frac{1}{5^3}\right) = 1.0985,$$

and the error

$$\rho_2 < \frac{1}{2\cdot 5\cdot 5^3\cdot 2\cdot 3} = \frac{1}{7500}.$$

Proceeding in the same way, we can compute the natural logarithm of any positive number with a sufficient accuracy.

(4) **Applying series for computing definite integrals.** Let, for instance, there be required to compute the integral

$$\int_0^1 \frac{\sin x}{x}\,dx\,*.$$

The corresponding indefinite integral

$$\int \frac{\sin x}{x}\,dx$$

cannot be expressed in terms of elementary functions, i.e. represents an "inexpressible" integral, and, consequently, the Newton-Leibniz formula is inapplicable here. Nevertheless, the initial definite integral can be computed approximately with the aid of series.

Dividing termwise the series for $\sin x$ by x, we shall have

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$

Hence, integrating termwise we get

$$\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 dx - \frac{1}{3!}\int_0^1 x^2\,dx + \frac{1}{5!}\int_0^1 x^4\,dx - \ldots =$$

$$= x\Big|_0^1 - \frac{1}{3!}\cdot\frac{x^3}{3}\Big|_0^1 + \frac{1}{5!}\cdot\frac{x^5}{5}\Big|_0^1 - \ldots =$$

$$= 1 - \frac{1}{3!\cdot 3} + \frac{1}{5!\cdot 5} - \ldots \approx 0.94611.$$

Since the series is alternating and the moduli of its terms decrease monotonically, confining ourselves to three terms, we shall be mistaken by less than $\frac{1}{7!\cdot 7} = \frac{1}{35\,280} < 0.00003$.

Sec. 194. Taylor's Series

In some cases the function $f(x)$ or its derivatives lose sense at $x = 0$, as, for instance, the function $f(x) = \ln x$ or $f(x) = \sqrt{x}$. Such functions cannot be expanded into Maclaurin's series. To expand functions of such kind, we may sometimes take advantage of more general power series arranged by increasing powers of the difference $x - a$, where a is a properly chosen constant number.

* Here the integrand $f(x) = \frac{\sin x}{x}$ is defined for $x \neq 0$. For $x = 0$, by continuity, we set $f(0) = 1$.

Suppose the given function $f(x)$ allows the expansion in increasing powers of the difference $x - a$:

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3 + \\ + A_4(x-a)^4 + \ldots, \quad (194\text{-}1)$$

which holds in a certain interval $|x-a| < R$.

We put $x - a = z$. Then expansion (194-1) will be rewritten in the form

$$F(z) \equiv f(z+a) = A_0 + A_1 z + A_2 z^2 + \ldots, \quad (194\text{-}2)$$

where $|z| < R$. Consequently, according to Sec. 191, expansion (194-2) is the Maclaurin series for the function $F(z)$. Since $F^{(n)}(z) = f^{(n)}(z+a)$ $(n = 1, 2, \ldots)$, hence we obtain

$$A_0 = F(0) = f(a), \quad A_1 = \frac{F'(0)}{1!} = \frac{f'(a)}{1!}, \quad A_2 = \frac{F''(0)}{2!} = \\ = \frac{f''(a)}{2!}, \ldots, \quad A_n = \frac{F^{(n)}(0)}{n!} = \frac{f^{(n)}(a)}{n!}, \ldots$$

Substituting these values of the coefficients into series (194-1), we shall have

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \\ \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots \quad (194\text{-}3)$$

This is just *Taylor's series.*

In particular, putting here $a = 0$, we shall obtain Maclaurin's series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

Confining ourselves in (194-3) only to a finite number of terms, instead of Taylor's series, we get *Taylor's polynomial*

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \quad (194\text{-}4)$$

(cf. Sec. 90, formula (90-1)). If series (194-3) converges in some neighbourhood U_a of the point a and its sum is equal to the function $f(x)$, then the polynomial $P_n(x)$ gives an approximate representation of the function $f(x)$ in the neighbourhood U_a.

Example 1. Expand the polynomial

$$f(x) = x^3 - 5x^2 + 8x + 3$$

in ascending powers of the difference $x - 2$.

Differentiating the function $f(x)$, we have

$$f'(x) = 3x^2 - 10x + 8, \quad f''(x) = 6x - 10, \quad f'''(x) = 6,$$
$$f^{(n)}(x) = 0 \quad \text{for} \quad n > 3.$$

Substituting $x = 2$, we get

$$f(2) = 7, \quad f'(2) = 0, \quad f''(2) = 2, \quad f'''(2) = 6,$$
$$f^{(n)}(2) = 0 \quad \text{for} \quad n > 3.$$

On the basis of Taylor's series (194-3), the expansion of the function $f(x)$ in ascending powers of the difference $x - 2$ has the form

$$f(x) = 7 + \frac{x-2}{1} \cdot 0 + \frac{(x-2)^2}{1 \cdot 2} \cdot 2 + \frac{(x-2)^3}{1 \cdot 2 \cdot 3} \cdot 6$$

or, finally,

$$f(x) = 7 + (x-2)^2 + (x-2)^3.$$

Example 2. Expand the function

$$f(x) = \ln x$$

in ascending powers of the difference $x - 1$.

We have

$$f'(x) = \frac{1}{x}, \quad f''(x) = -\frac{1}{x^2}, \quad f'''(x) = \frac{1 \cdot 2}{x^3}, \quad f^{IV}(x) =$$
$$= -\frac{1 \cdot 2 \cdot 3}{x^4}, \ \ldots$$

Hence,

$$f'(1) = 1, \quad f''(1) = -1, \quad f'''(1) = 1 \cdot 2, \quad f^{IV}(1) = -1 \cdot 2 \cdot 3, \ \ldots$$

Consequently,

$$\ln x = \ln 1 + 1 \cdot (x-1) - \frac{1}{1 \cdot 2} \cdot (x-1)^2 +$$
$$+ \frac{1 \cdot 2}{1 \cdot 2 \cdot 3} \cdot (x-1)^3 - \frac{1 \cdot 2 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} \cdot (x-1)^4 + \ldots$$

or

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

This expansion is true if $0 < x \leqslant 2$.

Note that this series could be obtained directly from the series for $\ln(1 + x)$ (see Sec. 190), setting $\ln x = \ln(1 + z)$, where $z = x - 1$.

Sec. 195. Series in a Complex Domain

In a number of cases we have to consider the series whose terms are complex numbers, i.e. the series of the form

$$(u_1 + iv_1) + (u_2 + iv_2) + \ldots + (u_n + iv_n) + \ldots, \qquad (195\text{-}1)$$

where u_n and v_n ($n = 1, 2, \ldots$) are real numbers and $i^2 = -1$.

Series (195-1) is called *convergent* if there converge separately both the series composed of the real parts of the terms of the given series:

$$u_1 + u_2 + \ldots + u_n + \ldots, \qquad (195\text{-}2)$$

and the series formed of the imaginary parts of these terms:

$$v_1 + v_2 + \ldots + v_n + \ldots \qquad (195\text{-}3)$$

If we denote the sum of the first n terms of (195-2) by S_n, and the sum of the first n terms of (195-3) by T_n, then in the case of the convergence of these series there exist

$$\lim_{n\to\infty} S_n = S \quad \text{and} \quad \lim_{n\to\infty} T_n = T.$$

In such a case the complex number $S + iT$ is called the *sum of series* (195-1).

There exists the following theorem.

Theorem. *If the series of the absolute values of the terms of series* (195-1) *converges, then series* (195-1) *also converges.*

Proof. Indeed, if the series

$$\sqrt{u_1^2+v_1^2} + \sqrt{u_2^2+v_2^2} + \ldots + \sqrt{u_n^2+v_n^2} + \ldots,$$

converges, then by virtue of the obvious inequalities

$$|u_n| \leqslant \sqrt{u_n^2+v_n^2}$$

and

$$|v_n| \leqslant \sqrt{u_n^2+v_n^2}$$

($n = 1, 2, \ldots$), by the comparison test (Sec. 184) and the theorem from Sec. 186, both series (195-2) and (195-3) will converge absolutely. Then, by definition, series (195-1) also converges. The theorem is proved.

Considered in the complex domain are also the power series

$$c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots,$$

where

$$c_n = a_n + ib_n, \quad z = x + iy \qquad (n = 0, 1, 2, \ldots).$$

By the preceding theorem, such a series will converge automat-

ically, if there converges the series of the moduli

$$|c_0| + |c_1|\,|z| + |c_2|\,|z|^2 + \ldots + |c_n|\,|z|^n + \ldots,$$

where

$$|c_n| = \sqrt{a_n^2 + b_n^2} \quad \text{and} \quad |z| = \sqrt{x^2 + y^2}$$

($n = 0, 1, 2, \ldots$). To investigate the convergence of the last series, we may apply all known tests, for instance, D'Alembert's test.

Sec. 196. Euler's Formulas

Let us apply the obtained expansions e^x, $\sin x$, $\cos x$ for deriving very important formulas relating these functions.

If x is a real number, then, as is known (see Sec. 192), there takes place the following expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots,$$

and this series converges for any value of x.

If $z = x + iy$, where x and y are real numbers and $i^2 = -1$, then, by definition, we set

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \tag{196-1}$$

Applying D'Alembert's test to the series of the moduli

$$1 + \frac{|z|}{1!} + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + \ldots,$$

we reveal that this series converges for any value of $|z|$ and, consequently, (196-1) converges as well. Hence, the exponential function e^z is defined for all complex values of z.

In particular, for $z = ix$, where x is a real number, we have

$$e^{ix} = 1 + \frac{(ix)}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \ldots$$

Since

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \text{ etc.},$$

substituting these values in the expansion for e^{ix}, we get

$$e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + \ldots$$

or, separating real and imaginary parts, we shall have

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots\right) + i\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right).$$

According to (192-2) and (192-3), the expression in the first parentheses is equal to $\cos x$, and the expression in the second parentheses to $\sin x$. Therefore, we arrive to the following remarkable formula:

$$e^{ix} = \cos x + i \sin x. \tag{196-2}$$

Replacing here x by $-x$ and taking into account that $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$, we find

$$e^{-ix} = \cos x - i \sin x. \tag{196-3}$$

Thus, we have obtained the well-known *Euler formulas.*

Resolving (196-2) and (196-3) with respect to $\cos x$ and $\sin x$, we shall have

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

In the general case, if $z = x + iy$, we can show that

$$e^{x+iy} = e^x \cdot e^{iy}.$$

Hence,

$$e^{x+iy} = e^x (\cos y + i \sin y). \tag{196-4}$$

Example.

$$e^{1+\frac{\pi}{2}i} = e\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = ei.$$

If $z = r(\cos\varphi + i\sin\varphi)$, i.e. a complex number in the trigonometric form (see Sec. 134), then, by (196-4), we get the *exponential form of a complex number*

$$z = re^{i\varphi}, \tag{196-5}$$

where $r = |z|$ and $\varphi = \operatorname{Arg} z$.

Sec. 197. Fourier Trigonometric Series

Let us recall (Sec. 63) that a function $f(x)$ is called *piecewise continuous* on a given interval $\langle a, b\rangle$, if this interval can be divided into a finite number of subintervals $\langle a_s, b_s\rangle$ $(s = 1, 2, \ldots, N)$ on each of which: (1) the function $f(x)$ is bounded and continuous at interior points; (2) at the end-points there exist finite one-sided limits

$$f(a_s + 0) = \lim_{x_s \to a_s + 0} f(x), \quad f(b_s - 0) = \lim_{x_s \to b_s - 0} f(x)$$

$(s = 1, 2, \ldots, N)$.

The following number is understood as the integral of the function

$$\int_a^b f(x)\,dx = \sum_{s=1}^{N} \int_{a_s}^{b_s} f(x)\,dx.$$

We can prove that for a function $f(x)$ which is piecewise continuous on $\langle a, b\rangle$ there exists a generalized antiderivative (see Sec. 124)

$$F(x) = \int_{x_0}^{x} f(x)\,dx$$

$(x \in [a, b], \quad x_0 \in [a, b])$

and, consequently,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Let $\varphi(x)$ and $\psi(x)$ be two real piecewise continuous functions on a given finite interval $\langle a, b\rangle$. By analogy with the corresponding operation of the vector algebra (see Sec. 157), the *scalar product* of the functions $\varphi(x)$ and $\psi(x)$ is understood to be the integral

$$(\varphi, \psi) \equiv \int_a^b \varphi(x)\,\psi(x)\,dx. \tag{197-1}$$

Note. It is not difficult to figure out that the product of two piecewise continuous on $\langle a, b\rangle$ functions is again a piecewise continuous function on $\langle a, b\rangle$ and, consequently, in our case integral (197-1) exists.

The number

$$\|\varphi\| = \sqrt{(\varphi, \varphi)} = \left[\int_a^b \varphi^2(x)\,dx\right]^{\frac{1}{2}} \tag{197-2}$$

is called the *norm* of the function $\varphi(x)$.

The functions $\varphi(x)$ and $\psi(x)$ are called *orthogonal* on a given interval $\langle a, b\rangle$ if

$$(\varphi, \psi) \equiv \int_a^b \varphi(x)\,\psi(x)\,dx = 0. \tag{197-3}$$

Consider the *principal system of the trigonometric functions*

$$\left\{\cos\frac{n\pi x}{l},\ \sin\frac{n\pi x}{l}\right\} \qquad (n = 0, 1, 2, \ldots) \tag{197-4}$$

having a common period $T = 2l$ (l semi-period). Physicists call the functions

$$u_n = a_n \cos \frac{n\pi x}{l} \qquad (n = 0, 1, 2, \ldots)$$

and

$$v_n = b_n \sin \frac{n\pi x}{l} \qquad (n = 1, 2, \ldots)$$

the *principal harmonics*, their graphs being sinusoids with the amplitudes a_n and b_n, respectively (the harmonic v_0 is not considered since $v_0 \equiv 0$).

Lemma. *Principal trigonometric functions* (197-4) *are pairwise orthogonal on any interval whose length is equal to the common period* $T = 2l$ *of these functions*, i.e. for a standard interval $\langle -l, l \rangle$ we have the following orthogonality conditions:

$$\text{I.} \quad \left(\cos \frac{m\pi x}{l}, \cos \frac{n\pi x}{l}\right) = \int_{-l}^{l} \cos \frac{m\pi x}{l} \cos \frac{n\pi x}{l}\, dx = 0 \quad \text{for } m \neq n.$$

$$\text{II.} \quad \left(\sin \frac{m\pi x}{l}, \sin \frac{n\pi x}{l}\right) = \int_{-l}^{l} \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l}\, dx = 0 \quad \text{for } m \neq n.$$

$$\text{III.} \quad \left(\cos \frac{m\pi x}{l}, \sin \frac{n\pi x}{l}\right) = \int_{-l}^{l} \cos \frac{m\pi x}{l} \sin \frac{n\pi x}{l}\, dx = 0$$

(m and n any integers).

Orthogonality conditions I, II, III are verified directly by computing the corresponding integrals. The following trigonometric formulas are used here:

(1) $\cos \alpha \cos \beta = \frac{1}{2} [\cos (\alpha - \beta) + \cos (\alpha + \beta)]$;

(2) $\sin \alpha \sin \beta = \frac{1}{2} [\cos (\alpha - \beta) - \cos (\alpha + \beta)]$;

(3) $\sin \alpha \cos \beta = \frac{1}{2} [\sin (\alpha - \beta) + \sin (\alpha + \beta)]$.

For instance, for $m \neq n$ we have

$$\text{I.} \quad \left(\cos \frac{m\pi x}{l}, \cos \frac{n\pi x}{l}\right) = \int_{-l}^{l} \cos \frac{m\pi x}{l} \cos \frac{n\pi x}{l}\, dx =$$

$$= \frac{1}{2} \int_{-l}^{l} \left[\cos \frac{(m-n)\pi x}{l} + \cos \frac{(m+n)\pi x}{l}\right] dx =$$

$$= \frac{1}{2} \left[\frac{l}{(m-n)\pi} \sin \frac{(m-n)\pi x}{l} + \frac{l}{(m+n)\pi} \sin \frac{(m+n)\pi l}{l}\right] \Bigg|_{-l}^{l} = 0,$$

since $\sin k\pi = 0$ for any integral k.

Relations II and III are left for the reader to get convinced in their validity independently.

Note. Let us compute the norms of the basic trigonometric functions.

(1) For $n = 0$ we have the 0th harmonic $\cos 0x = 1$. According to (197-2), we get

$$\| 1 \|^2 = \int_{-l}^{l} 1^2 \, dx = 2l,$$

i.e.

$$\| 1 \| = \sqrt{2l}. \tag{197-5}$$

(2) For $n > 1$ we have

$$\left\| \cos \frac{n\pi x}{l} \right\|^2 = \int_{-l}^{l} \cos^2 \frac{n\pi x}{l} \, dx =$$

$$= \frac{1}{2} \int_{-l}^{l} \left(1 + \cos \frac{2n\pi x}{l} \right) dx = \frac{1}{2} \left(x + \frac{l}{2n\pi} \sin \frac{2n\pi x}{l} \right) \Big|_{-l}^{l} = l,$$

i.e.

$$\left\| \cos \frac{n\pi x}{l} \right\| = \sqrt{l} \qquad (n = 1, 2, 3, \ldots). \tag{197-6}$$

(3) Analogously,

$$\left\| \sin \frac{n\pi x}{l} \right\|^2 = \int_{-l}^{l} \sin^2 \frac{n\pi x}{l} \, dx =$$

$$= \frac{1}{2} \int_{-l}^{l} \left(1 - \cos \frac{2n\pi x}{l} \right) dx = \frac{1}{2} \left(x - \frac{l}{2n\pi} \sin \frac{2n\pi x}{l} \right) \Big|_{-l}^{l} = l,$$

i.e.

$$\left\| \sin \frac{n\pi x}{l} \right\| = \sqrt{l} \quad (n = 1, 2, 3, \ldots). \tag{197-7}$$

Let $f(x)$ be a piecewise continuous periodic function with the period $T = 2l$. It is natural to try to represent this function in the form of a sum of a finite or infinite number of harmonics

$$u_n = a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}$$

($n = 0, 1, 2, \ldots$) (see Sec. 39, Chapter 6) of the same period $2l$ (*harmonic analysis of a function*). Thus we come to the *Fourier series*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \tag{197-8}$$

(here, for the convenience of further calculations, the coefficient of the 0th harmonic is taken with the factor $\frac{1}{2}$). Historically, this problem first arose during mathematical treatment of the results of observations of the height of tidal waves at a given place which is periodically repeated with time. The harmonic analysis of the height of the tidal wave made it possible to give long-term forecasts of its magnitude which is of high importance for navigation.

Let us assume that series (197-8) converges on the interval $\langle -l, l \rangle$ and allows term-by-term integration.

Integrating (197-8) termwise, we shall have

$$\int_{-l}^{l} f(x)\, dx =$$

$$= \frac{a_0}{2} \int_{-l_0}^{l} dx + \sum_{n=1}^{\infty} \left[a_n \int_{-l}^{l} \cos \frac{n\pi x}{l}\, dx + b_n \int_{-l}^{l} \sin \frac{n\pi x}{l}\, dx \right]. \tag{197-9}$$

Since

$$\int_{-l}^{l} \cos \frac{n\pi x}{l}\, dx = \int_{-l}^{l} \sin \frac{n\pi x}{l}\, dx = 0$$

for $n = 1, 2, \ldots$ (this also follows from the orthogonality conditions), we get

$$\int_{-l}^{l} f(x)\, dx = \frac{a_0}{2} \cdot 2l = a_0 l,$$

hence

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx. \tag{197-10}$$

Note that the free term of series (197-8)

$$\frac{a_0}{2} = \frac{1}{2l} \int_{-l}^{l} f(x)\, dx$$

represents the *mean value* of a periodic function $f(x)$.

Multiplying now both sides of equality (197-8) by

$$\cos\frac{m\pi x}{l} \qquad (m=1, 2, \ldots)$$

and integrating termwise, we shall have

$$\int_{-l}^{l} f(x)\cos\frac{m\pi x}{l}\,dx = \frac{a_0}{2}\int_{-l}^{l}\cos\frac{m\pi x}{l}\,dx + \\ + \sum_{n=1}^{\infty}\left[a_n\int_{-l}^{l}\cos\frac{n\pi x}{l}\cdot\cos\frac{m\pi x}{l}\,dx + b_n\int_{-l}^{l}\sin\frac{n\pi x}{l}\cdot\cos\frac{m\pi x}{l}\,dx\right]. \tag{197-11}$$

Hence, by orthogonality conditions I, III and formula (197-6), we get

$$\int_{-l}^{l} f(x)\cos\frac{m\pi x}{l}\,dx = a_m\int_{-l}^{l}\cos^2\frac{m\pi x}{l}\,dx = a_m l \tag{197-12}$$

and, consequently,

$$a_m = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{m\pi x}{l}\,dx \qquad (m=1, 2, 3, \ldots). \tag{197-13}$$

Analogously, multiplying both sides of equality (197-8) by

$$\sin\frac{m\pi x}{l} \qquad (m=1, 2, \ldots)$$

and integrating termwise, we find

$$\int_{-l}^{l} f(x)\sin\frac{m\pi x}{l}\,dx = \frac{a_0}{2}\int_{-l}^{l}\sin\frac{m\pi x}{l}\,dx + \\ + \sum_{n=1}^{\infty}\left[a_n\int_{-l}^{l}\cos\frac{n\pi x}{l}\cdot\sin\frac{m\pi x}{l}\,dx + \right. \\ \left. + b_n\int_{-l}^{l}\sin\frac{n\pi x}{l}\cdot\sin\frac{m\pi x}{l}\,dx\right]. \tag{197-14}$$

Hence, by orthogonality conditions II and III and formula (197-7), we have

$$\int_{-l}^{l} f(x)\sin\frac{m\pi x}{l}\,dx = b_m\int_{-l}^{l}\sin^2\frac{m\pi x}{l}\,dx = b_m l$$

and, hence,

$$b_m = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{m\pi x}{l}\,dx \qquad (m = 1, 2, 3, \ldots). \tag{197-15}$$

Replacing the letter m by the letter n (which is permissible by the sense of the formulas!), from (197-13) and (197-15) we shall obtain the following values for the coefficients of expansion (197-8):

$$\left.\begin{matrix} a_n \\ b_n \end{matrix}\right\} = \frac{1}{l}\int_{-l}^{l} f(x)\left\{\begin{matrix} \cos\dfrac{n\pi x}{l} \\ \sin\dfrac{n\pi x}{l} \end{matrix}\right. dx \tag{197-16}$$

($n = 0, 1, 2, \ldots$). Note that the coefficient a_0 is obtained, on the basis of (197-10), from (197-16) for $n = 0$. This explains why the constant term of series (197-8) is taken in the form $\frac{1}{2}a_0$. The numbers a_n, b_n ($n = 0, 1, 2, \ldots$) are called the *Fourier coefficients* of the function $f(x)$.

Definition. *A trigonometric series* (197-8) *whose coefficients are the Fourier coefficients* (197-16) *of a given periodic function* $f(x)$ *is called its Fourier series (more precisely, Fourier trigonometric series) irrespective of whether the sum of this series is equal to the function* $f(x)$ *or not.*

In this understanding we say that the function $f(x)$ generates a Fourier series and write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right), \tag{197-17}$$

where the symbol $\sim$ means "corresponds".

We are going now to indicate (without proof) the sufficient conditions for a periodic function to be expanded into a Fourier series.

Let us call a function $f(x)$ *piecewise smooth* on an interval $\langle a, b\rangle$ if it is piecewise continuous on $\langle a, b\rangle$ and has a piecewise continuous derivative $f'(x)$ on this interval.

Convergence theorem. *Let a periodic function* $f(x)$ *defined on* $(-\infty, +\infty)$, *except, perhaps, for the points of its discontinuity and having the period* $T = 2l > 0$, *be piecewise smooth in its principal domain*.*

Then: (1) *its Fourier series* (197-17) *converges for any value* $x \in (-\infty, +\infty)$, *i. e. there exists the sum of the Fourier series*

$$S(x) = \lim_{N\to\infty}\left[\frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)\right]; \tag{197-18}$$

* That is on any interval whose length is equal to the period of this function.

(2) *the sum $S(x)$ of the Fourier series is equal to the function $f(x)$ at points x of its continuity: $S(x) = f(x)$ and is equal to the arithmetic mean of the left-hand and right-hand limits of the function $f(x)$ at points x_0 of discontinuity of the function**, i.e.

$$S(x_0) = \frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)]. \tag{197-19}$$

Since for a point of discontinuity x of the function $f(x)$ we have

$$f(x) = f(x - 0) = f(x + 0),$$

in the general case we may write

$$S(x) = \frac{1}{2}[f(x - 0) + f(x + 0)]. \tag{197-20}$$

Henceforward, we shall assume that for a function $f(x)$ all conditions of the convergence theorem are fulfilled, and, instead of the sign of correspondence $\sim$ we shall write the sign of equality $=$ (thus ignoring the points of discontinuity of the function!).

Thus, for the Fourier series of the function $f(x)$ we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right), \tag{197-21}$$

where the coefficients a_n and b_n are determined by formula (197-16).

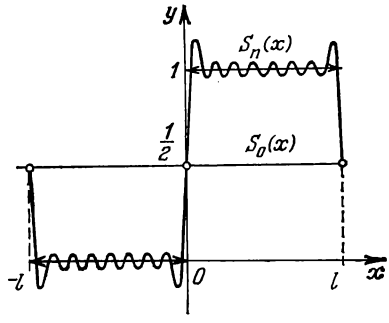


FIG. 220

Note. Formulas (197-21) and (197-16) are simplified if the period of the function $f(x)$ is equal to 2π. In this case $l = \pi$ and we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{197-21'}$$

where

$$\left.\begin{matrix} a_n \\ b_n \end{matrix}\right\} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \left\{ \begin{matrix} \cos nx \\ \sin nx \end{matrix} \right. dx \tag{197-16'}$$

$(n = 0, 1, 2, \ldots)$.

* There is only a finite number of such points in the principal domain of a periodic function $f(x)$.

Example. Write the Fourier series of the periodic function $f(x)$ of the period $T = 2l$ (Fig. 220) if

$$f(x) = \begin{cases} 0, & -l < x < 0, \\ 1, & 0 < x < l. \end{cases}$$

From (197-16) we obtain

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\left\{\int_{-l}^{0} 0\cdot dx + \int_{0}^{l} 1\cdot dx\right\} = 1,$$

i.e. $\frac{a_0}{2} = \frac{1}{2}$;

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx =$$

$$= \frac{1}{l}\left\{\int_{-l}^{0} 0\cdot\cos\frac{n\pi x}{l}\,dx + \int_{0}^{l} 1\cdot\cos\frac{n\pi x}{l}\,dx\right\} =$$

$$= \frac{1}{l}\cdot\frac{l}{n\pi}\sin\frac{n\pi x}{l}\Big|_0^l = 0 \qquad (n = 1, 2, \ldots);$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx =$$

$$= \frac{1}{l}\left\{\int_{-l}^{0} 0\cdot\sin\frac{n\pi x}{l}\,dx + \int_{0}^{l} 1\cdot\sin\frac{n\pi x}{l}\,dx\right\} =$$

$$= \frac{1}{l}\cdot\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)\Big|_0^l = -\frac{1}{n\pi}(\cos n\pi - 1) = -\frac{1}{n\pi}[(-1)^n - 1],$$

i.e.

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even}, \\ \frac{2}{n\pi}, & \text{if } n \text{ is odd}. \end{cases} \qquad (197\text{-}22)$$

Hence, since the function $f(x)$ is piecewise smooth in the interval $\langle -l, l\rangle$, the following expansion holds true

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \ldots\right)$$

$$(-l < x < l). \qquad (197\text{-}23)$$

Figure 220 represents the graphs of the partial sums of Fourier's series of the function $f(x)$: $S_0(x) = \frac{1}{2}$, $S_1(x) = \frac{1}{2} + \frac{2}{\pi}\sin\frac{\pi x}{l}$, and so on.

Sec. 198. The Fourier Series of Even and Odd Functions

Consider the symmetric integral

$$\int_{-l}^{l} f(x)\,dx = \int_{-l}^{0} f(x)\,dx + \int_{0}^{l} f(x)\,dx, \tag{198-1}$$

where $f(x)$ is a function continuous or piecewise continuous on the interval $[-l, l]$.

Substituting $x = -t$, $dx = -dt$ and taking into consideration the independence of the definite integral of the notation of the variable of integration, we get

$$\int_{-l}^{l} f(x)\,dx = -\int_{l}^{0} f(-t)\,dt + \int_{0}^{l} f(x)\,dx =$$

$$= \int_{0}^{l} f(-x)\,dx + \int_{0}^{l} f(x)\,dx = \int_{0}^{l} [f(-x) + f(x)]\,dx. \tag{198-2}$$

(1) Let $f(x)$ be an even function, i.e. $f(-x) = f(x)$. Then from (198-2) we have

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx. \tag{198-3}$$

Thus, *the symmetric integral of an even function is equal to the doubled integral of this function taken over half the interval of integration.*

(2) Let $f(x)$ be an odd function, i.e. $f(-x) = -f(x)$. In this case from (198-2) we get

$$\int_{-l}^{l} f(x)\,dx = 0. \tag{198-4}$$

And so, the *symmetric integral of an odd function is equal to zero.*

Note that assertions (1) and (2) are obvious from geometrical reasoning (Fig. 221, *a* and *b*).

Theorem. (1) *The Fourier series of an even periodic function contains only cosines of multiple arcs*, i.e. it comprises only even harmonics, including the constant term.

(2) *The Fourier series of an odd periodic function contains only sines of multiple arcs*, i.e. it comprises only odd harmonics.

Proof. (1) Let $f(x)$ be an even periodic function with the period $2l$ and a_n, b_n its Fourier coefficients. On the basis of (198-4) and taking into account that the harmonics $\sin\frac{n\pi x}{l}$ $(n = 1, 2, \ldots)$

are odd functions, we have

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = 0 \qquad (198\text{-}5)$$

($n = 1, 2, \ldots$).

Therefore,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad (198\text{-}6)$$

where from formula (198-3) we obtain

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx =$$

$$= \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \qquad (n = 0, 1, 2, \ldots). \qquad (198\text{-}7)$$

(2) Let now $f(x)$ be an odd periodic function with the period $2l$.

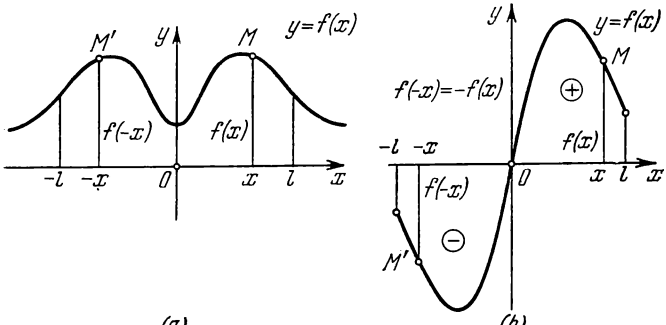


FIG. 221

Since $f(x) = \cos\frac{n\pi x}{l}$ ($n = 0, 1, 2, \ldots$) are odd functions, we have

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = 0$$

($n = 0, 1, 2, \ldots$).

Therefore,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad (198\text{-}8)$$

where, by (198-3), we have

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

($n = 1, 2, \ldots$).
The theorem is proved.

Sec. 199. The Fourier Series of Nonperiodic Functions

A piecewise smooth *nonperiodic* function $f(x)$ specified on an infinite axis $-\infty < x < +\infty$ *cannot* be represented by a Fourier series, since its sum, being the sum of harmonics with a common period T, is a periodic function with the same period T and, consequently, cannot be equal to the function $f(x)$ for all x's. But

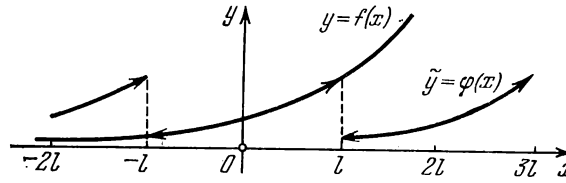


FIG. 222

it is possible to construct the representation of this function in the form of the corresponding Fourier series on any *finite interval*.

Let this interval be $\langle -l, l\rangle$, i.e. let it be symmetric about the origin (this can always be achieved by translating the x-axis).

Let us construct the function $\varphi(x)$ with the period $2l$ such that (Fig. 222)

$$\varphi(x) \equiv f(x) \quad \text{for } -l < x < l. \tag{199-1}$$

Assuming that the function $\varphi(x)$ satisfies the conditions of the convergence theorem (Sec. 197), we have

$$\varphi(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) \tag{199-2}$$

$(-\infty < x < +\infty)$,

where

$$\left.\begin{matrix} a_n \\ b_n \end{matrix}\right\} = \frac{1}{l}\int_{-l}^{l}\varphi(x)\left\{\begin{matrix} \cos\dfrac{n\pi x}{l} \\ \sin\dfrac{n\pi x}{l} \end{matrix}\right. dx \tag{199-3}$$

($n = 0, 1, 2, \ldots$).

Hence, by identity (199-1), we get

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

$$(-l<x<l), \tag{199-2'}$$

where

$$\left.\begin{matrix}a_n\\ b_n\end{matrix}\right\}=\frac{1}{l}\int_{-l}^{l}f(x)\left\{\begin{matrix}\cos\frac{n\pi x}{l}\\ \sin\frac{n\pi x}{l}\end{matrix}\right.dx \tag{199-3'}$$

($n=0,\ 1,\ 2,\ \ldots$).

Let us compute the sum $S(x)$ of series (199-2'), or the corresponding series (199-2) at the end-points $x=\pm l$. According to the general formula (Sec. 197), we have

$$S(l)=\frac{\varphi(l-0)+\varphi(l+0)}{2}. \tag{199-4}$$

But on the basis of identity (199-1) and $2l$-periodicity of the function $\varphi(x)$, geometrically it is obvious (Fig. 222) that

$$\varphi(l-0)=f(l-0),\ \varphi(l+0)=\varphi(-l+0)=f(-l+0).$$

Therefore, from (199-4) we get

$$S(l)=\frac{f(l-0)+f(l+0)}{2} \tag{199-5}$$

From the $2l$-periodicity of the sum $S(x)$ it follows that

$$S(-l)=S(l). \tag{199-6}$$

Example 1. The function $f(x)=e^x$ is expanded into a Fourier series on the interval $(-1,\ 1)$. Evaluate $S(1)$, where $S(x)$ is the sum of the Fourier series.

By (199-5), we have

$$S(1)=\frac{e^{1-0}+e^{-1+0}}{2}=\frac{e+e^{-1}}{2}=\cosh 1=1.54.$$

Suppose it is required now to represent a non-periodic function $f(x)$ by a Fourier series with the period $2l$ on the "semi-period" $0<x<l$.

Setting

$$\varphi(x)=\begin{matrix}f(x), & 0<x<l,\\ f_1(x), & -l<x<0,\end{matrix} \tag{199-7}$$

where $f_1(x)$ is an arbitrary piecewise smooth function, from (199-2) and (199-3) we obtain an infinite number of Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right) \qquad (0 < x < l), \quad (199\text{-}8)$$

representing the function $f(x)$ on the interval $(0, l)$.

In particular, putting $f_1(x) = f(-x)$ $(-l < x < 0)$ in formula (199-7) (an "even continuation"), we shall have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \qquad (0 < x < l), \quad (199\text{-}9)$$

where

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx \qquad (n = 0, 1, 2, \ldots). \quad (199\text{-}10)$$

Analogously, setting $f_1(x) = -f(-x)$ $(-l < x < 0)$ in formula (199-7) (an "odd continuation"), we shall obtain

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad (0 < x < l), \quad (199\text{-}11)$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad (199\text{-}12)$$

$(n = 1, 2, \ldots)$.

Thus, *a piecewise smooth function specified on a semi-period can be expanded into a corresponding Fourier series by an infinite number of methods. In particular, we can represent this function on the given semi-period*: (1) *in the form of the sum of even harmonics, or* (2) *in the form of the sum of odd harmonics.*

Example 2. Expand the function $f(x) = x$ in cosines of multiple arcs in the interval $(0, \pi)$.

Note that here the function $f(x)$ is odd and it is required to obtain its Fourier series containing only even harmonics. This can be done by reducing the interval of expansion.

Setting $l = \pi$, from (199-9) we shall have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \quad (199\text{-}13)$$

Using (199-10), we find

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx=\frac{2}{\pi}\int_0^{\pi} x\,dx=\frac{2}{\pi}\cdot\frac{x^2}{2}\Big|_0^{\pi}=\pi$$

and

$$a_n=\frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi} \overset{u}{x}\,d\Big(\overbrace{\frac{\sin nx}{n}}^{v}\Big)=$$

$$=\frac{2}{\pi}\left\{x\cdot\frac{\sin nx}{n}\Big|_0^{\pi}-\int_0^{\pi}\frac{\sin nx}{n}\,dx\right\}=$$

$$=\frac{2}{\pi}\cdot\frac{\cos nx}{n^2}\Big|_0^{\pi}=\frac{2}{n^2\pi}(\cos n\pi-1)=\frac{2}{n^2\pi}[(-1)^n-1]$$

($n=1,\ 2,\ 3,\ \ldots$). Hence

$$a_n=\begin{cases}0, & \text{if } n \text{ is even,}\\ -\dfrac{4}{n^2\pi}, & \text{if } n \text{ is odd.}\end{cases}$$

Thus, for $0\leqslant x\leqslant\pi$ we have

$$x=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right). \qquad (199\text{-}14)$$

Figure 223 depicts the graph of the function $y=x$ and the graph of the sum $\tilde{y}=S(x)$ of Fourier series (199-14). For $0\leqslant x\leqslant\pi$

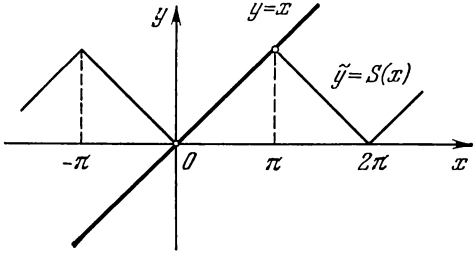


FIG. 223

they coincide, differing outside the interval $[0,\ \pi]$.

Putting $x=0$ in formula (199-14), we get a remarkable numerical series known as Euler's series

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}. \qquad (199\text{-}15)$$

Exercises

In Problems 1 to 6, applying the necessary condition and comparison test, investigate the convergence of the given series:

1. $1-1+1-1+\ldots$

2. $0.001+\sqrt{0.001}+\sqrt[3]{0.001}+\ldots$

3. $\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{9}\right)+\left(\frac{1}{8}+\frac{1}{27}\right)+\ldots$

4. $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 2^2}+\frac{1}{5\cdot 2^3}+\ldots$

5. $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots$

Hint. Compare with the harmonic series multiplied by $\frac{1}{2}$.

6. $1+\frac{1}{4\sqrt{2}}+\frac{1}{9\sqrt{3}}+\ldots$

In Problems 7 to 12, using D'Alembert's test find out the convergence of the given series:

7. $\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots$

8. $\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots$

9. $\frac{1}{1000}+\frac{2^2}{2000}+\frac{2^3}{3000}+\ldots$

10. $\frac{(1!)^2}{2!}+\frac{(2!)^2}{4!}+\frac{(3!)^2}{6!}+\ldots$

11. $\frac{1}{2}+\frac{1\cdot 3}{2\cdot 5}+\frac{1\cdot 3\cdot 5}{2\cdot 5\cdot 8}+\ldots$

12. $\frac{1!}{1^1}+\frac{2!}{2^2}+\frac{3!}{3^3}+\ldots$

In Problems 13 to 16 investigate the convergence of the given alternating series:

13. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\ldots$

14. $1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\ldots$

15. $1-\frac{2}{3}+\frac{3}{5}-\frac{4}{7}+\ldots$

16. $\frac{1}{2}+\frac{9}{4}-\frac{25}{8}-\frac{49}{16}+\frac{81}{32}+\frac{121}{64}-\ldots$

In Problems 17 to 20 determine the intervals of convergence of the given power series and investigate their behaviour at boundary points:

17. $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$

18. $1+\frac{x}{2!}+\frac{x^2}{4!}+\frac{x^3}{6!}+\ldots$

19. $1!\,x+2!\,x^2+3!\,x^3+\ldots$

20. $\frac{x-1}{1\cdot 2}+\frac{(x-1)^2}{2\cdot 2^2}+\frac{(x-1)^3}{3\cdot 2^2}+\ldots$

In Problems 21 and 22 expand the given functions into Maclaurin's series:

21. $f(x) = a^x$ $(a > 0)$.

22. $f(x) = \sin^2 x$.

23. Expand the function $f(x) = \sqrt{x}$ in integral increasing powers of the difference $x - 1$.

In Problems 24 to 27, using standard expansions, represent the given functions in the form of power series:

24. $f(x) = e^{-x^2}$. **25.** $f(x) = \cos^2 x$.

Hint. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

26. $f(x) = \frac{1}{(1-x)^2}$. **27.** $f(x) = \frac{1}{\sqrt{1-x^2}}$.

28. Taking advantage of expansions into series, approximate: (a) $\sqrt{e}$; (b) $\sin 18°$; (c) $\sqrt[5]{1.2}$.

29. Evaluate approximately the integral

$$\int_0^{\frac{1}{2}} e^{x^2}\,dx.$$

30. Derive an approximate formula for the function

$$f(x) = \int_0^x \cos\sqrt{x}\,dx,$$

if $|x|$ is a small quantity.

31. Expand the given functions into Fourier trigonometric series in the interval $(-\pi, \pi)$:

(a) $f(x) = \frac{1}{2}x$; (b) $f(x) = x^2$.

Chapter 22

Differential Equations

Sec. 200. Basic Concepts

In many problems of geometry, physics, mechanics, engineering, etc. a significant role is played by *differential equations*. This term is assigned to the equations relating the independent variable x, the desired function y and its derivatives of various orders with respect to x. The order of the higher derivative entering a given differential equation is called the *order* of this equation.

Thus, the general form of a differential equation of the nth order is as follows:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0. \tag{200-1}$$

In particular cases this equation may occur without x, y, and individual derivatives of the order lower than n. For instance, the equations

$$y' + \frac{2}{x} y = \sin x, \quad y'' + 4y' + 13y = 0, \quad y''' + yy' = 0$$

are of the first, second, and third order, respectively.

Differential equation (200-1) is said to be *linear* if its left-hand side is a polynomial of the first degree with respect to the unknown function y and its derivatives $y', y'', \ldots, y^{(n)}$ (and does not contain their products), i.e. if this equation has the form

$$a_0(x)\, y^{(n)} + a_1(x)\, y^{(n-1)} + \ldots + a_n(x)\, y = f(x). \tag{200-2}$$

Here, the functions $a_0(x)$, $a_1(x)$, $\ldots$, $a_n(x)$ usually defined and continuous in a certain common interval are called the *coefficients* of a linear equation, and the function $f(x)$ is termed its *right-hand member* or its *constant term*. If the right-hand member $f(x)$ of a linear equation (200-2) is identically equal to zero, then the equation is called *homogeneous* (or *without the right-hand member*); otherwise this equation is said to be *nonhomogeneous* (or *with the right-hand member*). Linear differential equations are widely used in applications.

Any function

$$y = \varphi(x),$$

which when substituted into equation (200-1) turns it into an identity is called the *solution* of this equation. *To solve*, or *to integrate* a given differential equation means to find all its solutions in the specified domain. The graph of the solution is called the *integral curve.*

Note that the fundamental task of integral calculus, i.e. finding the function y whose derivative is equal to a given continuous function $f(x)$, is reduced to the simplest differential equation

$$y' = f(x).$$

The general solution of this equation is

$$y = \int f(x)\,dx + C, \tag{200-3}$$

where C is an arbitrary constant and the integral is understood to be one of the antiderivatives of the function $f(x)$*.

Choosing the constant C in a proper way, we can obtain any solution of this simplest differential equation, provided the function $f(x)$ is continuous.

When integrating differential equations of higher orders, there appear several arbitrary constants.

Example 1. Consider the second-order equation

$$y'' = 0.$$

Since

$$y'' = (y')' = 0,$$

hence it follows

$$y' = C_1.$$

Integrating the last equality, we shall have

$$y = \int C_1\,dx + C_2 = C_1x + C_2. \tag{200-4}$$

---

* More precisely formula (200-3) should be written in the form

$$y = \int_{x_0}^{x} f(x)dx + c, \tag{200-3'}$$

where x_0 is a certain initial point of the given domain. In this form it is convenient for application, since it separates explicitly the arbitrary constant C. This footnote should not be forgotten.

Thus, solution (200-4) contains two arbitrary constants C_1 and C_2, i.e. the number of the arbitrary constants in formula (200-4) is exactly equal to the order of the equation. Such a solution is called the *general solution of an equation.* In the given case it represents the entire infinite set of solutions of the differential equation.

Definition 1. *The general solution of differential equation* (200-1) *is one of its solutions*

$$y = \varphi(x, C_1, C_2, \ldots, C_n)^*,$$

such that the number of the arbitrary constants contained in it $(C_1, C_2, \ldots, C_n)$ *is equal to the order of this equation.*

Arbitrary constants are called *independent* if the total number of the constants contained in the function φ cannot be reduced by introducing other arbitrary constants continuously dependent on the given constants.

If the general solution is given in the implicit form

$$\Phi(x, y, C_1, C_2, \ldots, C_n) = 0,$$

then it is usually called the *general integral.*

Definition 2. *Any solution of a differential equation obtained from the general solution by assigning some admissible concrete numerical values to the arbitrary constants contained in it is called a particular solution of this differential equation.*

Example 2. Consider the second-order equation

$$y'' + y = 0.$$

It is easy to figure out that the functions $\sin x$ and $\cos x$ are the solutions of this equation, since

$$(\sin x)'' = -\sin x \quad \text{and} \quad (\cos x)'' = -\cos x.$$

As it is not difficult to check directly, the function

$$y = C_1 \sin x + C_2 \cos x,$$

where C_1 and C_2 are independent arbitrary constants, is also a solution of our equation and, consequently, represents its general solution. If, for instance, we put

$$C_1 = 2 \quad \text{and} \quad C_2 = -5,$$

then we shall obtain the function

$$y_1 = 2 \sin x - 5 \cos x,$$

which is a particular solution of the given differential equation.

* The function φ is supposed to be continuously differentiable with respect to all its arguments a sufficient number of times.

If, as the result of solving a differential equation a certain function is found, then, substituting this function into the given equation, we can check the correctness of the solution.

Example 3. Show that the function

$$y = (C_1 + C_2 x)\, e^x$$

is the solution of the equation

$$y'' - 2y' + y = 0.$$

Indeed, here

$$y' = (C_1 + C_2 x)\, e^x + C_2 e^x = (C_1 + C_2 + C_2 x)\, e^x$$

and

$$y'' = (C_1 + C_2 + C_2 x)\, e^x + C_2 e^x = (C_1 + 2C_2 + C_2 x)\, e^x.$$

Hence,

$$y'' - 2y' + y = (C_1 + 2C_2 + C_2 x)\, e^x - 2\,(C_1 + C_2 + C_2 x)\, e^x + \\ + (C_1 + C_2 x)\, e^x \equiv 0,$$

which proves our statement.

Sec. 201. Differential Equations of the First Order

The general form of a first-order differential equation is as follows

$$F(x,\ y,\ y') = 0.$$

In simplest cases this equation can be resolved with respect to the derivative y':

$$y' = f(x,\ y). \tag{201-1}$$

The general solution of equation (201-1) has the form

$$y = \varphi(x,\ C), \tag{201-2}$$

where C is an arbitrary constant. Geometrically, the general solution (201-2) represents a *family of integral curves*, i.e. a set of lines corresponding to different values of the constant C (Fig. 224). The integral curves possess the following property: at each point $M(x, y)$ of the curve the slope of the tangent satisfies the condition

$$\tan \alpha = f(x,\ y).$$

If a point $M_0(x_0, y_0)$ is given through which an integral curve must pass, then this means that from an infinite family of integral curves, in the simplest case, a definite integral curve is singled out which corresponds to a particular solution of our differential equation.

Analytically, this requirement is reduced to the so-called *initial condition*: $y = y_0$ for $x = x_0$. If the general solution (201-2) is known, then we have

$$y_0 = \varphi(x_0, C).$$

From this condition, generally speaking, we can determine the arbitrary constant C and, consequently, find the corresponding particular solution. The problem of determining the particular solution specified by a given initial condition is referred to as *Cauchy's problem*.

Cauchy's problem. Find the solution $y = \varphi(x)$ of differential equation (201-1) satisfying the given initial condition: $y_0 = \varphi(x_0)$, i.e. attaining the given value $y = y_0$ for $x = x_0$.

Geometrically, Cauchy's problem is formulated as follows: *find the integral curve of differential equation* (201-1) *passing through a given point* $M_0(x_0, y_0)$.

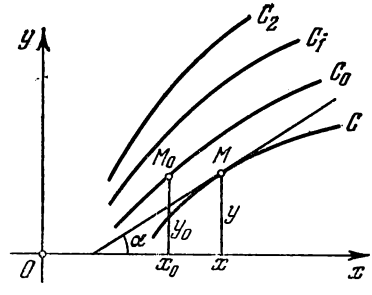


FIG. 224

We should like to note the following: differential equations are a mathematical means with the aid of which we can study the processes occurring in nature. If the initial conditions of a problem specify the process completely, then it must proceed *uniquely*, i.e. the solution of the differential equation establishing the law of development of the process must be *unique*. The general solution of a differential equation contains arbitrary constants and, consequently, gives no definite answer to the put question. Therefore, for solving concrete problems, besides a differential equation some additional conditions are needed. In the simplest case, these are initial conditions, and here we come to Cauchy's problem.

In some cases it seems to be advantageous to write first-order differential equation (201-1) in the form

$$\frac{dy}{dx} = f(x, y), \qquad (201\text{-}1')$$

or in the form

$$P(x, y)\,dx + Q(x, y)\,dy = 0, \qquad (201\text{-}3)$$

where $P(x, y)$ and $Q(x, y)$ are known functions. Form (201-3) is convenient due to the fact that the variables x and y are *equivalent* here, i.e. either of them may be considered as a function of the other. The solutions of equation (201-3) are understood in the general

case to be the functions $x = \varphi(t)$, $y = \psi(t)$ represented parametrically (t parameter) and satisfying equation (201-3).

There is no general method for integrating a first-order differential equation. It is common practice to consider only some individual types of such equations supplying each of them with a particular method for their solution.

Sec. 202. First-Order Equations with Variables Separable

Definition. *A first-order differential equation is said to be an equation with variables separable if it has the form*

$$X(x)\,Y(y)\,dx + X_1(x)\,Y_1(y)\,dy = 0, \tag{202-1}$$

where $X(x)$, $X_1(x)$ are functions only of the variable x and $Y(y)$, $Y_1(y)$ are functions of the variable y only.

To solve equation (202-1), we divide both its members by the product $Y(y)\,X_1(x)$, assuming that it is not equal to zero. Then, after obvious reductions, we get

$$\frac{X(x)}{X_1(x)}\,dx + \frac{Y_1(y)}{Y(y)}\,dy = 0. \tag{202-2}$$

In equation (202-2) dx is preceded by a function only of x and dy by a function only of y. In this case we say that the *variables are separated* in the sense that each variable only enters into the term of the equation containing its differential. Integrating the left-hand and right-hand members of equation (202-2), we shall have

$$\int \frac{X(x)}{X_1(x)}\,dx + \int \frac{Y_1(y)}{Y(y)}\,dy = C. \tag{202-3}$$

These integrals are understood to be some corresponding antiderivatives.

Relation (202-3) just represents a total integral of equation (202-1) expressed in an implicit form.

In the general case, dividing the equation by the product $X_1(x) \times Y(y)$, we run risks to lose the solutions of equation (202-1) which reduce this product to zero.

By a direct substitution it is easy to get convinced that the function

$$x = a, \tag{202-4}$$

where a is the root of the equation $X_1(x) = 0$, i.e. $X_1(a) = 0$, is a solution of equation (202-1). The function

$$y = b, \tag{202-5}$$

where b is the root of the equation $Y(y) = 0$, i.e. $Y(b) = 0$, is also a solution of equation (202-1).

Geometrically, solutions (202-4) and (202-5), if any, represent straight lines parallel to the y- and x-axes, respectively.

Example 1. Suppose we are given the equation

$$\frac{dy}{dx} = \frac{y}{x}. \tag{202-6}$$

Hence, we have

$$x\,dy = y\,dx.$$

Let us assume that $y \neq 0$. If we divide both members of this equation by xy, then the variables will become separated, and we shall obtain

$$\frac{dy}{y} = \frac{dx}{x}.$$

Integrating, we shall have

$$\int \frac{dy}{y} = \int \frac{dx}{x},$$

or

$$\ln y = \ln x + \ln C^*. \tag{202-7}$$

The arbitrary constant is taken here in the *logarithmic form* which is, of course, valid, since any positive or negative number C_1 can be represented as the logarithm of another number:

$$C_1 = \ln C,$$

where

$$C = e^{C_1}.$$

Resorting to involution, we finally get

$$y = Cx \qquad (x \neq 0,\ C \neq 0). \tag{202-8}$$

Setting now $xy = 0$ and taking into account that $x \neq 0$, we get the solution of equation (202-6) $y = 0$. Formally, this solution is obtained from (202-8) for $C = 0$.

Geometrically, the general solution (202-8) represents a family of rays ($0 \leqslant |C| < +\infty$) emanating from the origin (Fig. 225).

* Strictly speaking we must write

$$\ln |y| = \ln |x| + \ln \overline{C},$$

where $\overline{C} > 0$. But the "liberty" taken by us will not affect the final result if after involution the arbitrary constant C is regarded to be a real number. It should be taken into further consideration.

Example 2. Find the curve passing through the point $Q\ (-1,\ 4)$ and possessing the following property: its subnormal at any point has one and the same value equal to 4.

Let $y = f(x)$ be the required curve, MT the tangent to this curve at point $M\ (x,\ y)$, MN the normal (the perpendicular to the tangent

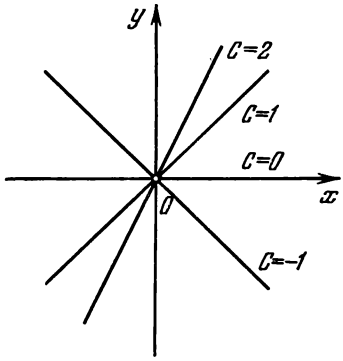


FIG. 225

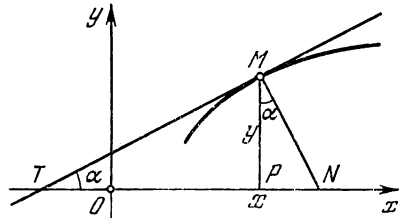


FIG. 226

at the point of tangency) (Fig. 226). The *subnormal* (PN) is the name given to the projection of the segment of the normal MN on the x-axis.

Since

$$PM = y$$

and

$$\angle NMP = \angle MTx = \alpha,$$

we have

$$PN = y \tan \alpha.$$

But, according to the geometrical meaning of the derivative

$$\tan \alpha = y';$$

therefore we finally have the following expression for the subnormal:

$$PN = yy'.$$

By the initial condition of the problem

$$yy' = 4, \quad \text{or} \quad y\frac{dy}{dx} = 4.$$

Separating the variables, we get

$$y\,dy = 4\,dx.$$

Taking integrals of the right- and left-hand members, we shall have

$$\int y\,dy = 4\int dx + C$$

or

$$\frac{y^2}{2} = 4x + C.$$

Hence,

$$y^2 = 8x + C_1. \qquad (202\text{-}9)$$

We have obtained a family of parabolas whose vertices lie on the x-axis.

We now determine the arbitrary constant C_1 from the condition that our parabola passes through the given point Q (—1, 4). Substituting the coordinates of the point Q instead of current coordinates into equation (202-8), we find

$$16 = -8 + C_1;$$

hence

$$C_1 = 24.$$

Consequently, the equation of the desired parabola has the form

$$y^2 = 8x + 24,$$

or

$$y^2 = 8(x + 3).$$

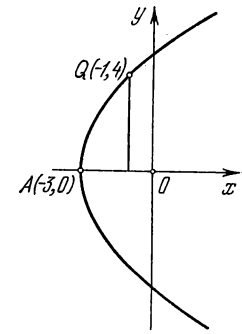


FIG. 227

The vertex of the parabola is found at the point A (—3, 0), the x-axis serving as its axis (Fig. 227).

Example 3. The rate of cooling of a body in air is proportional to the difference between the temperature of the body and air temperature. The air temperature is equal to 20°C. It is known that during 20 minutes the body is cooled from 100°C to 60°C. Determine the law of change of the body temperature with time.

If we denote the time by t and body temperature by U, then the rate of cooling of the body, or in other words, the rate of change of its temperature, will be equal to the derivative $\frac{dU}{dt}$. According to the initial condition, we have

$$\frac{dU}{dt} = k(U - 20),$$

where k is the proportionality factor. Separating the variables, we get

$$\frac{dU}{U-20} = k\,dt.$$

Taking integrals of the left- and right-hand members, we shall have

$$\int \frac{dU}{U-20} = k \int dt + \ln C^*,$$

or

$$\ln (U - 20) = kt + \ln C.$$

Hence

$$U - 20 = Ce^{kt}$$

and, consequently,

$$U = 20 + Ce^{kt}. \qquad (202\text{-}10)$$

To determine the constants C and k, let us take advantage of the conditions of the problem

$$U = 100° \quad \text{for} \quad t = 0$$

and

$$U = 60° \quad \text{for} \quad t = 20.$$

Substituting these values into equation (202-10), we shall have

$$\left.\begin{aligned} 100 &= 20 + C, \\ 60 &= 20 + Ce^{20k}. \end{aligned}\right\}$$

Hence

$$C = 80, \quad e^{20k} = \frac{1}{2}$$

and, consequently,

$$e^k = \left(\frac{1}{2}\right)^{\frac{1}{20}}.$$

Introducing these values into (202-10), we finally get

$$U = 20 + 80 \cdot \left(\frac{1}{2}\right)^{\frac{t}{20}}.$$

---

* Since we shall resort to involution, it is suitable here to write $\ln C$ instead of C.

Such is the law of change of temperature U with time t under the given conditions.

In Examples 2 and 3, in forming differential equations we dealt directly with the derivative of the desired function. We are now going to consider an example where it is more convenient to reason using the differentials of the required quantities.

Example 4. In a reservoir containing 10 kilograms of salt per 100 litres of mixture each minute 30 litres of water flows in and 20 litres of mixture runs out (Fig. 228, *a*). Determine the amount of salt which will remain in the reservoir in t minutes assuming that the mixture is stirred up instantaneously.

Let x be the amount of salt in the reservoir at the moment of time t and $x + dx$ the amount of salt at the moment $t + dt$. Since the

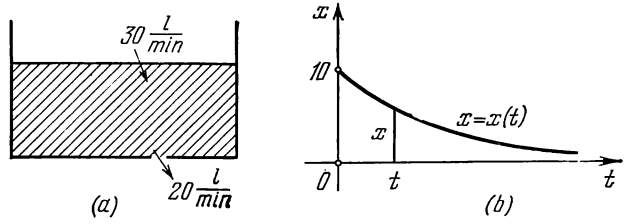


FIG. 228

mixture flows out, the amount of salt x reduces with time and, consequently, $dx < 0$ for $dt > 0$. The volume of the mixture in the reservoir at the moment of time t is, obviously, equal to

$$v = 100 + 30t - 20t = 100 + 10t,$$

therefore, the *concentration of salt* (i.e. the amount of salt per unit of the mixture volume) at the moment t will be equal to

$$\frac{x}{100+10t}. \tag{202-11}$$

The change in the amount of salt, dx, during an infinitely small interval of time, $[t, t + dt]$ is found by multiplying the volume of the mixture flown out during this interval, $20dt$, by the concentration of salt (202-11). Hence we have a differential equation

$$-dx = \frac{x}{100+10t} \cdot 20dt,$$

or

$$dx = -\frac{2x}{10+t}\,dt. \tag{202-12}$$

Besides, from the conditions of the problem there follows the initial condition

$$x\,|_{t=0} = 10. \tag{202-13}$$

Separating the variables in (202-12) and integrating, we get in succession

$$\frac{dx}{x} = -\frac{2}{10+t}\,dt$$

and

$$\int \frac{dx}{x} = -2\int \frac{dt}{10+t},$$

i.e.

$$\ln x = -2\ln(10+t) + \ln C$$

and, consequently,

$$x = \frac{C}{(10+t)^2}.$$

Setting $t = 0$, from the initial condition (202-13), we find

$$10 = \frac{C}{100},$$

i.e. $C = 1000$. Therefore, the law of change in the amount of salt x in kilograms depending on the time t in minutes elapsed (Fig. 228, *b*) is given by the formula

$$x = \frac{1000}{(10+t)^2}. \qquad (202\text{-}14)$$

Note that by this formula, knowing the amount of salt which remained in the reservoir (this is easy to determine by measuring the volume of the reservoir and the concentration of salt in it), we can find out how much time has elapsed from the beginning of the process. This idea underlies the computation of the age of seas and oceans.

Sec. 203. Homogeneous Differential Equations of the First Order

The notion of a homogeneous differential equation of the first order is linked with *homogeneous functions.*

The polynomial

$$P(x, y) = \sum_{i,j} a_{ij}x^i y^j$$

is called *homogeneous* of degree n if all its terms have one and the same order n, i.e. for each such term $a_{ij}x^i y^j$ we have

$$i + j = n.$$

For instance,

$$P(x, y) = 2x^2 - 3xy - 5y^2 \tag{203-1}$$

is a homogeneous polynomial of degree 2. Note that if the arguments x and y of a homogeneous polynomial of degree n are replaced by the proportional quantities kx and ky, then, as the result, this polynomial will be multiplied by the nth power of the proportionality factor k. Thus, for instance, for polynomial (203-1) we have

$$P(kx, ky) = 2(kx)^2 - 3(kx)(ky) - 5(ky)^2 =$$
$$= k^2(2x^2 - 3xy - 5y^2) = k^2P(x, y).$$

This property underlies the general definition of a homogeneous function.

Definition 1. *The function $P(x, y)$ is called homogeneous of degree n if for any number k the identity*

$$P(kx, ky) \equiv k^nP(x, y)$$

holds true.

Let us now consider the differential equation

$$P(x, y)\,dx + Q(x, y)\,dy = 0. \tag{203-2}$$

Definition 2. *A first-order differential equation* (203-2) *is called homogeneous if the coefficients $P(x, y)$ and $Q(x, y)$ of the differentials of the variables x and y are homogeneous functions of one and the same degree.*

It is possible to prove that with the aid of the substitution

$$u = \frac{y}{x} \qquad \left(\text{or } v = \frac{x}{y}\right), \tag{203-3}$$

where u is a new unknown function, the homogeneous differential equation (203-2) is reduced to an equation with variables separable.

Example. Solve the differential equation

$$(x + y)\,dx + x\,dy = 0. \tag{203-4}$$

Here

$$P = x + y \quad \text{and} \quad Q = x$$

are homogeneous functions of the first degree, therefore equation (203-4) is homogeneous. According to the hint, we put

$$\frac{y}{x} = u \tag{203-5}$$

and

$$y = xu,$$

where u is an unknown function.

Hence

$$dy = x\,du + u\,dx.$$

Substituting this expression into (203-4), we shall have

$$(x + xu)\,dx + x\,(x\,du + u\,dx) = 0$$

or

$$x\,du + (2u + 1)\,dx = 0.$$

Separating the variables, we get

$$\frac{du}{2u+1} = -\frac{dx}{x}.$$

For the sake of convenience, we multiply both members of the last equality by 2. Then, integrating termwise, we shall have

$$\int \frac{2du}{2u+1} = -2\int \frac{dx}{x};$$

hence we find

$$\ln(2u + 1) = -2\ln x + \ln C$$

and

$$2u+1 = \frac{C}{x^2}.$$

By (203-5), we have

$$\frac{2y}{x}+1 = \frac{C}{x^2}$$

and, consequently,

$$y = -\frac{x}{2}+\frac{C_1}{x}, \qquad (203\text{-}6)$$

where $C_1 = \frac{1}{2}C$ is an arbitrary constant.

In the process of solving we had to divide by the functions x and $2u + 1$. Equating them to zero, we get possible solutions:

(1) $x = 0$ and (2) $2u+1=0$; hence $y = -\frac{x}{2}$.

Both functions (1) and (2), as it is easy to get convinced by a check, satisfy the given equation (203-4); the latter is obtained from the general solution (203-6) for $C_1 = 0$.

Let now a differential equation have the form

$$y' = f(x, y), \qquad (203\text{-}7)$$

or

$$\frac{dy}{dx} = f(x, y).$$

Writing the last equation in differentials, we get

$$dy = f(x, y)\, dx.$$

We see that dy has a coefficient equal to 1, i.e. it is a homogeneous function of degree 0; consequently, $f(x, y)$ must also be a homogeneous function of degree 0.

Thus, *a differential equation* (203-7) *is homogeneous if and only if its right-hand member* $f(x, y)$ *is a homogeneous function of degree zero**.

Sec. 204. Linear Differential Equations of the First Order

A first-order differential equation has the form (see Sec. 200)

$$a(x)\, y' + b(x)\, y + c(x) = 0, \tag{204-1}$$

where $a(x)$, $b(x)$, $c(x)$ are given functions. If $a(x) \not\equiv 0$, then equation (204-1) can be written in the reduced form:

$$y' + p(x)\, y = f(x), \tag{204-2}$$

where $p(x) = b(x)/a(x)$ and $f(x) = -c(x)/a(x)$ ($f(x)$ is the *constant term* or the *right-hand member of the equation*). We shall assume that the coefficient $p(x)$ and the constant term $f(x)$ of equation (204-2) are continuous on a certain interval (a, b).

For solving equation (204-2) we represent the required function y in the form of the product of two factors:

$$y = uv, \tag{204-3}$$

where u is a certain non-zero solution of the corresponding homogeneous equation

$$u' + p(x)\, u = 0, \tag{204-4}$$

and v is a new unknown function. Since

$$y' = vu' + uv', \tag{204-5}$$

substituting expressions (204-3) and (204-5) into differential equation (204-2), we get

$$v[u' + p(x)\, u] + uv' = f(x) \tag{204-6}$$

* That is the right-hand member of this equation remains unchanged when x is replaced by kx and y by ky, where k is an arbitrary proportionality factor.

or, by (204-4), we have

$$uv' = f(x). \tag{204-7}$$

We should like to note here that actually the function u is chosen so that *the coefficient of v in equation* (204-6) *is equal to zero.*

The functions u and v are found in succession from equations (204-4) and (204-7). We take for u a concrete non-zero solution. Substituting the expressions obtained for the functions u and v into formula (204-3), we find the desired function y.

Note. Practically, there is no necessity to reduce linear equation (204-1) to the form (204-2); we may apply substitution (204-3) at once.

Example 1. Solve the equation

$$xy' + 2y = x^2. \tag{204-8}$$

This equation is obviously linear. We put

$$y = uv, \quad y' = vu' + uv'. \tag{204-9}$$

Substituting these expressions into equation (204-8), we get

$$v(xu' + 2u) + xuv' = x^2.$$

We choose the function u so that

$$xu' + 2u = 0; \tag{204-10}$$

then

$$xuv' = x^2. \tag{204-11}$$

From (204-10) we get in succession

$$x\frac{du}{dx} = -2u, \quad \frac{du}{u} = -2\frac{dx}{x};$$

integrating, we find

$$\int \frac{du}{u} = -2\int \frac{dx}{x},$$

i.e.

$$\ln u = -2\ln x + \ln C_0,$$

and, consequently, choosing $C_0 = 1$, we get

$$u = \frac{1}{x^2}. \tag{204-12}$$

Hence, from (204-11) we have

$$x \cdot \frac{1}{x^2}\frac{dv}{dx} = x^2, \quad \frac{dv}{dx} = x^3$$

and thus,

$$v = \int x^3\,dx = \frac{x^4}{4} + C, \tag{204-13}$$

where C is an arbitrary constant.

Thus, by (204-12) and (204-13), we finally find

$$y = uv = \frac{1}{x^2}\left(\frac{x^4}{4} + C\right), \quad \text{i.e.} \quad y = \frac{x^2}{4} + \frac{C}{x^2}.$$

Example 2. Find the solution of the equation

$$(x + y)\,y' = 1, \tag{204-14}$$

satisfying the initial condition: $y = 0$ for $x = -1$.

By the form, (204-14) is not a linear equation. But if x is considered as a function of y, then, taking into account that $y' = \frac{1}{x'}$, we shall obtain a linear equation

$$x' = x + y. \tag{204-15}$$

As usual, we put

$$x = uv, \quad x' = \frac{dx}{dy} = v\,\frac{du}{dy} + u\,\frac{dv}{dy}.$$

Substituting these expressions into equation (204-15), we shall have

$$v\,\frac{du}{dy} + u\,\frac{dv}{dy} = uv + y. \tag{204-16}$$

Hence, taking into account that, according to the choice of u,

$$\frac{du}{dy} = u, \tag{204-17}$$

we obtain

$$u\,\frac{dv}{dy} = y. \tag{204-18}$$

From (204-17) we find the partial solution

$$u = e^y. \tag{204-19}$$

Therefore, from (204-18) we get

$$e^y\,\frac{dv}{dy} = y, \quad dv = ye^{-y}\,dy$$

and, hence,

$$v = \int ye^{-y}\,dy = -ye^{-y} - e^{-y} + C \tag{204-20}$$

(here we took advantage of integration by parts!). From (204-19) and (204-20) we find the *general solution*

$$x = uv = -y - 1 + Ce^{y}. \tag{204-21}$$

Setting here $y = 0$ for $x = -1$, we get $-1 = -1 + C$, i.e. $C = 0$. Thus,

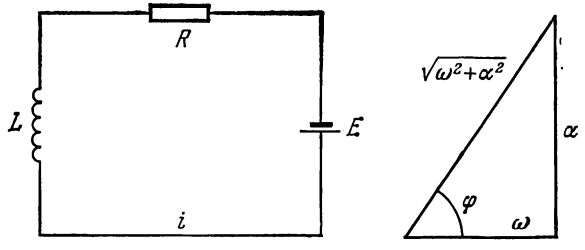
$$x = -y - 1, \quad \text{i.e.} \quad y = -(x + 1)$$

is the required *particular solution.*

Example 3. Current intensity in an electric circuit with a resistor R and self-induction L satisfies the differential equation

$$L\frac{di}{dt} + Ri = E, \tag{204-22}$$

where E is the electromotive force (Fig. 229, *a*). Find the current



(*a*) (*b*) FIG. 229

intensity i in t seconds after the moment of switching on if E changes according to the sinusoidal law:

$$E = E_0 \cos \omega t \tag{204-23}$$

and $i = 0$ for $t = 0$.

From (204-22) we have

$$\frac{di}{dt} + \alpha i = \frac{E_0}{L} \cos \omega t, \tag{204-24}$$

where, for brevity, we set $\alpha = R/L$.

Putting

$$i = uv,$$

we get in a usual way

$$\frac{du}{dt} + \alpha u = 0, \quad u\frac{dv}{dt} = \frac{E_0}{L} \cos \omega t. \tag{204-25}$$

Hence

$$\frac{du}{u} = -\alpha\, dt, \quad \ln u = -\alpha t$$

(we omit the integration constant here!) and

$$u = e^{-\alpha t}. \tag{204-26}$$

From (204-25) we have

$$e^{-\alpha t}\frac{dv}{dt} = \frac{E_0}{L}\cos \omega t,$$

$$dv = \frac{E_0}{L} e^{\alpha t}\cos \omega t\, dt$$

and

$$v = \frac{E_0}{L}(I + C), \tag{204-27}$$

where

$$I = \int e^{\alpha t}\cos \omega t\, dt \tag{204-28}$$

(one of the antiderivatives).

Integrating by parts twice, we find

$$I = \int e^{\alpha t} d\left(\frac{\sin \omega t}{\omega}\right) = e^{\alpha t}\frac{\sin \omega t}{\omega} - \int \frac{\sin \omega t}{\omega}\cdot \alpha e^{\alpha t}\, dt =$$

$$= e^{\alpha t}\frac{\sin \omega t}{\omega} + \frac{\alpha}{\omega}\int e^{\alpha t}\cdot d\left(\frac{\cos \omega t}{\omega}\right) =$$

$$= e^{\alpha t}\frac{\sin \omega t}{\omega} + \frac{\alpha}{\omega}\left[e^{\alpha t}\frac{\cos \omega t}{\omega} - \int \frac{\cos \omega t}{\omega}\cdot \alpha e^{\alpha t}\, dt\right] =$$

$$= \frac{1}{\omega} e^{\alpha t}\sin \omega t + \frac{\alpha}{\omega^2} e^{\alpha t}\cos \omega t - \frac{\alpha^2}{\omega^2} I;$$

hence, we get

$$I = \frac{\frac{1}{\omega^2} e^{\alpha t}(\omega \sin \omega t + \alpha \cos \omega t)}{1 + \frac{\alpha^2}{\omega^2}} = \frac{e^{\alpha t}(\omega \sin \omega t + \alpha \cos \omega t)}{\omega^2 + \alpha^2}. \tag{204-29}$$

Substituting this expression into (204-27), we find

$$v = \frac{E_0}{L}\left(e^{\alpha t}\cdot \frac{\omega \sin \omega t + \alpha \cos \omega t}{\omega^2 + \alpha^2} + C\right), \tag{204-30}$$

where C is an arbitrary constant.

Multiplying the function u by the function v ((204-26) by (204-30)), we get the *law of change of current intensity*

$$i = \frac{E_0}{L}\left(\frac{\omega \sin \omega t + \alpha \cos \omega t}{\omega^2 + \alpha^2} + Ce^{-\alpha t}\right). \tag{204-31}$$

For $t = 0$ we find from the initial condition

$$0 = \frac{E_0}{L}\left(\frac{\alpha}{\omega^2 + \alpha^2} + C\right),$$

i.e.

$$C = -\frac{\alpha}{\omega^2+\alpha^2}.$$

Consequently,

$$i = \frac{E_0}{L(\omega^2+\alpha^2)}(\omega \sin \omega t + \alpha \cos \omega t - \alpha e^{-\alpha t}). \tag{204-32}$$

If, t is sufficiently large then $e^{-\alpha t}$ is a small quantity ($\alpha > 0$) and we may ignore it in formula (204-32). Then we shall have

$$i \approx \frac{E_0}{L(\omega^2+\alpha^2)}(\omega \sin \omega t + \alpha \cos \omega t). \tag{204-33}$$

Setting (Fig. 229, *b*)

$$\omega = \sqrt{\omega^2+\alpha^2}\cos \varphi, \quad \alpha = \sqrt{\omega^2+\alpha^2}\sin \varphi,$$

from (204-33) we finally obtain

$$i \approx \frac{E_0}{L\sqrt{\omega^2+\alpha^2}}\sin(\omega t + \varphi) = \frac{E_0}{\sqrt{(L\omega)^2+R^2}}\sin(\omega t + \varphi),$$

where $\varphi = \arctan\frac{\alpha}{\omega}$ is the initial phase of the current.

Sec. 205. Euler's Method

In the previous sections we considered the simplest types of first-order differential equations which permit solutions in quadratures* (or, as we sometimes say, integrable in the finite form!). But there is no general method for finding the exact solution of an arbitrary differential equation of the first order. Therefore, *approximate methods of solving* differential equations are of great importance. This section is dedicated to one of them which is called *Euler's method.*

Suppose on a given segment $x_0 \leqslant x \leqslant X$ it is required to find the solution of the first-order differential equation

$$y' = f(x, y) \tag{205-1}$$

with a continuous right-hand member $f(x, y)$ satisfying the initial condition

$$y(x_0) = y_0. \tag{205-2}$$

Geometrically, this means that for differential equation (205-1) we have to construct an integral curve $y = y(x)$ passing through

* That is whose solutions are expressed with the aid of indefinite integrals.

the point $M_0(x_0, y_0)$ (Fig. 230, *a*). From the geometrical meaning of the derivative it follows that at each point $M(x, y)$ of the integral curve its *inclination* (i.e. slope) satisfies the condition

$$k = \tan \alpha = f(x, y). \tag{205-3}$$

Since, by the assumption, the right-hand member of differential equation (205-1) is continuous, we may consider that on a small

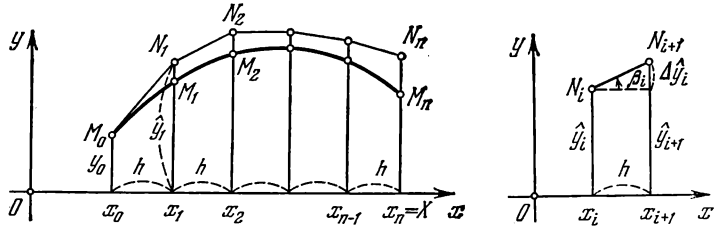


FIG. 230

part of the integral curve its inclination is *constant*, i.e. this curve may be approximately replaced by a polygonal line.

In practice this is done in the following way: we divide the interval $[x_0, X]$ into sufficiently small parts: $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, X]$, whose number is equal to n, and let

$$h_i = x_{i+1} - x_i \qquad (i = 0, 1, \ldots, n-1)$$

be the lengths of the corresponding small intervals. For the sake of simplicity, we shall consider them to be equal (though, not necessarily!). Then

$$h_i = \frac{X - x_0}{n} = h = \text{const}. \tag{205-4}$$

The quantity h is called the *pitch of a process*.

Let us replace the curve $M_0M_1M_2 \ldots M_n$ with vertices $M_i(x_i, y_i)$ by the polygonal line $N_0N_1N_2 \ldots N_n$ with vertices $N_i(x_i, \hat{y}_i)$ $(i = 0, 1, 2, \ldots, n-1, \hat{y}_0 = y_0)$, where $N_0 = M_0$, and successive slopes

$$\tan \beta_i = f(x_i, \hat{y}_i) = \hat{y}_i' \tag{205-5}$$

$$(i = 0, 1, 2, \ldots, n-1, \quad \hat{y}_0' = y_0' = f(x_0, y_0))$$

(*Euler's polygon*) (Fig. 230, *a*). From Fig. 230, *b* we have the following computational formulas:

$$\left.\begin{aligned} &x_i = x_0 + ih, \\ &\Delta\hat{y}_i = h \tan \beta_i = hf(x_i, \hat{y}_i), \\ &\hat{y}_{i+1} = \hat{y}_i + \Delta\hat{y}_i \end{aligned}\right\} \tag{205-6}$$

$$(i = 0, 1, 2, \ldots, n-1, \ \hat{y}_0 = y_0).$$

Note that from the mechanical point of view we replace the continuous process described by differential equation (205-1) by a *pulsing process* taking place at a constant rate on the elementary intervals $[x_i, x_{i+1}]$ $(i = 0, 1, 2, \ldots, n - 1)$ whose rate changes by jumps when passing over to the next interval.

These are the drawbacks of this method: (1) low accuracy at a considerable pitch h, a large amount of work at a small pitch, (2) systematic accumulation of errors.

Euler's method serves as a conceptual base for other, more perfect methods for approximate solution of differential equations.

Example. Using Euler's method on the interval [0, 0.5] find the solution of the differential equation

$$y' = x + y, \quad y(0) = 1. \tag{205-7}$$

Let us take the pitch $h = 0.1$. The results of computation (with an accuracy to 10^{-3}) are tabulated below:

x	$\hat{y}$	$\hat{y}' = x + \hat{y}$	$\Delta\hat{y} = \hat{y}'h$
0	1.000	1.000	0.100
0.1	1.100	1.200	0.120
0.2	1.220	1.420	0.142
0.3	1.362	1.662	0.166
0.4	1.528	1.928	0.193
0.5	1.721		

Thus, $\hat{y}(0.5) = 1.721$. It is not difficult to find the exact solution ((205-7) is a linear equation!): $y = 2e^x - (x + 1)$, hence, $y(0.5) = 2\sqrt{e} - 1.5 \approx 2 \cdot 1.645 - 1.500 = 1.790$.

Sec. 206. Differential Equations of the Second Order

The general form of a second-order differential equation resolved with respect to the higher derivative is as follows:

$$y'' = f(x, y, y'). \tag{206-1}$$

The general solution

$$y = \varphi(x, C_1, C_2) \tag{206-2}$$

of this equation contains two independent arbitrary constants C_1 and C_2. Geometrically, general solution (206-2) represents an infinite set of integral curves which depends on two independent parameters

C_1 and C_2. Generally speaking, through each point $M_0(x_0, y_0)$ of the xy-plane there passes a *pencil of integral curves* (Fig. 231). Therefore, to single out one definite integral curve Γ from our family of integral curves, it is not sufficient to indicate the point $M_0(x_0, y_0)$ through which this curve must pass. One should indicate the direction in which the curve Γ passes through the point M_0, i.e. give the tangent of the angle α_0 formed by the tangent line to this curve at the point M_0 with the positive direction of the x-axis. Analytically, if we denote

$$\tan \alpha_0 = y'_0,$$

we come to the following *initial conditions*: $y = y_0$, $y' = y'_0$ for $x = x_0$. By (206-2), we have

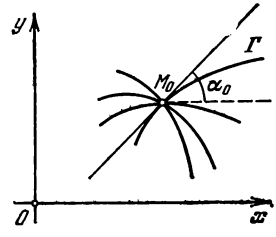


FIG. 231

$$\left.\begin{aligned} y_0 &= \varphi(x_0, C_1, C_2), \\ y'_0 &= \varphi'_x(x_0, C_1, C_2). \end{aligned}\right\} \qquad (206\text{-}3)$$

From system (206-3) we can, generally speaking, determine the constants C_1 and C_2 and, thus, find the *particular solution*

$$y = \varphi(x),$$

satisfying our equation (206-1) and the given initial conditions

$$y|_{x=x_0} = y_0 \quad \text{and} \quad y'|_{x=x_0} = y'_0 \qquad (206\text{-}4)$$

(*Cauchy's problem*). Note that in solving concrete physical problems, as a rule, along with a differential equation there appear certain initial conditions (206-4), since the solution of such a problem must be *unique*.

Again, with the aid of a second-order differential equation we write the *basic dynamical equation*.

Let a material point of mass m move along the x-axis under the action of a variable force F. If we denote by j the acceleration of this point, then, according to Newton's law, we have

$$mj = F. \qquad (206\text{-}5)$$

In the most general case the force F depends on the time t, the coordinate x (characterizing the position of the material point on the x-axis), and the velocity $\frac{dx}{dt}$ of this point. Consequently,

$$F = F\left(t, x, \frac{dx}{dt}\right).$$

On the other hand, as is known (Sec. 83), for a rectilinear motion the acceleration j is equal to the second derivative of the distance covered

with respect to the time elapsed, i.e.

$$j = \frac{d^2x}{dt^2}.$$

Substituting these quantities in (206-5), we get the *differential equation of motion of a mass point*

$$m\frac{d^2x}{dt^2} = F\left(t,\ x,\ \frac{dx}{dt}\right).$$

In order to completely describe the motion of a point, we have to specify additionally the initial position of the point

$$x\,|_{t=t_0} = x_0$$

and the initial velocity of the point

$$\left.\frac{dx}{dt}\right|_{t=t_0} = v_0$$

(*the initial conditions*).

Sec. 207. Integrable Types of Second-Order Differential Equations

In the general case a differential equation of the second order cannot be solved in a finite form. We shall consider here some simple cases when a second-order equation is solved with the aid of *quadratures*, i.e. applying the operations of indefinite integration.

Type I. Let

$$y'' = f(x). \tag{207-1}$$

Integrating, we shall have

$$y' = \int f(x)\,dx + C_1.$$

Integrating once again, we finally get

$$y = \int dx \int f(x)\,dx + C_1x + C_2,$$

where C_1 and C_2 are arbitrary constants, and the indefinite integrals are treated as certain antiderivatives of the corresponding functions.

Type II. Let

$$y'' = f(y). \tag{207-2}$$

We put

$$y' = p.$$

Hence, regarding p as a function of y, we shall have

$$y'' = \frac{dy'}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p \frac{dp}{dy}.$$

Hence, equation (207-2) will attain the form

$$p \frac{dp}{dy} = f(y).$$

Separating the variables, we obtain

$$p\, dp = f(y)\, dy.$$

Integrating the last equation, we find

$$\frac{p^2}{2} = \int f(y)\, dy + \frac{C_1}{2}$$

or

$$p = \pm \sqrt{2 \int f(y)\, dy + C_1}.$$

Since

$$p = \frac{dy}{dx},$$

the preceding equation may be written as follows:

$$\frac{dy}{dx} = \pm \sqrt{2 \int f(y)\, dy + C_1}.$$

Hence, separating the variables once again and integrating, we shall finally have

$$\int \frac{dy}{\sqrt{2 \int f(y)\, dy + C_1}} = \pm (x + C_2).$$

The reader should not memorize this complicated formula of the general solution of type-II equation but should understand the method of integration.

Type III. Let

$$y'' = f(y'). \qquad (207\text{-}3)$$

We put

$$y' = p.$$

Then

$$y'' = \frac{dp}{dx}.$$

Equation (207-3) will take the form

$$\frac{dp}{dx} = f(p).$$

Separating the variables and integrating, we shall have in succession

$$\frac{dp}{f(p)} = dx$$

and

$$\int \frac{dp}{f(p)} = x + C_1.$$

On determining the quantity $p = \frac{dy}{dx}$, we shall be able to find y as well, using the method of secondary integration.

Example 1. Determine the motion of a material point of mass m thrown vertically upwards with the initial velocity v_0.

Let us take the vertical straight line which is the path of the moving point for the x-axis, the upward direction of the x-axis being established as its positive direction. The initial position of our material point is taken for the origin.

If air resistance is ignored, then the only force acting on our point is the gravity force numerically equal to mg and directed vertically downwards. According to Newton's law, we have the following differential equation of motion:

$$m \frac{d^2x}{dt^2} = -mg$$

or

$$\frac{d^2x}{dt^2} = -g. \tag{207-4}$$

Besides, we must observe the initial conditions:

$$x|_{t=0} = 0 \quad \text{and} \quad \left.\frac{dx}{dt}\right|_{t=0} = v_0. \tag{207-5}$$

Substituting

$$\frac{dx}{dt} = v \quad \text{and} \quad \frac{d^2x}{dt^2} = \frac{dv}{dt},$$

from (207-4) we obtain

$$\frac{dv}{dt} = -g$$

or

$$dv = -g\,dt.$$

Integrating, we shall have

$$v = C_1 - gt.$$

Setting here $t = 0$ and using the second condition (207-5), we find

$$C_1 = v_0.$$

Hence,

$$v = v_0 - gt \tag{207-6}$$

or

$$\frac{dx}{dt} = v_0 - gt.$$

Integrating once again, we shall have

$$x = C_2 + v_0 t - \frac{gt^2}{2}.$$

To determine the constant C_2, let us note that, by virtue of first condition (207-5), $x = 0$ for $t = 0$. Substituting these values in our last equation, we get

$$C_2 = 0.$$

Consequently,

$$x = v_0 t - \frac{gt^2}{2}. \tag{207-7}$$

Such is the law of motion of a point thrown vertically upwards with the initial velocity v_0 (with air resistance ignored).

In particular, for the highest point reached by our mass point there must be

$$v = 0.$$

Hence, from (207-6) we determine the *time of climb*

$$T = \frac{v_0}{g},$$

and from (207-7) the corresponding *altitude of climb*

$$H = \frac{v_0^2}{2g}.$$

Example 2. Solve the equation

$$y'' = y^{-3}.$$

We put here

$$y' = p,$$

hence,

$$y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} \cdot p.$$

Substituting into the differential equation, we shall obtain

$$p \frac{dp}{dy} = y^{-3}.$$

Separating the variables and integrating, we shall have in succession

$$p\, dp = y^{-3}\, dy$$

and

$$\frac{p^2}{2} = \frac{y^{-2}}{-2} + \frac{C_1}{2}.$$

Hence,

$$p = \pm \sqrt{C_1 - y^{-2}} \qquad (C_1 > 0)$$

and, consequently

$$\frac{dy}{dx} = \pm \frac{\sqrt{C_1 y^2 - 1}}{y}.$$

This is a first-order equation. Separating the variables, we have

$$\frac{y\, dy}{\sqrt{C_1 y^2 - 1}} = \pm dx.$$

Multiplying both members by C_1, we get

$$\frac{C_1 y\, dy}{\sqrt{C_1 y^2 - 1}} = \pm C_1\, dx.$$

On integrating we shall have

$$\int \frac{C_1 y\, dy}{\sqrt{C_1 y^2 - 1}} = \pm (C_1 x + C_2).$$

Let us compute the integral constituting the left-hand member of this equation. Noting that

$$C_1 y\, dy = \frac{1}{2} d(C_1 y^2 - 1),$$

we shall have

$$\int \frac{C_1 y\, dy}{\sqrt{C_1 y^2 - 1}} = \frac{1}{2} \int \frac{d(C_1 y^2 - 1)}{\sqrt{C_1 y^2 - 1}} =$$

$$= \frac{1}{2} \int (C_1 y^2 - 1)^{-\frac{1}{2}} d(C_1 y^2 - 1) = \frac{\frac{1}{2} (C_1 y^2 - 1)^{\frac{1}{2}}}{\frac{1}{2}} = \sqrt{C_1 y^2 - 1}.$$

Thus, we find

$$\sqrt{C_1y^2-1}=\pm(C_1x+C_2),$$

or finally

$$C_1y^2 - 1 = (C_1x + C_2)^2.$$

Example 3. Find the solution of the equation

$$2y'y'' = 1,$$

that satisfies the initial conditions: $y = 0$, $y' = 1$ for $x = 1$.

Setting

$$y' = p \quad \text{and} \quad y'' = \frac{dp}{dx},$$

we have

$$2p\frac{dp}{dx}=1.$$

Separating the variables, we then obtain

$$2p\,dp = dx,$$

or after integration

$$p^2 = x + C_1.$$

For determining the constant C_1 we take advantage of the initial condition $p = y' = 1$ for $x = 1$. We have

$$1 = 1 + C_1;$$

hence $C_1 = 0$ and, consequently,

$$p^2 = x.$$

Taking the root, we get

$$p=\frac{dy}{dx}=\sqrt{x},$$

the plus sign being taken before the radical, since for $x = 1$ we must have $p = 1$.

Separating the variables and integrating, we find

$$y=\int\sqrt{x}\,dx=\frac{2}{3}\,x^{\frac{3}{2}}+C_2.$$

To determine the constant C_2, we put $x = 1$ and $y = 0$; then

$$0=\frac{2}{3}+C_2, \text{ i.e. } C_2=-\frac{2}{3}.$$

Thus, the required solution is

$$y=\frac{2}{3}\,(x^{\frac{3}{2}}-1).$$

Sec. 208. Reducing the Order of a Differential Equation

We are going to indicate two cases when a second-order differential equation

$$y'' = f(x, y, y') \tag{208-1}$$

is reduced to a differential equation of the first order.

Case 1. Let the left-hand member of differential equation (208-1) not explicitly contain x, i.e. the equation has the form

$$y'' = f(y, y'). \tag{208-2}$$

Setting here

$$y' = p \quad \text{and} \quad y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = \frac{dp}{dy} p,$$

we get a first-order differential equation

$$p \frac{dp}{dy} = f(y, p),$$

where the role of an independent variable is played by y.

Case 2. Let the left-hand member of differential equation (208-1) not explicitly contain y, i.e. the equation has the form

$$y'' = f(x, y'). \tag{208-3}$$

Putting

$$y' = p \quad \text{and} \quad y'' = \frac{dp}{dx},$$

we obtain a first-order equation

$$\frac{dp}{dx} = f(x, p)$$

with an unknown function p.

Note that types I and III considered in the preceding section are particular cases of equations (208-2) and (208-3), respectively.

Example 1. Solve the equation

$$y'' = \frac{y'^2}{y}. \tag{208-4}$$

According to Case 1, we put

$$y' = p \quad \text{and} \quad y'' = p \frac{dp}{dy}.$$

Then equation (208-4) will take the form

$$p\frac{dp}{dy}=\frac{p^2}{y}.$$

Hence: (1) $p=0$, i.e. $y=C$; or (2) $\frac{dp}{dy}=\frac{p}{y}$, i.e. $\frac{dp}{p}=\frac{dy}{y}$ and

$$\ln p = \ln y + \ln C_1.$$

Resorting to involution, we shall have

$$p=\frac{dy}{dx}=C_1y$$

and, consequently,

$$\frac{dy}{y}=C_1\,dx.$$

On integrating, we get

$$\ln y = C_1x + \ln C_2$$

and, hence,

$$y = C_2e^{C_1x},$$

where C_1 and C_2 are arbitrary constants.

Example 2. Find the solution of the equation

$$xy'' = 2x - y', \tag{208-5}$$

satisfying the initial conditions: $y=\frac{1}{2}$ and $y'=1$ for $x=1$.

In equation (208-5) we put

$$y'=p \quad \text{and} \quad y''=\frac{dp}{dx}.$$

Then

$$x\frac{dp}{dx}=2x-p$$

or

$$\frac{dp}{dx}=2-\frac{p}{x}. \tag{208-6}$$

The obtained equation is *homogeneous**, therefore we set

$$\frac{p}{x}=u,$$

consequently,

$$p=xu \quad \text{and} \quad \frac{dp}{dx}=x\frac{du}{dx}+u.$$

* Equation (208-6) may also be regarded as *linear*.

Substituting into equation (208-6), we shall have

$$x\frac{du}{dx}+u=2-u;$$

hence

$$\frac{du}{dx}=\frac{2-2u}{x}$$

and

$$\frac{du}{u-1}=-\frac{2dx}{x}.$$

Integrating, we get

$$\ln(u-1)=-2\ln x+\ln C_1$$

and, consequently,

$$u-1=\frac{C_1}{x^2},$$

i.e.

$$\frac{p}{x}=1+\frac{C_1}{x^2} \quad \text{and} \quad p=x+\frac{C_1}{x}.$$

To find the constant C_1, we use the initial conditions: $p = y' = 1$ for $x = 1$. We obtain

$$1=1+C_1,$$

i.e. $C_1 = 0$ and, thus,

$$p=\frac{dy}{dx}=x.$$

Hence, we have

$$dy = x\,dx$$

and

$$y=\int x\,dx=\frac{x^2}{2}+C_2. \tag{208-7}$$

The constant C_2 is determined from the initial conditions. Setting $x = 1$ and $y = \frac{1}{2}$ in (208-1), we get

$$\frac{1}{2}=\frac{1}{2}+C_2,$$

i.e. $C_2 = 0$. Consequently, the desired particular solution is

$$y=\frac{x^2}{2}.$$

Sec. 209. Integrating Differential Equations with the Aid of Power Series

For the sake of simplicity, we shall explain this method by considering the following first-order differential equation

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{209-1}$$

where $f(x, y)$ is an *infinitely differentiable* function, i.e. it has derivatives of all orders.

We shall look for the solution of problem (209-1) in the form of Taylor's series (see Sec. 194)

$$y = y_0 + y_0'(x - x_0) + \frac{y_0''}{2!}(x - x_0)^2 + \ldots, \tag{209-2}$$

where, for the sake of brevity we set

$$y_0^{(n)} = y^{(n)}(x_0) \qquad (n = 1, 2, \ldots).$$

The constant term y_0 is determined from the initial condition (209-1). The coefficient y_0' is found from differential equation (209-1)

$$y_0' = f(x_0, y_0).$$

To find the coefficient y_0'', let us differentiate equation (209-1) with respect to x, assuming that y is a function of x. We have

$$y'' = \frac{d}{dx}[f(x, y)],$$

hence,

$$y_0'' = \left\{\frac{d}{dx}[f(x, y)]\right\}\Big|_{\substack{x=x_0 \\ y=y_0}}$$

and so on. We leave the complicated problem about the convergence of series (209-2) without consideration.

This method, with obvious alterations, is also applicable to the second-order equation

$$y'' = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = y_0'. \tag{209-3}$$

Example. With the aid of power series, find the solution of the differential equation

$$y' = xy + y^2, \quad y(0) = 1. \tag{209-4}$$

We put

$$y = y_0 + y_0'x + \frac{y_0''}{2!}x^2 + \frac{y_0'''}{3!}x^3 + \ldots \tag{209-5}$$

From conditions (209-4) we have

$$y_0 = 1, \quad y_0' = 0 \cdot 1 + 1^2 = 1.$$

Differentiating the right-hand member of the equation (209-4) as a composite function, we get

$$y'' = (y + xy') + 2yy'; \tag{209-6}$$

hence

$$y''_0 = (1 + 0) + 2 \cdot 1 \cdot 1 = 3.$$

Differentiating then equation (209-6), we shall have

$$y''' = (2y' + xy'') + 2(y'^2 + yy'')$$

and, consequently,

$$y'''_0 = (2 + 0) + 2 \cdot (1 + 1 \cdot 3) = 10$$

and so on.

Thus, from (209-5) we have

$$y = 1 + x + \frac{3}{2}x + \frac{5}{3}x^3 + \ldots \tag{209-7}$$

The results of computations are usually tabulated:

x	0	0.1	0.2	0.3	...
y	1	1.117	1.273	1.480	...

Sec. 210. Common Properties of the Solutions of Second-Order Linear Homogeneous Differential Equations

Consider the second-order linear homogeneous differential equation

$$y'' + p(x)\, y' + q(x)\, y = 0, \tag{210-1}$$

whose coefficients $p(x)$ and $q(x)$ are continuous.

Let

$$y_1 = y_1(x) \quad \text{and} \quad y_2 = y_2(x)$$

be particular solutions of equation (210-1)*.

Definition. *Two solutions y_1 and y_2 are called linearly dependent if it is possible to choose constant numbers a_1 and a_2 not equal both to zero such that a linear combination of these functions is identically*

* The word "particular" is understood here in the sense that these solutions do not contain arbitrary constants.

equal to zero, i.e.

$$a_1y_1 + a_2y_2 \equiv 0. \tag{210-2}$$

Otherwise (if it is impossible to choose such numbers), the solutions y_1 and y_2 are termed *linearly independent*. In other words, if the functions y_1 and y_2 are linearly independent and the above identity holds true, then $a_1 = a_2 = 0$.

Obviously, *the solutions y_1 and y_2 will be linearly dependent if and only if they are proportional to each other*, i.e. if

$$y_2 = ay_1 \tag{210-3}$$

(or vice versa), where a is a constant proportionality factor.

Indeed, if this condition is fulfilled, then we may write

$$a_1y_1 + a_2y_2 \equiv 0,$$

where $a_1 = a$ and $a_2 = -1 \neq 0$, and, consequently, these solutions are linearly dependent.

Conversely, if the solutions y_1 and y_2 are linearly dependent, then we have the identity

$$a_1y_1 + a_2y_2 \equiv 0,$$

where at least one constant, a_1 or a_2, is non-zero. Setting, for instance, $a_2 \neq 0$ and $a = -a_1/a_2$, we obtain $y_2 = ay_1$.

The concept of linear dependence is applicable to any pair of functions. Linear dependence or linear independence of several functions is determined in a similar way.

Example. The function e^{k_1x} and e^{k_2x} for $k_2 \neq k_1$ are linearly independent.

Indeed, let us assume that there takes place the relation

$$a_1e^{k_1x} + a_2e^{k_2x} \equiv 0,$$

in which at least one of the coefficients, a_1 or a_2, for instance a_2, is non-zero. Then we shall get the identity

$$e^{(k_2-k_1)x} \equiv -\frac{a_1}{a_2},$$

which is impossible, since the left-hand member of this equality varies with x, and the right-hand member is constant.

Knowing two particular linearly independent solutions y_1 and y_2 of equation (210-1), we can obtain easily the general solution of this equation. And namely, there exists the following theorem.

Theorem. *If y_1 and y_2 are linearly independent particular solutions of a second-order linear homogeneous equation* (210-1), *then the general solution of the equation is a linear combination of these particular*

solutions, i.e. the general solution of equation (210-1) *has the form*

$$y = C_1y_1 + C_2y_2, \tag{210-4}$$

where C_1 *and* C_2 *are arbitrary constants* ($-\infty < C_1 < +\infty$, $-\infty < C_2 < +\infty$).

Proof. Indeed, since y_1 and y_2 are the solutions of equation (210-1), we have

$$y_1'' + p(x)\, y_1' + q(x)\, y_1 \equiv 0 \tag{210-5}$$

and

$$y_2'' + p(x)\, y_2' + q(x)\, y_2 \equiv 0. \tag{210-6}$$

Substituting (210-4) into the left-hand side of equation (210-1), by virtue of (210-5) and (210-6), we get

$$(C_1y_1 + C_2y_2)'' + p(x)\,(C_1y_1 + C_2y_2)' + q(x)\,(C_1y_1 + C_2y_2) =$$

$$= C_1y_1'' + C_2y_2'' + p(x)\,C_1y_1' + p(x)\,C_2y_2' + q(x)\,C_1y_1 + q(x)\,C_2y_2 =$$

$$= C_1[y_1'' + p(x)\,y_1' + q(x)\,y_1] + C_2[y_2'' + p(x)\,y_2' + q(x)\,y_2] =$$

$$C_1 \cdot 0 + C_2 \cdot 0 \equiv 0.$$

Hence, it follows that the function

$$y = C_1y_1 + C_2y_2 \tag{210-7}$$

will be the solution of equation (210-1) at *any choice* of the constants C_1 and C_2.

If the solutions y_1 and y_2 are linearly independent, then the obtained solution (210-7) will be the *general solution* of differential equation (210-1), since it contains two arbitrary constants C_1 and C_2 which in this case cannot be reduced to one, i.e. they are independent.

We can prove that formula (210-7) yields *all solutions* of the corresponding linear differential equation (210-1).

Note. But if the particular solutions y_1 and y_2 are linearly dependent, then (210-4) will not be the general solution. Indeed, let y_1 and y_2 be linearly dependent, i.e. let there exist the relation

$$y_2 = ay_1,$$

where a is a certain constant. Substituting y_2 into expression (210-4), we shall have

$$y = C_1y_1 + C_2ay_1,$$

or

$$y = Cy_1, \tag{210-8}$$

where

$$C = C_1 + aC_2.$$

This solution contains only one arbitrary constant C and therefore it will not be general.

Thus, to find the general solution of equation (210-1), it is sufficient to know two of its particular, linearly independent solutions, y_1 and y_2.

Sec. 211. Second-Order Linear Homogeneous Differential Equations with Constant Coefficients

Let the linear homogeneous differential equation

$$y'' + py' + qy = 0 \tag{211-1}$$

have constant coefficients p and q.

We shall look for the particular solution of equation (211-1) in the form

$$y = e^{kx}, \tag{211-2}$$

where k is a constant number to be determined. From (211-2) we have

$$y' = ke^{kx} \quad \text{and} \quad y'' = k^2e^{kx}.$$

Substituting y, y' and y'' into (211-1), we get

$$k^2e^{kx} + pke^{kx} + qe^{kx} \equiv 0$$

or, reducing by the factor e^{kx}, which is not equal to zero, we find

$$k^2 + pk + q = 0. \tag{211-3}$$

Quadratic equation (211-3) from which the number k is determined is called the *characteristic equation* of the given linear equation of the second order with constant coefficients (211-1). Note that to write characteristic equation (211-3), it is sufficient to replace in differential equation (211-1) the derivatives y'', y', and the function y by the corresponding powers of the quantity k, considering the function y as the derivative of order zero.

Solving characteristic equation (211-3), we get

$$k_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}. \tag{211-4}$$

Three cases are possible here.

Case 1. If

$$\frac{p^2}{4}-q>0, \tag{211-5}$$

then, according to formula (211-4), characteristic equation (211-3) has two real and different roots k_1 and k_2. Consequently, linear equation (211-1) allows two different particular solutions

$$y_1 = e^{k_1 x} \quad \text{and} \quad y_2 = e^{k_2 x}.$$

Since $k_1 \neq k_2$, these solutions, as we saw in the preceding section (Example 1) are linearly independent. Hence, the general solution for Case 1 has the form

$$y = C_1 e^{k_1 x} + C_2 e^{k_2 x}. \tag{211-6}$$

Example 1. Let

$$y'' - 2y' - 8y = 0. \tag{211-7}$$

Solving the characteristic equation

$$k^2 - 2k - 8 = 0,$$

we find its roots

$$k_1 = 4, \quad k_2 = -2.$$

The general solution of equation (211-7) has the form

$$y = C_1 e^{4x} + C_2 e^{-2x}.$$

Case 2. If

$$\frac{p^2}{4}-q=0, \tag{211-8}$$

then, by (211-4), characteristic equation (211-3) has the only root

$$k_1 = k_2 = -\frac{p}{2}.$$

Such root is called *multiple*. Therefore, one particular solution of equation (211-1) will be

$$y_1 = e^{-\frac{p}{2}x}.$$

Any other particular solution y_2, linearly independent of y_1 must necessarily have the form

$$y_2 = y_1 \cdot z\,(x), \tag{211-9}$$

where $z = z(x)$ is a certain function of x which is not identically constant. Hence,

$$y_2 = e^{-\frac{p}{2}x} z.$$

Differentiating, we find

$$y_2' = e^{-\frac{p}{2}x} z' + e^{-\frac{p}{2}x} \cdot \left(-\frac{p}{2}\right) z = e^{-\frac{p}{2}x}\left(z' - \frac{p}{2} z\right)$$

and

$$y_2'' = e^{-\frac{p}{2}x}\left(z'' - \frac{p}{2} z'\right) - \frac{p}{2} e^{-\frac{p}{2}x}\left(z' - \frac{p}{2} z\right) =$$

$$= e^{-\frac{p}{2}x}\left(z'' - pz' + \frac{p^2}{4} z\right).$$

Substituting y_2, y_2', and y_2'' into equation (211-1), on reducing by a common factor $e^{-\frac{px}{2}}$, we get

$$z'' - pz' + \frac{p^2}{4} z + pz' - \frac{p^2}{2} z + qz = 0,$$

or

$$z'' + \left(q - \frac{p^2}{4}\right) z = 0.$$

Finally by (211-8) we shall have

$$z'' = 0.$$

Hence

$$z' = a \quad \text{and} \quad z = ax + b,$$

where a and b are arbitrary constants.

Consequently,

$$y_2 = (ax + b)\, e^{-\frac{px}{2}}. \tag{211-10}$$

Since we are interested only in a particular solution, we may set $a = 1$ and $b = 0$. Then

$$y_2 = xe^{-\frac{px}{2}}.$$

Thus, in Case 2 the general solution of equation (211-1) will be

$$y = e^{-\frac{p}{2}x}(C_1 + C_2 x). \tag{211-11}$$

Note that, generally speaking, formula (211-10) already yielded this general solution, since any solution of equation (211-1) can be represented in the form (211-10).

Example 2. Let

$$y'' - 6y' + 9y = 0.$$

Solving the characteristic equation

$$k^2 - 6k + 9 = 0,$$

we find the multiple root

$$k_{1,2} = 3 \pm \sqrt{9-9} = 3.$$

Hence, the general solution will be written in the form

$$y = e^{3x}(C_1 + C_2 x).$$

Case 3. If

$$\frac{p^2}{4} - q < 0,$$

then, by (211-4), characteristic equation (211-3) has two conjugate complex roots

$$k_1 = \alpha + \beta i \quad \text{and} \quad k_2 = \alpha - \beta i,$$

where $\alpha = -\frac{p}{2}$ and $\beta = \sqrt{q - \frac{p^2}{4}}$.

Then the particular solutions y_1 and y_2 of equation (211-1) will be:

$$y_1 = e^{(\alpha+\beta i)x} \quad \text{and} \quad y_2 = e^{(\alpha-\beta i)x}*. \tag{211-12}$$

Hence, the general solution of equation (211-1) can be formally written in the following way:

$$y = \overline{C}_1 e^{(\alpha+\beta i)x} + \overline{C}_2 e^{(\alpha-\beta i)x}$$

or

$$y = e^{\alpha x}(\overline{C}_1 e^{i\beta x} + \overline{C}_2 e^{-i\beta x}), \tag{211-13}$$

where $\overline{C}_1$ and $\overline{C}_2$ are certain complex constants chosen so that expression (211-13) is real.

In (211-13) we can get rid of imaginary quantities. According to Euler's formulas (see Sec. 196), we have

$$e^{i\beta x} = \cos \beta x + i \sin \beta x \quad \text{and} \quad e^{-i\beta x} = \cos \beta x - i \sin \beta x.$$

---

* The *derivative of the complex function*

$$f_1(x) + if_2(x)$$

of a real variable x, where $f_1(x)$ and $f_2(x)$ are real functions of x and i is an imaginary unit, is understood, by definition, to be the expression

$$[f_1(x) + if_2(x)]' = f_1'(x) + if_2'(x).$$

Using formula (196-4) from Chapter 21, it is easy to check that if $k = \alpha + i\beta$, then $(e^{kx})' = ke^{kx}$. Consequently, the functions y_1 and y_2, as also any of their linear combinations, satisfy our differential equation.

Hence

$$y = e^{\alpha x}\,[(\overline{C}_1 + \overline{C}_2)\cos\beta x + i\,(\overline{C}_1 - \overline{C}_2)\sin\beta x].$$

Setting

$$\overline{C}_1 + \overline{C}_2 = C_1 \quad \text{and} \quad i\,(\overline{C}_1 - \overline{C}_2) = C_2,$$

we finally get

$$y = e^{\alpha x}\,(C_1\cos\beta x + C_2\sin\beta x), \tag{211-14}$$

where C_1 and C_2 are arbitrary (in view of arbitrariness of the constants $\overline{C}_1$ and $\overline{C}_2$) constant real numbers. This is just the general solution in the real form of equation (211-1) for Case 3.

In particular, if in characteristic equation (211-3) we have $p = 0$ and $q = \beta^2$, then the roots $k_{1,2} = \pm\beta i$ will be pure imaginary ($\alpha = 0$) and for the corresponding differential equation

$$y'' + \beta^2 y = 0$$

we shall obtain its general solution in the form:

$$y = C_1\cos\beta x + C_2\sin\beta x. \tag{211-15}$$

Note. In applications, another form of formula (211-14) is sometimes used. And namely, setting

$$C_1 = A\sin\varphi \quad \text{and} \quad C_2 = A\cos\varphi, \tag{211-16}$$

where A and φ are new arbitrary ($A \geqslant 0$) constants, we shall have

$$y = Ae^{\alpha x}\sin(\beta x + \varphi). \tag{211-17}$$

From (211-16) we get

$$A = \sqrt{C_1^2 + C_2^2}, \qquad \tan\varphi = \frac{C_1}{C_2}.$$

If x is interpreted as time, then, from the physical point of view, function (211-17) describes an oscillating process which is damped for $\alpha < 0$ and increases without bound for $\alpha > 0$.

Example 3. We have the equation

$$y'' - 6y' + 13y = 0.$$

Solving the characteristic equation

$$k^2 - 6k + 13 = 0,$$

we obtain the complex roots $k_{1,2} = 3 \pm 2i$. Here $\alpha = 3$ and $\beta = 2$. Hence, on the basis of (211-14), the general solution will be written in the form

$$y = e^{3x}\,(C_1\cos 2x + C_2\sin 2x).$$

Example 4. A material point of mass m is attracted to a fixed centre O with a force proportional to the distance x of the point from

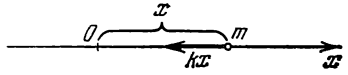


FIG. 232

the attracting centre (*elastic force*) (Fig. 232). Find the law of motion of this point (ignoring the resistance of the medium).

According to Newton's law, we have

$$m\frac{d^2x}{dt^2} = -kx,$$

where k is the proportionality factor, the minus sign meaning that the direction of the acting force is opposite to the sign of the displacement x. Hence,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

where

$$\omega = \sqrt{\frac{k}{m}}.$$

We have obtained a second-order linear equation with constant coefficients. The roots of the corresponding characteristic equation

$$k^2 + \omega^2 = 0$$

are pure imaginary:

$$k_{1,2} = \pm\omega i.$$

Therefore, by (211-14) ($\alpha = 0$, $\beta = \omega$), we have

$$x = C_1 \cos \omega t + C_2 \sin \omega t.$$

We may put

$$C_1 = A \sin \varphi, \quad C_2 = A \cos \varphi,$$

where A and φ are certain other arbitrary constants. Hence

$$x = A \sin (\omega t + \varphi),$$

i.e. under the given conditions the material point performs periodic harmonic oscillations about the attracting centre with *amplitude* A and *initial phase* φ.

Sec. 212. Second-Order Linear Nonhomogeneous Differential Equations with Constant Coefficients

Consider the second-order linear nonhomogeneous equation

$$y'' + py' + qy = f(x), \tag{212-1}$$

where p and q are given constant numbers and $f(x)$ (the right-hand member) is a known function of x. The following theorem holds true.

Theorem. *The general solution of nonhomogeneous equation* (212-1) *is equal to the sum of the general solution of the corresponding homogeneous equation*

$$\bar{y}'' + p\bar{y}' + q\bar{y} = 0 \tag{212-2}$$

and a particular solution of the given nonhomogeneous equation.

Proof. Let

$$\bar{y} = C_1 y_1 + C_2 y_2 \tag{212-3}$$

be the general solution of the equation without the right-hand member (212-2) and z a certain particular solution of the corresponding equation with a right-hand member (212-1). Obviously, we have

$$\bar{y}'' + p\bar{y}' + q\bar{y} = 0 \quad \text{and} \quad z'' + pz' + qz = f(x).$$

Adding these equations termwise and taking into consideration that the derivative of a sum is equal to the sum of derivatives, we get

$$(\bar{y} + z)'' + p(\bar{y} + z)' + q(\bar{y} + z) = f(x).$$

Hence, it is clear that the function

$$y = \bar{y} + z \tag{212-4}$$

is the general solution of equation (212-1), since it contains, by (212-3), two arbitrary independent constants C_1 and C_2.

Since we already know how to find the general solution $\bar{y}$, we shall concentrate on the method for finding the particular solution z of the corresponding nonhomogeneous equation (212-1), where p and q are constants.

In considering this problem we shall confine ourselves to simplest right-hand members $f(x)$. In these cases, for finding the particular solution of equation (212-1) the so-called *method of undetermined coefficients* is usually used.

Case 1. The right-hand member of equation (212-1) is an *exponential function*

$$f(x) = ae^{mx} \qquad (a \neq 0).$$

We look for the particular solution z also in the form of an exponential function

$$z = Ae^{mx}, \tag{212-5}$$

where A is an undetermined coefficient. Hence,

$$z' = Ame^{mx} \quad \text{and} \quad z'' = Am^2e^{mx}.$$

Substituting $f(x)$ and the expressions for z and its derivatives into equation (212-1), on reducing by e^{mx}, we shall have

$$A(m^2 + pm + q) = a. \tag{212-6}$$

Two cases are possible: (1) m is not a root of the characteristic equation, i.e.

$$m^2 + pm + q \neq 0.$$

Then

$$A = \frac{a}{m^2+pm+q}$$

and, consequently,

$$z = \frac{ae^{mx}}{m^2+pm+q}.$$

(2) The number m is the root of the characteristic equation i.e.

$$m^2 + pm + q = 0. \tag{212-7}$$

Then equation (212-6) is contradictory and, hence, differential equation (212-1) has no particular solution in the form (212-5).

In this case: (a) if m is a simple root of the characteristic equation (i.e. the other root of this equation is different from m), then the particular solution of equation (212-1) should be taken in the form

$$z = Axe^{mx},$$

and (b) if m is a multiple root of the characteristic equation, then the particular solution of equation (212-1) should be taken in the form

$$z = Ax^2e^{mx}.$$

This recommendation can be checked directly.

Example 1. Let

$$y'' - 5y' + 6y = e^x.$$

Let us first solve the equation without the right-hand member:

$$\bar{y}'' - 5\bar{y}' + 6\bar{y} = 0.$$

The characteristic equation here has the form

$$k^2 - 5k + 6 = 0.$$

Hence, its roots will be

$$k_1 = 3, \quad k_2 = 2.$$

Consequently, the general solution of the equation without the right-hand member is:

$$\bar{y} = C_1 e^{3x} + C_2 e^{2x}.$$

Since $m = 1$ is not the root of the characteristic equation, we look for the particular solution of the equation with the right-hand member in the following form:

$$z = Ae^x,$$

where A is an undetermined coefficient. Differentiating, we shall have

$$z' = Ae^x, \quad z'' = Ae^x.$$

Substituting these expressions in our nonhomogeneous equation we get

$$Ae^x - 5Ae^x + 6Ae^x = e^x \quad \text{or} \quad 2A = 1.$$

Hence,

$$A = \frac{1}{2}.$$

Thus, the particular solution of the equation with the right-hand member is

$$z = \frac{1}{2} e^x.$$

And the general solution of this equation, by the proved theorem has the form

$$y = C_1 e^{3x} + C_2 e^{2x} + \frac{1}{2} e^x.$$

Case 2. The right-hand member of nonhomogeneous equation (212-1) is the *trigonometric polynomial*

$$f(x) = M \cos \omega x + N \sin \omega x. \tag{212-8}$$

We look now for the particular solution z of this equation also in the form of the trigonometric polynomial

$$z = A \cos \omega x + B \sin \omega x,$$

where A and B are undetermined coefficients.

Differentiating, we get

$$z' = -A\omega \sin \omega x + B\omega \cos \omega x \quad \text{and} \quad z'' = -A\omega^2 \cos \omega x - B\omega^2 \sin \omega x.$$

Hence, substituting these expressions into equation (212-1) and collecting the terms containing $\cos \omega x$ and $\sin \omega x$, we shall have

$$(-A\omega^2 + Bp\omega + Aq) \cos \omega x + (-B\omega^2 - Ap\omega + Bq) \sin \omega x \equiv$$
$$\equiv M \cos \omega x + N \sin \omega x.$$

Since the last equality is an identity, the coefficients of $\cos \omega x$ and $\sin \omega x$ in the left- and right-hand members of this equality must be equal to each other, respectively, and we shall obtain

$$A(q - \omega^2) + Bp\omega = M, \quad -Ap\omega + B(q - \omega^2) = N. \qquad (212\text{-}9)$$

From this system, generally speaking, we shall be able to determine the coefficients A and B. The only case when system (212-9) is incompatible is

$$p = 0, \quad q = \omega^2$$

(i.e. when $\pm i\omega$ are the roots of the characteristic equation). Then the particular solution z should be taken in the form:

$$z = x(A \cos \omega x + B \sin \omega x).$$

Example 2. Let

$$y'' - 4y' + 4y = \cos x. \qquad (212\text{-}10)$$

The corresponding homogeneous equation will be

$$\bar{y}'' - 4\bar{y}' + 4\bar{y} = 0. \qquad (212\text{-}11)$$

Solving the characteristic equation

$$k^2 - 4k + 4 = 0,$$

we find the multiple root

$$k_{1,2} = 2 \pm \sqrt{4-4} = 2.$$

Hence, the general solution of homogeneous equation (212-11) is

$$\bar{y} = e^{2x}(C_1 + C_2 x).$$

We are going to look for the particular solution of equation (212-10) in such a form

$$z = A \cos x + B \sin x,$$

where A and B are undetermined coefficients.

Differentiating, we get

$$z' = -A \sin x + B \cos x, \quad z'' = -A \cos x - B \sin x.$$

Substituting z, z', and z'' into equation (212-10), we shall have

$$-A\cos x - B\sin x + 4A\sin x - 4B\cos x + \\ + 4A\cos x + 4B\sin x = \cos x.$$

Equating the coefficients of $\cos x$ and $\sin x$ on the right and on the left, we get the system

$$3A - 4B = 1, \quad 4A + 3B = 0.$$

Solving these equations simultaneously, we find

$$A = \frac{3}{25} \quad \text{and} \quad B = -\frac{4}{25}$$

and, consequently

$$z = \frac{3}{25}\cos x - \frac{4}{25}\sin x.$$

Hence, the general solution of equation (212-10) will have the form

$$y = e^{2x}(C_1 + C_2 x) + \frac{3}{25}\cos x - \frac{4}{25}\sin x.$$

Example 3. Study the oscillations of a material point of mass m being acted by an elastic force whose magnitude is proportional

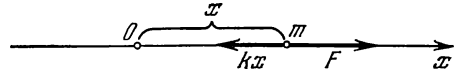


FIG. 233

to the deviation x of the point from the equilibrium position in the presence of a periodic disturbing force numerically equal to

$$F = F_0 \sin pt$$

(F_0, p constants). The resistance of medium is to be ignored.

According to Newton's law, the differential equation of motion of a point has the form (Fig. 233)

$$m\frac{d^2x}{dt^2} = -kx + F_0 \sin pt$$

(k proportionality factor) or

$$\frac{d^2x}{dt^2} + \omega^2 x = a\sin pt, \tag{212-12}$$

where

$$\omega = \sqrt{\frac{k}{m}} \quad \text{and} \quad a = \frac{F_0}{m}.$$

The general solution of the homogeneous equation

$$\frac{d^2\bar{x}}{dt^2}+\omega^2\bar{x}=0,$$

as is known (see Sec. 209, Example 5), has the form (*free oscillations of a point*)

$$\bar{x} = C_1 \cos \omega t + C_2 \sin \omega t, \tag{212-13}$$

where C_1 and C_2 are arbitrary constants.

In finding the particular solution z of nonhomogeneous equation (212-12) the following two cases should be distinguished.

Case 1. Let $p \neq \omega$, i.e. the frequency of the external force does not coincide with the frequency of free oscillations (212-13).

We put

$$z = A \cos pt + B \sin pt, \tag{212-14}$$

where A and B are undetermined coefficients.

Substituting (212-14) into (212-12), we shall have

$$-Ap^2 \cos pt - Bp^2 \sin pt + \omega^2 (A \cos pt + B \sin pt) = a \sin pt$$

or

$$A (\omega^2 - p^2) \cos pt + B (\omega^2 - p^2) \sin pt = a \sin pt.$$

Hence

$$A (\omega^2 - p^2) = 0, \quad B (\omega^2 - p^2) = a$$

and, consequently,

$$A=0, \quad B=\frac{a}{\omega^2-p^2}.$$

Thus,

$$z=\frac{a}{\omega^2-p^2}\sin pt.$$

The general solution of non-homogeneous equation (212-12) (*forced oscillations of a point*) is given by the formula

$$x=C_1\cos\omega t+C_2\sin\omega t+\frac{a}{\omega^2-p^2}\sin pt \tag{212-15}$$

and represents the superposition of two oscillations with frequencies ω and p, the oscillations being *bounded*.

Case 2. Let $p = \omega$, i.e. the frequency of the external force coincides with the frequency of free oscillations (212-13).

In this case formula (212-15), obviously, loses its sense. We put

$$z = t (A \cos \omega t + B \sin \omega t);$$

hence

$$z' = (A \cos \omega t + B \sin \omega t) + t(-A\omega \sin \omega t + B\omega \cos \omega t)$$

and

$$z'' = 2(-A\omega \sin \omega t + B\omega \cos \omega t) + t(-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t).$$

Substituting these expressions into equation (212-12), we shall have

$$2(-A\omega \sin \omega t + B\omega \cos \omega t) - \omega^2 t(A \cos \omega t + B \sin \omega t) + \omega^2 t(A \cos \omega t + B \sin \omega t) = a \sin \omega t$$

or

$$2(-A\omega \sin \omega t + B\omega \cos \omega t) = a \sin \omega t.$$

Hence, for determining the coefficients A and B we have the system

$$-2A\omega = a, \quad 2B\omega = 0.$$

Consequently,

$$A = -\frac{a}{2\omega}, \quad B = 0$$

and, hence,

$$z = -\frac{at}{2\omega} \cos \omega t$$

(Fig. 234). The forced oscillations of the point here will be

$$x = C_1 \cos \omega t + C_2 \sin \omega t - \frac{at}{2\omega} \cos \omega t. \qquad (212\text{-}16)$$

Formula (212-16) shows that the amplitude of oscillations x increases infinitely with time t. Thus, in Case 2 even insignificantly small external force causes unbounded oscillations of the system.

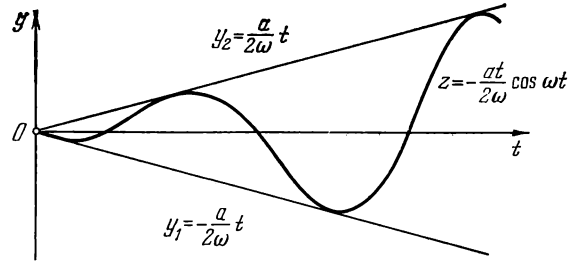


FIG. 234

This phenomenon is called *resonance*. The physical effect of resonance is disturbance of operation and even destruction of an elastic system. For instance, we know accidents when rhythmic knocks of a train moving through a railway bridge caused the destruction of the bridge.

Case 3. The right-hand member of linear equation (212-1) represents a polynomial, for instance, of second degree

$$f(x) = ax^2 + bx + c \qquad (a \neq 0). \tag{212-17}$$

We look for the particular solution z of this equation also in the form of the second-degree polynomial

$$z = Ax^2 + Bx + C,$$

where A, B, and C are undetermined coefficients.

Differentiating, we shall have

$$z' = 2Ax + B \quad \text{and} \quad z'' = 2A.$$

Hence, substituting z', z', and z'' into equation (212-1), we get

$$2A + p(2Ax + B) + q(Ax^2 + Bx + C) \equiv ax^2 + bx + c$$

or

$$Aqx^2 + (2Ap + Bq)x + (2A + Bp + Cq) \equiv ax^2 + bx + c.$$

Since two polynomials are identically equal to each other if and only if the coefficients of equal powers of the variable x are respectively equal, we have for determining the coefficients A, B, and C the following system

$$Aq = a, \quad 2Ap + Bq = b, \quad 2A + Bp + Cq = c. \tag{212-18}$$

If $q \neq 0$, this system yields definite numerical values for the coefficients A, B, and C. Thus, the particular solution z will be quite determined.

But if $q = 0$ (the characteristic equation has a simple zero root), then system (212-18) is incompatible. In this case, assuming that $p \neq 0$, the particular solution should be looked for in the form

$$z = x(Ax^2 + Bx + C).$$

If $f(x)$ is a polynomial of some other degree, we proceed in a similar way.

Example 4. Let

$$y'' - 4y' + 13y = 2x + 1.$$

Here the equation without the right-hand member will be

$$\bar{y}'' - 4\bar{y}' + 13\bar{y} = 0.$$

The characteristic equation has the form

$$k^2 - 4k + 13 = 0.$$

Hence we determine its roots

$$k_{1,2} = 2 \pm 3i.$$

The general solution of the homogeneous equation will be written as follows:

$$\bar{y} = e^{2x} (C_1 \cos 3x + C_2 \sin 3x).$$

We look for a particular solution of the nonhomogeneous equation in such a form:

$$z = Ax + B.$$

Hence

$$z' = A \quad \text{and} \quad z'' = 0.$$

Substituting these expressions into the nonhomogeneous equation, we get

$$-4A + 13 (Ax + B) \equiv 2x + 1.$$

Equating the coefficients of equal powers of x in the left- and right-hand members of this identity, we shall have

$$13A = 2; \quad -4A + 13B = 1.$$

Solving this system of simultaneous equations, we obtain

$$A = \frac{2}{13}, \quad B = \frac{21}{169}.$$

Consequently, the particular solution of the nonhomogeneous equation is

$$z = \frac{2}{13} x + \frac{21}{169}.$$

Therefore its general solution has the form

$$y = e^{2x} (C_1 \cos 3x + C_2 \sin 3x) + \frac{2}{13} x + \frac{21}{169}.$$

The arbitrary constants entering into the general solution can be determined from the initial conditions.

Example 5. Find the solution $y = y(x)$ of the equation

$$y'' = x^2 + y \tag{212-19}$$

such that

$$y(0) = -2, \quad y'(0) = 1. \tag{212-20}$$

Let us write equation (212-19) in the standard form

$$y'' - y = x^2. \tag{212-21}$$

Here is the homogeneous equation:

$$\bar{y}'' - \bar{y} = 0.$$

The characteristic equation

$$k^2 - 1 = 0$$

has the roots $k_1 = 1$ and $k_2 = -1$. Consequently, the general solution of the homogeneous equation is

$$\bar{y} = C_1 e^x + C_2 e^{-x},$$

where C_1 and C_2 are constants.

To find the particular solution z of nonhomogeneous equation (212-21), we set

$$z = Ax^2 + Bx + C.$$

Substituting this function into equation (212-21), we shall have

$$2A - (Ax^2 + Bx + C) \equiv x^2.$$

Hence,

$$\left.\begin{aligned} -A &= 1, \\ -B &= 0, \\ 2A - C &= 0; \end{aligned}\right\}$$

consequently,

$$A = -1, \quad B = 0, \quad C = -2 \quad \text{and} \quad z = -x^2 - 2.$$

The general solution of nonhomogeneous equation (212-19) has the form

$$y = \bar{y} + z,$$

or

$$y = C_1 e^x + C_2 e^{-x} - (x^2 + 2). \tag{212-22}$$

Differentiating, we find

$$y' = C_1 e^x - C_2 e^{-x} - 2x. \tag{212-23}$$

Setting $x = 0$ in (212-22) and (212-23) and using the initial conditions (212-20), we get the following system for determining the constants C_1 and C_2

$$\left.\begin{aligned} -2 &= C_1 + C_2 - 2, \\ 1 &= C_1 - C_2 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} C_1 + C_2 &= 0 \\ C_1 - C_2 &= 1. \end{aligned}\right\}$$

Hence,

$$C_1 = \frac{1}{2}, \quad C_2 = -\frac{1}{2}.$$

Substituting these values in (212-22), we get the required solution

$$y = \frac{1}{2}(e^x - e^{-x}) - (x^2 + 2), \quad \text{i.e.} \quad y = \sinh x - (x^2 + 2).$$

Sec. 213. Differential Equations Containing Partial Derivatives

Let the function u describe a certain physical process. Any process takes place in space, whose points can be characterized by rectangular Cartesian coordinates (x, y, z) and in time t. Therefore, in the general case the function u is a function of four variables: $u = u(x, y, z, t)$*. Differentiating the function u, we obtain partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, etc. In a given process these derivatives are connected by known relations, and thus we come to differential equations containing partial derivatives**.

Of special interest for physical applications are differential equations containing derivatives of the second order (so-called *differential equations of the second order*). They comprise the equations of gas dynamics, hydrodynamical equations, mathematical equations of electromagnetism (the Maxwell equations) and many others. Therefore, differential equations with partial derivatives of the second order are called the *equations of mathematical physics.*

Let us set forth the most important types of such equations for the case of two independent variables.

I. *One-dimensional wave equation*

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

This equation is applied when studying a number of oscillation (vibration) processes (transverse oscillation of an elastic string, longitudinal vibration of a rod, vibration of gas in a tube, and so on).

II. *Heat conduction equation* (the Fourier equation)

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2},$$

describing transient heat conditions of a rod.

* In certain cases we may confine ourselves to considering a plane or a straight line; the number of the independent variables of the function u is then reduced accordingly.

** Soviet mathematicians have made a valuable contribution to the theory of differential equations with partial derivatives. The works by Academicians M. V. Keldysh, M. A. Lavrentiev, I. G. Petrovsky, S. L. Sobolev, A. N. Tikhonov and others are of special importance.

The problem on propagation of electric oscillations in the mains is also linked with this equation.

III. *Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,$$

yielding a stationary distribution of temperature in a homogeneous plate, etc.

For solving these equations under various conditions special methods were developed (the so-called "methods of mathematical physics").

For the sake of simplicity, let us confine ourselves to the case of two independent variables x and y:

$$u = u\,(x,\ y)$$

and introduce brief notation: $\frac{\partial u}{\partial x}=u_x$, $\frac{\partial u}{\partial y}=u_y$, $\frac{\partial^2 u}{\partial x\,\partial y}=u_{xy}$ and so on. Then the general form of a second-order differential equation for the unknown function u is as follows:

$$F\,(x,\ y,\ u,\ u_x,\ u_y,\ u_{xx},\ u_{xy},\ u_{yy})=0, \tag{213-1}$$

where F is a known function.

Any function $u=\varphi\,(x,\ y)$ which turns equation (213-1) into an identity is called its *solution*, the graph of the solution is termed an *integral surface.*

Example 1. Find $u = u\,(x,\ y)$ if

$$\frac{\partial^2 u}{\partial y^2}=0. \tag{213-2}$$

This equation can be written in the following form:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=0. \tag{213-3}$$

Hence, it follows that $\frac{\partial u}{\partial y}$ is independent of y, i.e. it is a function only of the variable x. Thus, from (213-3) it follows that

$$\frac{\partial u}{\partial y}=C_1\,(x), \tag{213-4}$$

where $C_1\,(x)$ is an arbitrary function.

Integrating equation (213-4) with respect to the variable y, we get

$$u=\int C_1\,(x)\,dy^*, \tag{213-5}$$

or

$$u = C_1\,(x)\,y + C_2\,(x), \tag{213-6}$$

* In integral (213-5) the variable x is assumed to be constant, each fixed x being assigned its own arbitrary constant C_2. Therefore, $C_2 = C_2\,(x)$.

where $C_1(x)$ and $C_2(x)$ are arbitrary functions. With the aid of differentiation it is easy to get convinced that the general-form solution (213-6) containing the arbitrary functions $C_1(x)$ and $C_2(x)$ yields here the set of *all* solutions of differential equation (213-2). Thus, the solution of differential equation (213-2) is an arbitrary function, linear with respect to the variable y.

Note that general solutions of ordinary differential equations contain arbitrary constants; for differential equations with partial derivatives their general-form solutions contain arbitrary functions.

Rendering the functions $C_1(x)$ and $C_2(x)$ concrete in formula (213-6), we obtain particular solutions of equation (213-2). For instance, setting $C_1(x) = \sin x$, $C_2(x) = \cos x$, we shall have a particular solution $u = y \sin x + \cos x$, and so on.

Differential equations with partial derivatives, as a rule, have an infinite number of solutions (see, for instance, Example 1). But the solution of a physical problem described by a differential equation must be *unique* by its meaning, otherwise it provides no possibility to predict the corresponding physical phenomenon and, consequently, is of low value for practical purposes. Therefore, in solving physical problems, besides a differential equation we have to use additional conditions which enable us to single out of an infinite set of solutions its unique solution yielding the law governing the physical process under consideration. In the simplest case we are given the so-called *initial* and *boundary conditions*. Roughly speaking, the former characterize the given process at the initial (starting) moment of time, and the latter describe the behaviour of the process on the boundary of the domain (region) in question.

If in equation (213-1) the variable y is interpreted as time, then the simplest initial conditions for an unknown function u have the form

$$u(x, y_0) = f(x), \quad u_y(x, y_0) = f_1(x), \tag{213-7}$$

where $f(x)$ and $f_1(x)$ are given functions. Finding the function u satisfying differential equation (213-1) and initial conditions (213-7) is called *Cauchy's problem.*

Example 2. Find the solution $u = u(x, y)$ of the equation $u_{yy} = 0$ satisfying the initial conditions

$$u(x, 1) = x^2, \quad u_y(x, 1) = x.$$

By (213-6), we shall have

$$u = C_1(x)\, y + C_2(x), \quad u_y = C_1(x). \tag{213-8}$$

Putting $y = 1$ in (213-8), we get

$$x^2 = C_1(x) + C_2(x), \quad x = C_1(x); \tag{213-9}$$

hence

$$C_1(x) = 0, \quad C_2(x) = x^2 - x$$

and, consequently,

$$u = xy + (x^2 - x). \tag{213-10}$$

The solution u is unique.

A physical problem described by a differential equation with partial derivatives and boundary conditions is said to be *correctly formulated* if: (1) this problem has a solution; (2) the solution is unique; (3) the solution of the problem depends continuously on the boundary conditions.

Indeed, prior to solving a problem we have to get convinced whether it is solvable at all. Science history knows numerous examples when scientists spent much effort and time trying to find the solution of a problem having no solution. For instance, during about two thousand years many mathematicians tried to solve the problem on squaring the circle, i.e. with the aid of compasses and a rule to construct a square equivalent to a given circle. Only at the close of the nineteenth century it was proved that this is impossible. Analogously, in chemistry "philosophers' stone" which allegedly converts base metals into precious ones has been never found, all efforts in search of it turned out to be in vain. The solvability of a problem under consideration is guaranteed by the "existence theorem".

As far as the second requirement is concerned, as it was stated above, ambiguous solutions of a problem are of little use for practical purposes. The uniqueness of a solution is ensured by the "uniqueness theorem".

Finally, the violation of the third condition leads to undesirable consequences. From the point of view of practice, it is no good if negligible changes of the initial or boundary conditions (in reality they are known only approximately) cause a considerable alteration in the solution of a problem in a given region! Therefore, the "solution smoothness theorem" is needed here.

Recently, many mathematicians have shown interest for incorrectly formulated problems. In this field basic results were obtained by A. N. Tikhonov.

Sec. 214. Linear Differential Equations with Partial Derivatives

Definition. *A differential equation is called linear (more precisely, completely linear) if it is an integral polynomial of the first degree with respect to an unknown function and its derivatives and, in particular, does not contain their products.*

Thus, the general form of a second-order differential equation is as follows:

$$A(x, y)\, u_{xx} + B(x, y)\, u_{xy} + C(x, y)\, u_{yy} + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u = f(x, y), \quad (214\text{-}1)$$

where $A(x, y)$, $B(x, y)$, $C(x, y)$, $a(x, y)$, $b(x, y)$, $c(x, y)$ are known coefficients and $f(x, y)$ is a given constant term. If $f(x, y) \equiv 0$, then linear equation (214-1) is called *homogeneous* (without the constant term); otherwise equation (214-1) is said to be *nonhomogeneous*.

On introducing brief notation

$$L[u] = A(x, y)\, u_{xx} + B(x, y)\, u_{xy} + C(x, y)\, u_{yy} + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u$$

(here L is the so-called *linear differential operator*) equation (214-1) can be written in a compact form

$$L[u] = f(x, y). \quad (214\text{-}1')$$

The linear homogeneous differential equation

$$L[u] = 0 \quad (214\text{-}2)$$

possesses the following important property: *any linear combination with constant coefficients of the solutions of a linear homogeneous differential equation is also a solution of this equation* (cf. Sec. 210). In particular, *the sum of any number of solutions of a linear homogeneous differential equation is also a solution of this equation* (*the principle of superposition of solutions*).

Without presenting the proof in the general form, we shall confine ourselves to an example explaining its idea. Let there be given the homogeneous equation

$$L[u] \equiv u_{xx} - xu_{yy} = 0 \quad (214\text{-}3)$$

with u_1 and u_2 as its solutions, i.e.

$$L[u_1] = 0, \quad L[u_2] = 0.$$

Consider, for instance, the function

$$\hat{u} = 2u_1 - 3u_2. \quad (214\text{-}4)$$

From (214-3) we have

$$L[\hat{u}] = \frac{\partial^2}{\partial x^2}(2u_1 - 3u_2) - x\frac{\partial^2}{\partial y^2}(2u_1 - 3u_2) =$$

$$= 2\left(\frac{\partial^2 u_1}{\partial x^2} - x\frac{\partial^2 u_1}{\partial y^2}\right) - 3\left(\frac{\partial^2 u_2}{\partial x^2} - x\frac{\partial^2 u_2}{\partial y^2}\right) = 2L[u_1] - 3L[u_2] = 0.$$

Thus, $\hat{u}$ is the solution of equation (214-3).

Sec. 215. Deriving the Heat Conduction Equation

Consider a homogeneous rod* of a constant cross-sectional area S and length l which is heat insulated from its sides. We take its axis for the x-axis (Fig. 235). Let us denote by $u = u(x, t)$ $(0 \leqslant x \leqslant l,\ 0 \leqslant t < +\infty)$ the temperature of the rod in the section with the abscissa x at the moment of time t**.

Let $\rho = \text{const}$ be density of the rod, $c = \text{const}$ its specific heat, $k = \text{const}$ the thermal conductivity coefficient, $\Phi(x, t)$ intensity of the heat source found in the section x at the moment of time t referred to a mass unit and unit time*** (for instance, the apparatus operating in a spaceship may be regarded as a heat source). According to Fourier's law, the amount of heat flowing in the direction of the x-axis during an infinitely small interval of time dt through the section S with the abscissa x is equal to

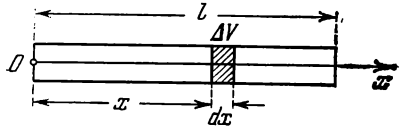


FIG. 235

$$dQ_x = -k \frac{\partial u}{\partial x} S\, dt, \tag{215-1}$$

where k is the thermal conductivity coefficient $\left(\frac{\partial u}{\partial x}\right.$ represents here the magnitude of temperature (u) gradient$\left.\right)$. The right-hand member of (215-1) is preceded by the minus sign, since for $\frac{\partial u}{\partial x} > 0$, i.e. with an increase of temperature u together with x, the heat flow is directed in the opposite side, and vice versa.

Let us set up heat balance for an element of the rod ΔV contained between two infinitely close sections I and II with the abscissas x and $x + dx$, respectively. Let us assume for definiteness that the rod temperature u increases in the direction of the x-axis. Then heat flows out through section I (—) and comes in through section II (+). Let dQ be the amount of heat accumulated by the element ΔV during the time interval dt.

Then, taking into consideration that the amount of heat generated by the heat sources concentrated in the element ΔV during the time dt is equal to

$$\Phi(x, t) \cdot \rho S\, dx \cdot dt \tag{215-2}$$

* Generally, the *rod* in mechanics is understood to be a body with one prevailing linear dimension. For instance, a rocket may be regarded as a rod with a variable section.

** It is assumed that the rod has one and the same temperature at all points of any cross section.

*** That is the amount of heat generated by this heat source per mass unit during unit time.

and applying formula (215-1), we shall have

$$dQ = -k\frac{\partial u}{\partial x}\Big|_x \cdot S\,dt + k\frac{\partial u}{\partial x}\Big|_{x+dx} \cdot S\,dt + \rho S\Phi(x, t)\,dx\,dt. \qquad (215\text{-}3)$$

Applying the formula (see Sec. 99, Chapter 12)

$$f(x + dx) \approx f(x) + f'(x)\,dx \qquad (215\text{-}4)$$

with an accuracy to infinitesimals of the highest order of smallness, we get

$$\frac{\partial u}{\partial x}\Big|_{x+dx} = \frac{\partial u}{\partial x}\Big|_x + \frac{\partial^2 u}{\partial x^2}\Big|_x dx. \qquad (215\text{-}5)$$

Therefore, formula (215-3) takes the form

$$dQ = k\frac{\partial^2 u}{\partial x^2} S\,dx\,dt + \rho S\Phi(x, t)\,dx\,dt. \qquad (215\text{-}6)$$

On the other hand, $\frac{\partial u}{\partial t}$ is the rate of change in temperature of the element ΔV and, therefore $\Delta u = \frac{\partial u}{\partial t}\,dt$ represents the change of its temperature. Since the mass of the element ΔV is equal to $\rho S\,dx$, the amount of heat accumulated is

$$dQ = c\rho S\,dx\frac{\partial u}{\partial t}\,dt \qquad (215\text{-}7)$$

Equating expressions (215-7) and (215-6) to each other, on reducing by the common factor $S\,dx\,dt$, we shall obtain

$$c\rho\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \rho\Phi(x, t) \qquad (215\text{-}8)$$

or, introducing the conventional notation

$$\frac{k}{c\rho} = a^2, \qquad (215\text{-}9)$$

we shall finally have

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{c}\Phi(x, t). \qquad (215\text{-}10)$$

Differential equation (215-10) describing the distribution of temperature u in a rod is called the *heat conduction equation* (or the *Fourier equation*).

If there are no heat sources, then (215-10) takes the form

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}. \qquad (215\text{-}11)$$

A similar equation is also true for body temperature.

The heat conduction equation finds its application in physics, chemistry, astronomy, building practice, and other fields of science and engineering.

Sec. 216. The Problem on Temperature Distribution in a Limited Rod

According to the preceding section, the temperature $u = u(x, t)$ of a homogeneous rod (Fig. 236) in the section x at the moment of time t in the absence of heat sources satisfies the heat conduction equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < l). \tag{216-1}$$

We shall assume that we are given the initial condition

$$u(x, 0) = f(x) \qquad (0 \leqslant x \leqslant l). \tag{216-2}$$

Let us also assume that the end-points of the rod $x = 0$ and $x = l$ have a constant temperature equal to the ambient temperature

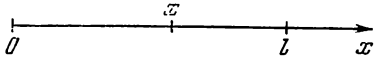


FIG. 236

which we shall conventionally consider to be equal to zero. Thus, we have the simplest boundary conditions

$$u(0, t) = 0, \qquad u(l, t) = 0 \tag{216-3}$$

for any $t \geqslant 0$.

Under the given conditions it is required to find the distribution of the temperature $u = u(x, t)$ in the rod for the subsequent moments of time $t \geqslant 0$.

For equation (216-1) we first shall look for the non-zero solutions of the *special form*:

$$u = X(x)\, T(t), \tag{216-4}$$

where $X(x)$ is a function of the variable x only, and $T(t)$ is a function of only the variable t. Since

$$\frac{\partial u}{\partial t} = XT', \quad \frac{\partial^2 u}{\partial x^2} = X''T,$$

substituting these expressions into (216-1), we get

$$XT' \equiv a^2 X''T.$$

Hence, separating the variables, we shall have

$$\frac{X''}{X} \equiv \frac{T'}{a^2 T}. \tag{216-5}$$

The left-hand member of (216-5) depends only on x, and the right-hand member only on t. Since x and t are independent variables, this is possible only when both members of identity (216-5) are

equal to a certain constant quantity. Denoting this constant by $-\lambda^2$ (for the convenience of further computation*), we get

$$\frac{X''}{X} = -\lambda^2, \quad \frac{T'}{a^2T} = -\lambda^2. \tag{216-6}$$

Hence we shall have two equations

$$X'' + \lambda^2 X = 0, \quad T' + a^2\lambda^2 T = 0. \tag{216-7}$$

The first of equations (216-7) is a linear homogeneous equation with constant coefficients; the roots of its characteristic equation $k^2 + \lambda^2 = 0$ are

$$k_{1,2} = \pm\lambda i.$$

According to the known formulas (see Sec. 214), its general solution has the form

$$X(x) = A \sin \lambda x + B \cos \lambda x, \tag{216-8}$$

where A and B are arbitrary constants.

The second equation of (216-3) is easily solved by the method of separation of the variables; and namely, we find

$$T(t) = Ce^{-a^2\lambda^2 t}, \tag{216-9}$$

where C is an arbitrary constant.

Multiplying (216-8) by (216-9) we shall have

$$u = e^{-a^2\lambda^2 t}(A \sin \lambda x + B \cos \lambda x), \tag{216-10}$$

and we put here $C = 1$ which is equivalent to the replacement of AC by A and BC by B.

Function (216-10) satisfies the heat conduction equation at any choice of the constants A, B and λ. Let us require that it also satisfy boundary conditions (216-3). Setting $x = 0$, we get

$$0 = e^{-a^2\lambda^2 t}B;$$

hence, $B = 0$ and, consequently,

$$u = Ae^{-a^2\lambda^2 t} \sin \lambda x. \tag{216-10'}$$

Putting now $x = l$, by condition (216-3), we shall have

$$0 = Ae^{-a^2\lambda^2 t} \sin \lambda l. \tag{216-11}$$

But $A \neq 0$, since, otherwise, we would have a zero solution $u \equiv 0$. Therefore

$$\sin \lambda l = 0 \tag{216-12}$$

---

* We can get directly convinced that this constant taken with the plus sign will not yield necessary solutions.

and

$$\lambda l = n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{216-13}$$

Hence

$$\lambda_n = \frac{n\pi}{l} \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{216-14}$$

The numbers λ_n are called the *characteristic numbers* of the problem and their set the *spectrum of the problem.* To each of the characteristic numbers λ_n there corresponds a particular solution of the heat conduction equation:

$$u_n = A_n e^{-b^2 n^2 t} \sin \frac{n\pi x}{l}, \tag{216-15}$$

where, for the sake of brevity, we put

$$b = \frac{a\pi}{l}.$$

Note that it is sufficient to take only positive integers for n ($n = 1, 2, \ldots$), since for $n = 0$ we have $u \equiv 0$ which is contradictory, and for $n < 0$ we obtain solutions of the same nature as for the corresponding $n' = -n > 0$.

Thus, formula (216-15) yields a complete set of linearly independent particular solutions of the form (216-4) of the heat conduction equation (216-1) which satisfy boundary conditions (216-3). Physically, the functions u_n represent *temperature waves* whose graphs are sinusoids damping as $t \to \infty$ (Fig. 237, *a*, *b*).

Lastly, we have to ensure initial condition (216-2). Since equation (216-1) is linear and homogeneous, we may apply the principle of superposition of solutions (see Sec. 214). Hence, we shall have

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-b^2 n^2 t} \sin \frac{n\pi x}{l} \tag{216-16}$$

and if series (216-16) converges, then under certain conditions function (216-16) is the solution of equation (216-1). Setting $t = 0$ in (216-16), by virtue of the initial condition (216-2), we shall have

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}. \tag{216-17}$$

This series represents an expansion of the function $f(x)$ on the closed interval $[0, l]$ into Fourier's series in sines of multiple arcs. For the expansion coefficients the following formulas hold true (see

Sec. 199)

$$A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx \qquad (216\text{-}18)$$

($n = 1, 2, 3, \ldots$).

Thus, the solution of the problem under consideration is given by series (216-16) whose coefficients are determined by formula (216-18). For common engineering practice it is sufficient to take several terms of this series.

It should be noted that the obtained solution is of formal character, since the series was not investigated for convergence. But we can show that if the function $f(x)$ is sufficiently smooth on the

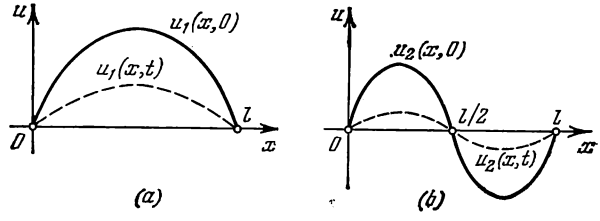


FIG. 237

interval $[0, l]$, then series (216-16) converges and its sum $u(x, t)$ satisfies both differential equation (216-1) and initial condition (216-2), as also boundary conditions (216-3), i.e. $u(x, t)$ is the solution of our problem in the usual sense.

The method applied for solving this problem is usually called the *Fourier method* (or the method of separating the variables).

Exercises

1. Show that the function

$$y = Ce^{-x^2},$$

where C is an arbitrary constant, is the solution of the equation

$$y' + 2xy = 0.$$

2. Show that the function

$$y = e^x (C_1 \cos x + C_2 \sin x),$$

where C_1 and C_2 are arbitrary constants, is the solution of the equation

$$y'' - 2y' + 2y = 0.$$

3. Find the integral curve of the equation

$$xy' = 2y,$$

passing through the point M_0 (2, 3).

4. Integrate the given equations with variables separable:

(a) $x\,dx + y\,dy = 0$;

(b) $y\,dx + x\,dy = 0$;

(c) $dx - x\,dy = 0$;

(d) $y' = 2 + y$;

(e) $y' = e^{x+y}$.

5. Solve the following differential equations:

(a) $(x^2 + y^2)\,dx - xy\,dy = 0$;

(b) $y' = \frac{y}{x} \ln \frac{y}{x}$.

6. Find the integral curve of the equation

$$xy' = y + \sqrt{x^2 + y^2},$$

passing through the point (1, 0).

7. Solve the linear differential equations:

(a) $xy' = x^3 + y$;

(b) $(x + y^2)\,y' = 1$.

8. Find the solution of the equation

$$y' \cos x + y \sin x = 1$$

satisfying the initial condition $y = 0$ for $x = 0$.

9. Find the curve at each point of which the tangent is perpendicular to the polar radius of the point of tangency.

10. Find the curve passing through the point A (2, 1) and having a constant subtangent (i.e. the projection on the x-axis of the segment of the tangent from the point of tangency to the point of intersection with the x-axis) equal to 4.

11. The rate of decay of radium at each moment of time is proportional to its amount available. Find the law of decay of radium if the initial amount of radium is equal to Q_0 and it is known that in 1600 years (half-life) only half of this amount will remain.

12. A chemical reaction turning substance A into substance B runs its course so that at each moment of time the rate of reduction of the amount of substance A is proportional to the product of the available amounts of substances A and B.

At the initial moment the retort contained 800 grams of substance A and 200 grams of substance B; in two hours there remained 400 grams of substance A. How much substance A will be in the retort in four hours?

13. Using Euler's method, find $y(2)$ if $y' = x - y$ and $y(1) = 0.370$ ($h = 0.2$).

In Problems 14 to 16 integrate the given second-order equations:

14. $y'' = \sin x$.

15. $y'' = -y$.

16. $2yy'' = 1 + y'^2$.

17. Find the integral curve of the equation

$$y'' = x,$$

passing through the point $M_0(0, 1)$ and touching at it the straight line

$$y = \frac{x}{2} + 1.$$

18. With the aid of power series, integrate the equations:

(a) $y' = y + \frac{x^2}{4}$, $y(0) = 1$;

(b) $y'' = xy$, $y(0) = 0$, $y'(0) = 1$.

In Problems 19 to 27 integrate the given linear equations with constant coefficients:

19. $y'' + y' - 2y = 0$. **20.** $y'' + 2y' + 2y = 0$.

21. $y'' + y' + y = 0$. **22.** $y'' = y' - 0.25y$.

23. $y'' + y = 0$, if $y = 2$ and $y' = -1$ for $x = 0$.

24. $y'' - y = e^{2x}$. **25.** $y'' + 4y = \sin x$. **26.** $y'' - 5y' + 6y = x^2$.

27. $y'' - 2y' + 2y = 2x$, if $y = 0$ and $y' = 0$ for $x = 0$.

28. A point of mass m is acted upon by an attracting elastic force proportional to the distance of this point from the equilibrium position and by the force of resistance of medium proportional to the velocity of the point. Find the law of motion of the point, assuming that the resistance of medium is small as compared with the elastic force.

Chapter 23

Line Integrals

Sec. 217. The Line Integral of the First Kind

Let K be a certain smooth (or piecewise smooth*) plane curve

$$x = x(t), \quad y = y(t), \qquad (t \in [\alpha, \beta]),$$

where t is a parameter and

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'^2(t) + y'^2(t)}\,|\,dt\,|$$

its **arc** differential. Here, if $\alpha \leqslant \beta$, then $dt > 0$ and $ds = +\sqrt{x'^2 + y'^2}\,dt$; and if $\alpha \geqslant \beta$, then $dt < 0$ and $ds = -\sqrt{x'^2 + y'^2}\,dt$. If $f(x, y)$ is a function continuous on the curve K, then its *line integral of the first kind* taken over the curve K is understood to be the integral

$$\int_K f(x, y)\,ds = \int_\alpha^\beta f(x(t), y(t))\sqrt{x'^2(t) + y'^2(t)}\,|\,dt\,|. \tag{217-1}$$

If the curve K is specified by the equation

$$y = y(x) \qquad (a \leqslant x \leqslant b),$$

then, regarding x as a parameter, we shall obtain

$$\int_K f(x, y)\,ds = \int_a^b f(x, y(x))\sqrt{1 + y'^2(x)}\,dx.$$

Suppose K is a material curve, i.e. it has some mass. Let Δs be a certain arc of the curve K containing a point M, and Δm the mass of this arc. Then the ratio $\Delta m/\Delta s$ is called the *mean density* of the arc Δs, and

$$\mu(M) = \lim_{\Delta s \to M} \frac{\Delta m}{\Delta s},$$

* That is the derivatives of its coordinates, perhaps, allow a finite number of points of discontinuity of the first kind.

i.e. the limit of the mean density of the arc, provided the arc Δs contracts into the point M, is termed the *line density* of the arc at the point M.

If $\mu = f(x, y)$ is considered as the line density of the arc at its current point $M(x, y)$, then

$dm = \mu\, ds$

is the mass of an infinitely small arc ds (*elementary mass*) and the integral

$$m = \int_K \mu\, ds \tag{217-2}$$

represents the mass of the line (physical meaning of the line integral of the first kind).

The line integral of the first kind possesses the following obvious properties.

(1) With a change in the direction of integration the line integral of the first type does not change its value (Fig. 238), i.e.

$$\int_{K^+} = \int_{K^-},$$

where K^+ is the curve K integrated in a given direction (for instance,

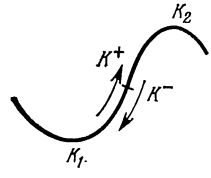


FIG. 238

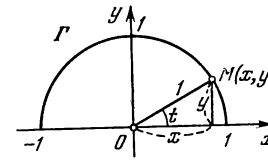


FIG. 239

in the direction of increase in parameter t), and K^- is the curve K integrated in the opposite direction (in which t decreases).

(2) If the curve of integration K is divided into two parts by a certain point: $K = K_1 \cup K_2$ (Fig. 238), then

$$\int_{K_1 \cup K_2} = \int_{K_1} + \int_{K_2}$$

Example. Find the mass of the semicircle $x^2 + y^2 = 1$, $y \geqslant 0$ (Γ) (Fig. 239) if its line density at a current point $M(x, y)$ is proportional to the ordinate y.

Taking the polar angle (Fig. 239) as the parameter t, we get the parametric equations of the semicircle

$$x = \cos t, \quad y = \sin t \qquad (0 \leqslant t \leqslant \pi). \tag{217-3}$$

The elementary mass

$$dm = \mu\, ds = ky \sqrt{x'^2 + y'^2}\, dt, \tag{217-4}$$

where k is the proportionality factor

Since

$$x' = -\sin t, \quad y' = \cos t$$

and

$$ds = \sqrt{x'^2 + y'^2}\, dt = dt,$$

from (217-4) we have

$$dm = k \sin t\, dt.$$

Hence the mass of the line Γ will be equal to

$$m = k \int_0^{\pi} \sin t\, dx = k\,(-\cos t)\,|_0^{\pi} = 2k.$$

The line integral of the first kind of the function $f(x, y, z)$ taken over a piecewise smooth space curve K is determined in a similar manner:

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad (t \in [\alpha, \beta]):$$

$$\int_K f(x, y, z)\, ds = \int_{\alpha}^{\beta} f(x(t), y(t), z(t)) \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\, |\, dt\, |,$$

where

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\, |\, dt\, |$$

is the differential of the space curve K.

Sec. 218. The Line Integral of the Second Kind

Let

$$x = x(t), \quad y = y(t) \qquad (t \in [\alpha, \beta])$$

be a smooth (or piecewise smooth) curve K with a chosen direction (for the sake of brevity, this line will be called the *path*) and $X(x, y)$, $Y(x, y)$ a pair of functions continuous on the curve K. Taking into account that the differentials of the current coordinates x and y of the curve K have the form

$$dx = x'(t)\, dt, \quad dy = y'(t)\, dt \tag{218-1}$$

the *line integral of the second kind* of the pair of functions X and Y taken over the curve K is understood to be the integral

$$\int_K X(x, y)\,dx + Y(x, y)\,dy =$$

$$= \int_\alpha^\beta [X(x(t), y(t))\,x'(t) + Y(x(t), y(t))\,y'(t)]\,dt \quad (218\text{-}2)$$

(traditionally, for the expression on the left the brackets are not written and it is assumed that the integral sign $\int_K$ refers to the entire sum).

If the path K is specified by the equation

$$y = y(x) \qquad (x \in [a, b]),$$

then formula (218-2) takes the form

$$\int_K X(x, y)\,dx + Y(x, y)\,dy =$$

$$= \int_a^b [X(x, y(x)) + Y(x, y(x))\,y'(x)]\,dx. \quad (218\text{-}3)$$

Analogously, if K is given by the equation

$$x = x(y) \qquad (y \in [A, B]),$$

then

$$\int_K X(x, y)\,dx + Y(x, y)\,dy =$$

$$= \int_A^B [X(x(y), y)\,x'(y) + Y(x(y), y)]\,dy. \quad (218\text{-}4)$$

The line integral of the second kind possesses the following properties.

(1) When the path of integration is traversed in the opposite direction, the line integral of the second kind reverses its sign, i.e.

$$\int_{K^-} = -\int_{K^+}. \quad (218\text{-}5)$$

Indeed, a change of the direction in which the path of integration is traversed is equivalent to a permutation of the limits of integration α and β in definite integral (218-2); and this causes a reversal of sign of the definite integral (see Sec. 117, Chapter 14).

(2) If the path of integration K consists of two parts $K = K_1 \cup K_2$, then

$$\int\limits_{K_1 \cup K_2} = \int\limits_{K_1} + \int\limits_{K_2}. \tag{218-6}$$

Example 1. Find the values of the integral

$$I = \int\limits_K y\,dx - x\,dy$$

over the indicated paths: (1) OA (a straight line); (2) OmA (a parabola with the vertex O and axis Oy); (3) OBA (a polygonal line), and (4) OCA (a polygonal line) (Fig. 240).

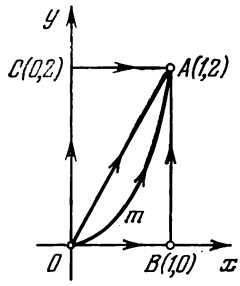


FIG. 240

Solution. (1) The equation of the straight line OA is

$$y = 2x \quad (0 \leqslant x \leqslant 1).$$

Hence, $dy = 2dx$ and, consequently,

$$I_1 = \int\limits_0^1 (2x\,dx - x \cdot 2\,dx) = 0.$$

(2) The equation of the parabola OmA has the form

$$y = kx^2.$$

Since the parabola passes through the point A (1, 2), we have $2 = k \cdot 1^2$, and, hence, $k = 2$, i.e. $y = 2x^2$.

Hence $dy = 4x\,dx$ and

$$I_2 = \int\limits_0^1 (2x^2\,dx - x \cdot 4x\,dx) = -\int\limits_0^1 2x^2\,dx = -\frac{2}{3}\,x^3\Big|_0^1 = -\frac{2}{3}.$$

(3) By property 2, we have

$$I_3 = \int\limits_{OB} (y\,dx - x\,dy) + \int\limits_{BA} (y\,dx - x\,dy). \tag{218-7}$$

Since the equation of OB is

$$y = 0 \qquad (0 \leqslant x \leqslant 1),$$

we have $dy = 0$.

Then, the equation of BA is written as follows

$$x = 1 \qquad (0 \leqslant y \leqslant 2);$$

therefore $dx = 0$. From (218-7) we get

$$I_3 = \int_0^1 (0 - x \cdot 0)\, dx + \int_0^2 (y \cdot 0 - 1)\, dy = -2.$$

(4) Analogously

$$I_4 = \int_{OC} (y\, dx - x\, dy) + \int_{CA} (y\, dx - x\, dy) =$$

$$= \int_0^2 (y \cdot 0 - 0)\, dy + \int_0^1 (2 - x \cdot 0)\, dx = 2.$$

Note that here, with the end-points of the path of integration K fixed, the integral I is path-dependent.

Example 2. Find

$$I = \int_K y\, dx + x\, dy$$

over the lines K indicated in Example 1.

Solution. Taking advantage of the above written equations of the line K, we have in succession

$$I_1 = \int_{OA} (y\, dx + x\, dy) = \int_0^1 (2x\, dx + x \cdot 2\, dx) = 4 \int_0^1 x\, dx = 2x^2 \Big|_0^1 = 2,$$

$$I_2 = \int_{OmA} (y\, dx + x\, dy) = \int_0^1 (2x^2\, dx + x \cdot 4x\, dx) = 6 \int_0^1 x^2\, dx = 2x^3 \Big|_0^1 = 2.$$

$$I_3 = \int_{OB} (y\, dx + x\, dy) + \int_{BA} (y\, dx + x\, dy) =$$

$$= \int_0^1 (0 + x \cdot 0)\, dx + \int_0^2 (y \cdot 0 + 1)\, dy = 2,$$

$$I_4 = \int_{OC} (y\, dx + x\, dy) + \int_{CA} (y\, dx + x\, dy) =$$

$$= \int_0^2 (y \cdot 0 + 0)\, dy + \int_0^1 (2 + x \cdot 0)\, dx = 2.$$

Thus, here the integral I has one and the same value for different paths, joining the points O and A. The fundamental difference between Examples 1 and 2 will be explained in Sec. 220.

If

$$x = x(t), \quad y = y(t), \quad z = z(t) \qquad (t \in [\alpha, \beta])$$

is a piecewise smooth space curve K and $X(x, y, z)$, $Y(x, y, z)$, $Z(x, y, z)$ are the three functions continuous on the curve K, then the corresponding line integral of the second type is understood to be the integral

$$\int_K X(x, y, z)\,dx + Y(x, y, z)\,dy + Z(x, y, z)\,dz =$$

$$= \int_\alpha^\beta [X(x(t), y(t), z(t))\,x'(t) + Y(x(t), y(t), z(t))\,y'(t) +$$

$$+ Z(x(t), y(t), z(t))\,z'(t)]\,dt.$$

Sec. 219. The Physical Meaning of the Line Integral of the Second Kind

Let $\boldsymbol{F} = \{X(x, y), Y(x, y)\}$ be a continuously changing variable force and

$$x = x(t), \quad y = y(t) \qquad (t \in [\alpha, \beta])$$

the path K traversed by the point of its application (Fig. 241). Let us denote by $\boldsymbol{ds} = \overrightarrow{MM'}$ the infinitely small vector of displacement from the current point $M(x, y)$ of the curve K to an infinitely

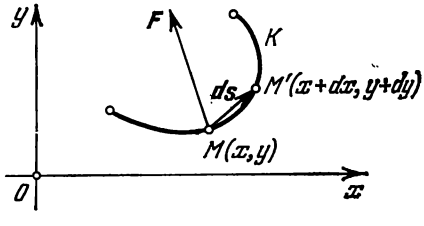


FIG. 241

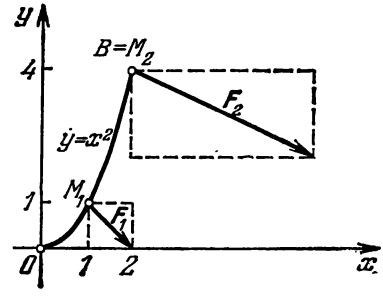


FIG. 242

close point $M'\{x + dx, y + dy\}$ (the infinitesimals of higher order than $\boldsymbol{ds}$ are neglected here). We have $\boldsymbol{ds} = \{dx, dy\}$. Since on the infinitesimal path $\boldsymbol{ds}$ a continuous force $\boldsymbol{F}$ may be regarded as constant, the elementary work performed by this force (see

Secs. 156 and 157) is equal to

$$dA = \boldsymbol{F}\,d\boldsymbol{s} = X\,dx + Y\,dy. \tag{219-1}$$

Integrating expression (219-1) over the curve K, we shall obtain the work performed by the force

$$A = \int_K X\,dx + Y\,dy. \tag{219-2}$$

Expression (219-2) is, obviously, the corresponding line integral of the second kind.

Thus, *the line integral of the second kind represents the work performed by a variable force over the path of integration whose projections on the coordinate axes are the corresponding coefficients of the differentials of the variables.*

Example. Find the work A performed by a variable force $\boldsymbol{F} = \{y, -x\}$, the point of application of which describes the parabola OB (Fig. 242):

$$y = x^2 \qquad (0 \leqslant x \leqslant 2). \tag{219-3}$$

By (219-2), we have

$$A = \int_{OB} X\,dx + Y\,dy = \int_{OB} y\,dx - x\,dy.$$

From equation (219-3) we get

$$dy = 2x\,dx,$$

therefore

$$A = \int_0^2 x^2\,dx - x \cdot 2x\,dx = -\int_0^2 x^2\,dx = -\frac{x^3}{3}\Big|_0^2 = -2\frac{2}{3}\text{units}.$$

Analogously, the work performed by the space force

$$\boldsymbol{F} = \{X(x, y, z), Y(x, y, z), Z(x, y, z)\}$$

along the path K:

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

is expressed by the line integral of the second kind

$$A = \int_K X\,dx + Y\,dy + Z\,dz.$$

Sec. 220. Condition Under Which the Line Integral of the Second Kind Is Independent of Path

Let $X = X(x, y)$ and $Y = Y(x, y)$ be continuous functions in a domain G (Fig. 243). Consider two arbitrary points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ of the domain and all possible paths $M_1\alpha M_2$, $M_1\beta M_2$, $M_1\gamma M_2$, . . ., connecting these points (M_1 the beginning of the path, M_2 the end of the path) and lying wholly in G. It may happen that

$$\int_{M_1\alpha M_2} X\,dx + Y\,dy = \int_{M_1\beta M_2} X\,dx + Y\,dy = \int_{M_1\gamma M_2} X\,dx + Y\,dy = \ldots \qquad (220\text{-}1)$$

Then, the line integral of the second kind

$$I = \int_{\overset{\frown}{M_1M_2}} X\,dx + Y\,dy \qquad (220\text{-}2)$$

is said to be independent of the choice of the path of integration in a given domain G.

If conditions (220-1) are fulfilled, then there is no need to indicate the path of integration for integral (220-2). In this case it is sufficient to mark its initial point $M_1(x_1, y_1)$ and its terminal point $M_2(x_2, y_2)$ of the path. Therefore, the following notation is used here

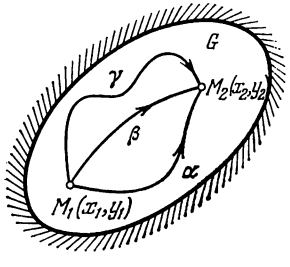


FIG. 243

$$I = \int_{(x_1, y_1)}^{(x_2, y_2)} X\,dx + Y\,dy. \qquad (220\text{-}3)$$

The following theorem holds true:

Theorem. *If in a domain G the integrand $X\,dx + Y\,dy$ is the total differential* of a certain function $U = U(x, y)$,* i.e.

$$dU = X\,dx + Y\,dy \quad \text{for} \quad (x, y) \in G, \qquad (220\text{-}4)$$

then line integral (220-2) *does not depend on the path of integration in the domain G.*

Proof. et

$$x = \varphi(t), \quad y = \psi(t) \qquad (t \in [t_1, t_2]) \qquad (220\text{-}5)$$

* For the condition (test) of the total differential see Sec. 177 (Chapter 20).

be an arbitrary path K in the domain G connecting the points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$, and

$$\left.\begin{aligned} \varphi(t_1) &= x_1, \quad \psi(t_1) = y_1; \\ \varphi(t_2) &= x_2, \quad \psi(t_2) = y_2. \end{aligned}\right\} \tag{220-6}$$

From (220-4) we have

$$X\,dx + Y\,dy = dU[\varphi(t), \psi(t)]. \tag{220-7}$$

Hence we get

$$I = \int_{t_1}^{t_2} dU[\varphi(t), \psi(t)] = U[\varphi(t), \psi(t)]\Big|_{t_1}^{t_2} =$$

$$= U[\varphi(t_2), \psi(t_2)] - U[\varphi(t_1), \psi(t_1)]. \tag{220-8}$$

Further, using (220-6), we shall have

$$I = U(x_2, y_2) - U(x_1, y_1) = U(M_2) - U(M_1). \tag{220-9}$$

Thus, the value of the integral I is one and the same for any choice of the functions $\varphi(t)$, $\psi(t)$ and, consequently, the integral I does not depend on the path connecting the points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$.

Corollary 1. *If relation* (220-4) *is fulfilled, then, by* (220-9), *we have*

$$\int_{(x_1, y_1)}^{(x_2, y_2)} X\,dx + Y\,dy = U(x_2, y_2) - U(x_1, y_1) \tag{220-10}$$

(*the Newton-Leibniz generalized formula*).

Corollary 2. *If the integrand* $X\,dx + Y\,dy$ *is a total differential and the path of integration* K *is closed, then*

$$\oint_K X\,dx + Y\,dy = 0$$

(the small circle on the integral sign means integration over a closed path).

Example. Find

$$I = \int_{(1, 2)}^{(3, 4)} y\,dx + x\,dy.$$

Since

$$y\,dx + x\,dy = d(xy),$$

then, independently of the path joining the points M_1 (1, 2) and M_2 (3, 4), we have

$$I = \int_{(1, 2)}^{(3, 4)} d(xy) = xy\Big|_{(1, 2)}^{(3, 4)} = 3 \cdot 4 - 1 \cdot 2 = 10.$$

Sec. 221. The Work Performed by a Potential Force

The theorem proved in the preceding section has a physical meaning. Let in a domain G there be defined a force field

$$\boldsymbol{F} = \{X(x, y), Y(x, y)\}.$$

The force field can be exemplified by a gravitational field at Earth's surface where any material point of mass m is acted upon by the gravity force numerically equal to mg (where g is acceleration due to gravity). A more general example of force field is a gravitational field generated by mass M. Here, according to Newton's law, a material point of mass m situated at a distance r from the attracting centre is acted upon by a force numerically equal to $k\frac{mM}{r^2}$ (k gravity constant) and directed towards the attracting centre. Another example of a force field is the Coulomb electric field.

If there exists a function $U = U(x, y)$ such that

$$X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y},$$

then the *field* is said *to be potential* (in other words, $\boldsymbol{F}$ is a potential force), and the function U is called the *field potential*.

In this case, obviously,

$$X\,dx + Y\,dy = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy = dU.$$

Hence, for the work A performed by the potential force $\boldsymbol{F}$ along the path connecting the points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$, we have

$$A = \int_{(x_1, y_1)}^{(x_2, y_2)} X\,dx + Y\,dy = \int_{(x_1, y_1)}^{(x_2, y_2)} dU = U(x_2, y_2) - U(x_1, y_1),$$

i.e. *the work performed by a potential force does not depend on the path and is equal to the difference between the force potentials for the terminal and initial points of the path.*

In particular, if the path is closed, then the work $A = 0$.

Example. Find the work A done by the gravity force when displacing in the vertical plane Oxy (near Earth's surface) a point of

mass m from the position $M_1(x_1, y_1)$ to the position $M_2(x_2, y_2)$ (Fig. 244).

If the x-axis is horizontal and the y-axis is vertical, then the projections of the gravity force acting upon the material point of mass m are equal to

$$X = 0, \quad Y = -mg.$$

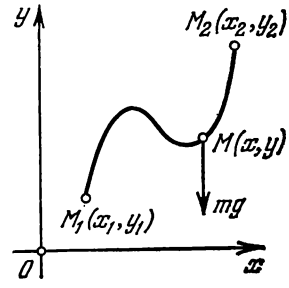


FIG. 244

We have

$$X\,dx + Y\,dy = -mg\,dy = d(-mgy).$$

Therefore, we may take

$$U = -mgy,$$

for the potential of the gravity field.

Hence, the work performed by the gravity force, independently on the path $\overset{\frown}{M_1M_2}$ will be equal to

$$A = -mgy\Big|_{(x_1, y_1)}^{(x_2, y_2)} = -mg(y_2 - y_1).$$

Note. Analogous results are valid for the line integral taken over a space curve. In particular, if

$$X\,dx + Y\,dy + Z\,dz = dU(x, y, z),$$

then

$$\int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} X\,dx + Y\,dy + Z\,dz = U(x_2, y_2, z_2) - U(x_1, y_1, z_1).$$

Exercises

1. Evaluate the given line integrals of the first kind:

(a) $\int\limits_K (x+y)\,ds$, where K is a line segment

$y = 2x - 1 \qquad (-1 \leqslant x \leqslant 2)$;

(b) $\int\limits_K x\,ds$, where K is a parabola arc

$y = \frac{x^2}{2} \qquad (0 \leqslant x \leqslant 1)$;

(c) $\int\limits_K x^2y^2\,ds$, where K is a circle

$x = R\cos t, \quad y = R\sin t \qquad (0 \leqslant t \leqslant 2\pi)$;

(d) $\int\limits_K \arctan \frac{y}{x}\, ds$, where K is an arc of the cardioid

$r = a(1 + \cos\varphi) \qquad \left(0 \leqslant \varphi \leqslant \frac{\pi}{2}\right)$.

2. Find the area of the "fence" constructed along the perimeter K of a square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, the height of which at point $(x, y) \in K$ is equal to $z = x^2 + y^2$.

3. Determine the mass of the circle

$x^2 + y^2 = R^2$,

if its density at point (x, y) is equal to $\rho = \frac{y^2}{R^2}$.

4. Determine the coordinates of the centre of gravity $C(x_0, y_0)$ of the homogeneous semicircle K:

$x^2 + y^2 = R^2, \qquad y \geqslant 0.$

Hint. It is proved in mechanics that the coordinates of the centre of gravity of a homogeneous curve K are expressed by the formulas

$$x_0 = \frac{1}{L}\int\limits_K x\, ds, \quad y_0 = \frac{1}{L}\int\limits_K y\, ds,$$

where L is the arc length of the curve.

5. Find the moment of inertia I_y of the arc of the parabola

$y = x^{\frac{3}{2}} \qquad \left(0 \leqslant x \leqslant \frac{4}{3}\right)$

relative to the y-axis.

Hint.

$$I_y = \int\limits_K x^2\, ds.$$

6. Find the moment of inertia I_x of the arc of the cycloid K:

$x = a(t - \sin t), \quad y = a(1 - \cos t) \qquad (0 \leqslant t \leqslant 2\pi)$

relative to the x-axis.

Hint.

$$I_x = \int\limits_K y^2\, ds.$$

7. Evaluate the indicated line integrals of the second kind:

(a) $\int\limits_K y^2\, dx - x^2\, dy$, where the curve K is the arc of the parabola $y = 1 - x^2$ from the point $M(-1, 0)$ to the point $N(1, 0)$;

(b) $\int\limits_K \frac{dx-dy}{\sqrt{xy}}$, where K is a segment of the straight line

$x+y=1 \qquad (0 \leqslant x \leqslant 1);$

(c) $\oint\limits_K \frac{x\,dy-y\,dx}{x^2+y^2}$, where K is the circle

$x = a\cos t, \quad y = a\sin t \qquad (0 \leqslant t \leqslant 2\pi).$

8. Evaluate

$$\oint\limits_K y\,(y\,dx - x\,dy),$$

where K is the contour of a triangle with the vertices O (0, 0), A (2, 1), and B (1, 2) traversed anticlockwise.

9. Compute the following path-independent line integrals of the second kind:

(a) $\int\limits_{(1,\,2)}^{(-3,\,4)} x\,dx - y\,dy$; (c) $\int\limits_{(1,\,2)}^{(2,\,1)} \frac{y\,dx-x\,dy}{y^2} \quad (y>0)$;

(b) $\int\limits_{(2,\,3)}^{(1,\,6)} y\,dx + x\,dy$; (d) $\int\limits_{(0,\,0)}^{(1,\,1)} e^{x+y}\,(dx+dy)$.

10. Find the work performed by the force with the projections

$X = y, \quad Y = -x$

along the ellipse

$x = a\cos t, \quad y = b\sin t \qquad (0 \leqslant t \leqslant 2\pi).$

11. Find the work performed by the force $\boldsymbol{F} = \{-kx, -ky\}$ when displacing the point of its application from M_1 $(a, 0)$ to M_2 $(0, b)$.

12. Find the work performed by the force with the projections

$X = \sin(x + y), \quad Y = 0$

over the contour of a triangle with the vertices

$O(0, 0), \quad M\left(\frac{\pi}{2}, 0\right), \quad N\left(0, \frac{\pi}{2}\right)$

when it is traversed anticlockwise.

13. Find the work performed by the force with the projections

$$X = \frac{x}{r^2}, \quad Y = \frac{y}{r^2},$$

where $r = \sqrt{x^2 + y^2}$, when displacing a point from the position $M(a, 0)$ to the position $M(0, b)$ $(a > 0, b > 0)$.

Hint. Use the formula $x\,dx + y\,dy = r\,ar$.

14. Evaluate the integral

$$\int_{(1, 2, 3)}^{(4, 5, 6)} x\,dx + z\,dy + y\,dz.$$

Chapter 24

Double and Triple Integrals

Sec. 222. Double Integrals

When studying the definite integral, we introduced the notion of the integral sum to determine the area of a curvilinear trapezoid. The definite integral was defined as the limit of the integral sum (see Sec. 121, Chapter 14). In solving the problem of determining the volume of a solid we arrive at the notion of the two-dimensional integral sum whose limit is called the *double integral.*

Problem. Find the volume of a solid bounded from above by a continuous surface $z = f(x, y)$ $(f(x, y) \geqslant 0)$, from below by a finite closed domain S of the xy-plane, and from the sides by a right cylindrical surface constructed on the boundary of the domain S and having elements perpendicular to the plane Oxy (Fig. 245).

For the sake of brevity, the above defined curvilinear cylinder is called the *cylindroid.* In a particular case, when the upper base of the cylindroid is a plane parallel to its lower base, the cylindroid is called the *cylinder.* The simplest example of the cylinder is a right circular cylinder dealt with at high school. Generalizing the reasoning usually used for finding the volume of a circular cylinder, it is not difficult to prove that the volume V of a cylinder with the area of its base S and altitude H is equal to $V = SH$.

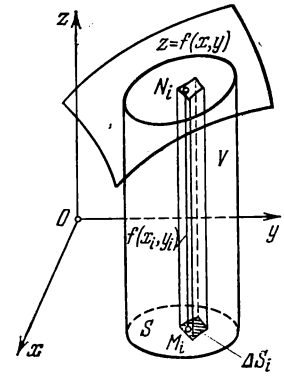


FIG. 245

To compute the volume V of the given cylindroid, we break up its base (i.e. the domain S) into a finite number of elementary cells $\Delta S_1, \Delta S_2, \ldots, \Delta S_n$ (generally speaking, into n closed curvilinear subdomains of arbitrary shape). Next, in each of these cells ΔS_i we take a point $M_i(x_i, y_i) \in \Delta S_i$ $(i = 1, 2, \ldots, n)$ and construct a right cylindrical column with the base ΔS_i and altitude $M_iN_i = f(x_i, y_i)$ equal to the z-coordinate of the surface at the chosen

point. By the formula for the volume of cylinder, the volume of such a column is, obviously, equal to

$$f(x_i, y_i)\,\Delta S_i, \tag{222-1}$$

where ΔS_i is the area of the corresponding cell*. The sum of the volumes of these cylindrical columns represents the volume of the stepped solid approximately replacing the given curvilinear solid. The approximation is, generally speaking, the more accurate, the less the diameters of the cells ΔS_i. Therefore the volume of our cylindroid will be approximately expressed by the sum

$$V \approx \sum_{i=1}^{n} f(x_i, y_i)\,\Delta S_i. \tag{222-2}$$

Formula (222-2) enables us to find the volume V with any degree of accuracy if the number of cells ΔS_i is sufficiently large and their linear dimensions are very small. Let us denote by d_i the *diameter* of the cell ΔS_i, i.e. its greatest linear dimension. Strictly speaking, the diameter d of a bounded closed (i.e. with the boundary adjoined) figure Φ (arc length, area element, etc.) is understood to be the length of its greatest chord AB, where $A \in \Phi$ and $B \in \Phi$ (Fig. 246)**. It follows from this definition that the figure Φ, having the diameter d, is wholly enclosed inside a circle of radius d described from its arbitrary point C as centre. Therefore, if $d \to 0$, the figure Φ "shrinks into a point". The diameter of a space solid is determined in a similar way.

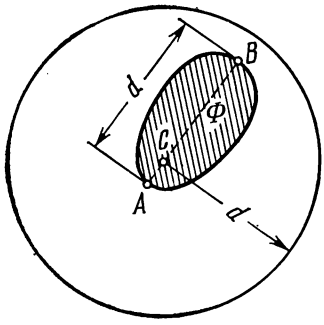


FIG. 246

Let $d = \max d_i$ be the greatest of the diameters of the cells ΔS_1, ΔS_2, . . ., ΔS_n. Assuming that in formula (222-2) the number of cells n increases without bound ($n \to \infty$), and the diameter of the greatest of them becomes arbitrarily small ($d \to 0$), in the limit we shall obtain a precise formula for the volume of the cylindroid

$$V = \lim_{d \to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta S_i^{***}. \tag{222-3}$$

The expression standing in the right-hand side of formula (222-3) is called the *double integral of the function* $f(x, y)$ *over the domain* S,

* For the sake of convenience, here we denote the cells and their areas by the same letters. The difference between them is seen from the context.

** The cells ΔS_i may be assumed to be closed.

*** Strictly speaking, by definition, the *volume* of the cylindroid is understood to be the limit (222-3), provided it exists.

and is denoted in the following way:

$$\lim_{d\to 0}\sum_{i=1}^{n} f(x_i, y_i)\,\Delta S_i = \iint_S f(x, y)\,dS. \tag{222-4}$$

Therefore, for the volume of the cylindroid we finally have

$$V = \iint_S f(x, y)\,dS. \tag{222-5}$$

Generalizing the scheme applied for computing the volume of the cylindroid, we come to the following definitions.

Definition 1. *The two-dimensional integral sum* (222-2) *of a given function* $f(x, y)$ *over a given domain* S *is the sum of the products of the areas of the elementary cells* ΔS_i *of the domain* S *by the values* $f(x_i, y_i)$ *of the function* $f(x, y)$ *at the chosen points of these cells* (Fig. 247).

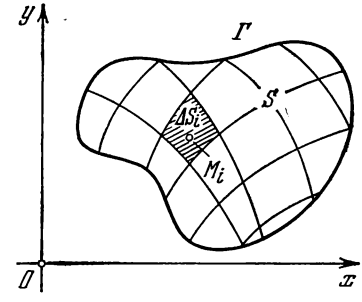


FIG. 247

Definition 2. *The double integral* (222-4) *of a function* $f(x, y)$ *over a given domain* S *is the limit of the corresponding two-dimensional integral sum* (222-2) *as the number* n *of elementary cells* ΔS_i *increases without bound and their greatest diameter* d *tends to zero, provided this limit exists and does not depend on the method of partition of the domain* S *into elementary cells* ΔS_i *and the choice of points in them.*

In (222-4), $f(x, y)$ is called the *integrand*, S the *domain* (*region*) *of integration*, and dS the *element of area.*

The following theorem holds:

Theorem. *If a domain* S *with a piecewise smooth boundary* Γ *is bounded and closed**, *and a function* $f(x, y)$ *is continuous in the domain* S, *then the double integral*

$$\iint_S f(x, y)\,dS = \lim_{d\to 0}\sum_{i=1}^{n} f(x_i, y_i)\,\Delta S_i, \tag{222-6}$$

i.e. the limit of the corresponding two-dimensional integral sum exists and is independent of the way in which the domain S *is divided into the elementary cells* ΔS_i *and of the choice of the points* M_i *in the cells.*

Henceforward we shall assume that the conditions of this theorem are fulfilled.

* That is, the boundary Γ is attached to the domain S (see Sec. 179).

In formula (222-6), there is no necessity to indicate that $n \to \infty$, since from $d \to 0$ it obviously follows that $n \to \infty$.

If $f(x, y) \geqslant 0$, then double integral (222-6) represents the volume of a right cylindroid constructed on the domain S as on the base and bounded from above by the surface $z = f(x, y)$ (*geometrical meaning of the double integral*).

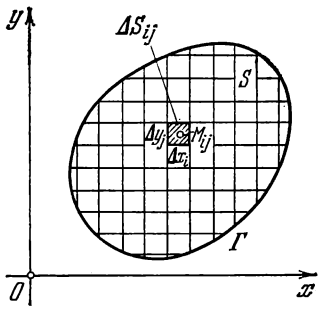


FIG. 248

Since the value of a double integral does not depend on the form of the elementary cells, we shall take advantage of this circumstance and choose the most suitable grids. Very frequent use is made of a rectangular grid formed by intersection of two systems of straight lines parallel to the x- and y-axes, respectively (Fig. 248). In this case the elementary cells ΔS_{ij}* are rectangles with the sides equal to Δx_i and Δy_j excepting, perhaps, the cells adjoining the boundary Γ. To indicate the use of a rectangular grid, in the notation of integral (222-4) we set

$$dS = dx\, dy \tag{222-7}$$

(*the two-dimensional element of area in rectangular coordinates*), and

$$\iint_S f(x, y)\, dx\, dy = \lim_{\substack{\max|\Delta x_i| \to 0 \\ \max|\Delta y_j| \to 0}} \sum_i \sum_j f(x_i, y_j)\, \Delta x_i \Delta y_j, \tag{222-8}$$

where $(x_i, y_j) \in \Delta S_{ij}$ and the above sum is extended to all values of i and j for which $\Delta S_{ij} = \Delta x_i\, \Delta y_j$ (we can show that the nonrectangular cells adjoining the piecewise smooth boundary Γ do not affect the value of the limit (222-8)).

The basic methods of evaluating double integrals will be dealt with in the subsequent sections.

Sec. 223. The Double Integral in Rectangular Cartesian Coordinates

For the sake of definiteness, let us assume that the domain of integration S is a curvilinear trapezoid (Fig. 249):

$$a \leqslant x \leqslant b, \quad y_1(x) \leqslant y \leqslant y_2(x), \tag{223-1}$$

* Here we use a system of two subscripts for each cell, thus indicating separately the number of the vertical strip (i) and the number of the horizontal strip (j) which contain a given cell, just as a cinema ticket indicates the number of row and the number of seat.

where $y_1 = y_1(x)$ and $y_2 = y_2(x)$ are single-valued continuous functions on the segment $[a, b]$. Such a domain will be called *standard* with respect to the y-axis. Note that the vertical line passing through the point x of the x-axis for $a < x < b$ intersects the boundary Γ of the domain S only at two points: $M_1(x, y_1)$ (the "point of entrance") and $M_2(x, y_2)$ (the "point of exit").

Let $f(x, y)$ be a function continuous in the domain S and

$$I = \iint_S f(x, y)\, dx\, dy \qquad (223\text{-}2)$$

its double integral.

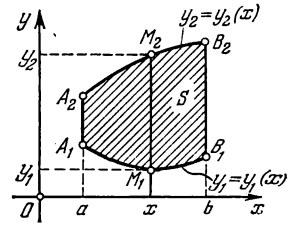


FIG. 249

(1) Let us first assume that $f(x, y) \geqslant 0$ in the domain S. Then the double integral I represents the volume of a cylindroid (Fig. 250) bounded by the domain S from below, by the surface $z = f(x, y)$ from above, and by a right cylindrical surface from the sides.

To compute the volume I, let us use the method of sections (see Sec. 129, Chapter 15). Namely, let $\sigma(x)$ be the area of the section

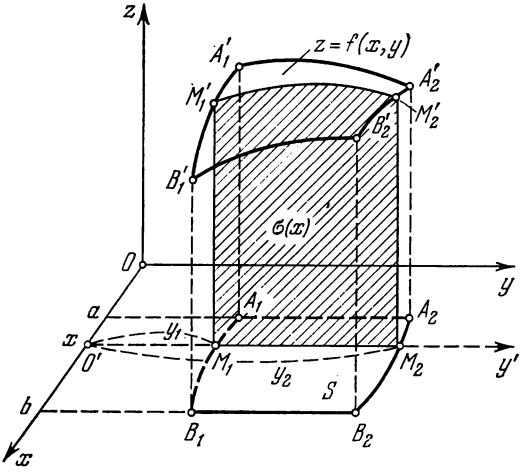


FIG. 250

of the cylindroid by the plane $M_1M_2M_2'M_1'$ perpendicular to the x-axis at its point $x \in [a, b]$ (Fig. 250).

Then we have

$$I = \int_a^b \sigma(x)\, dx. \qquad (223\text{-}3)$$

But $\sigma(x)$ represents the area of a curvilinear trapezoid bounded from below by the segment $y_1 \leqslant y \leqslant y_2$ of the axis $O'y'$ parallel to the y-axis, and from above by the curve $z = f(x, y)$, $x = \text{const}$. Therefore,

$$\sigma(x) = \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy. \tag{223-4}$$

We can prove that under our conditions $\sigma(x)$ is continuous for $x \in [a, b]$.

Substituting expression (223-4) into formula (223-3), we finally obtain

$$\iint_S f(x, y)\, dx\, dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy. \tag{223-5}$$

Thus, the double integral is equal to the corresponding *iterated integral* (223-5), i.e. evaluation of the double integral is reduced to two quadratures. Note that when evaluating the inner integral in (223-5), x is considered to be a constant.

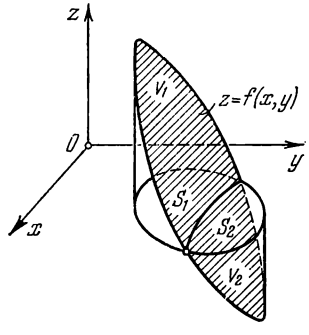


FIG. 251

(2) In the case of an alternating function $z = f(x, y)$, for instance, if $f(x, y) \geqslant 0$ for $(x, y) \in S_1$ and $f(x, y) < 0$ for $(x, y) \in S_2$ ($S_1 \cup S_2 = S$), double integral (223-2) is equal to an *algebraic sum* of the volumes V_1 and V_2 of the cylindroids constructed on the bases S_1 and S_2, respectively (Fig. 251), i.e.

$$\iint_S f(x, y)\, dx\, dy = V_1 - V_2. \tag{223-6}$$

We can prove that formula (223-5) is valid in this case as well.

Let us consider an important particular case: let S be a rectangle $a \leqslant x \leqslant b$, $A \leqslant y \leqslant B$ (Fig. 252) and $f(x, y) = X(x)\, Y(y)$, where $X(x)$ is a function continuous on $[a, b]$ and dependent only on x, and $Y(y)$ a function continuous on $[A, B]$ and dependent only on y. By (223-5), we have

$$\iint_S X(x)\, Y(y)\, dx\, dy = \int_a^b dx \int_A^B X(x)\, Y(y)\, dy =$$

$$= \int_a^b X(x)\, dx \int_A^B Y(y)\, dy. \tag{223-7}$$

But the inner integral in formula (223-7) is a constant, therefore it may be taken outside the sign of the outer integral, and we get

$$\iint\limits_S X(x)\,Y(y)\,dx\,dy = \int\limits_a^b X(x)\,dx \cdot \int\limits_A^B Y(y)\,dy, \tag{223-8}$$

i.e. double integral (223-8) is equal to the product of two single integrals.

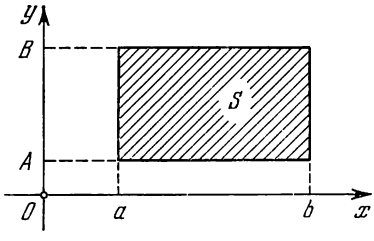


FIG. 252

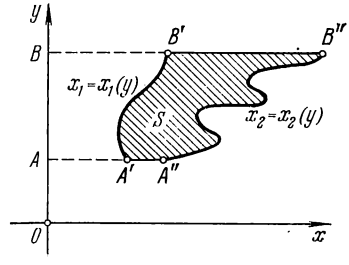


FIG. 253

Note 1. If the domain S is standard with respect to the x-axis, i.e. (Fig. 253)

$$A \leqslant y \leqslant B, \quad x_1(y) \leqslant x \leqslant x_2(y),$$

then, by analogy with formula (223-5) we obtain

$$\iint\limits_S f(x, y)\,dx\,dy = \int\limits_A^B dy \int\limits_{x_1(y)}^{x_2(y)} f(x, y)\,dx. \tag{223-9}$$

In particular, if the domain S is a rectangle: $a \leqslant x \leqslant b$, $A \leqslant y \leqslant B$, then we have

$$\iint\limits_S f(x, y)\,dx\,dy = \int\limits_a^b dx \int\limits_A^B f(x, y)\,dy$$

and

$$\iint\limits_S f(x, y)\,dx\,dy = \int\limits_A^B dy \int\limits_a^b f(x, y)\,dx.$$

Hence we get

$$\int\limits_a^b dx \int\limits_A^B f(x, y)\,dy = \int\limits_A^B dy \int\limits_a^b f(x, y)\,dx,$$

i.e. *if the limits of integration in the iterated integral of a continuous function are finite and constant, then the result of integration does not depend on the order of integration.*

Note 2. If the domain S is non-standard, then it is split (if it is possible) into a finite number of domains S_1, S_2, . . ., S_p standard with respect to the x- or y-axis, and on the basis of the property of the limits, we assume

$$\iint_S = \iint_{S_1} + \iint_{S_2} + \ldots + \iint_{S_p},$$

and then apply formula (223-5) or (223-9), respectively.

Example 1. Find

$$I = \iint_S (x^2 + y^2)\, dx\, dy,$$

where S is a square: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

Setting the limits of integration, we shall have

$$I = \int_0^1 dx \int_0^1 (x^2 + y^2)\, dy = \int_0^1 \left(x^2 y + \frac{y^3}{3} \right) \Big|_{y=0}^{y=1} dx =$$

$$= \int_0^1 \left(x^2 + \frac{1}{3} \right) dx = \left(\frac{x^3}{3} + \frac{1}{3} x \right) \Big|_{x=0}^{x=1} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Geometrically, I represents the volume of a cylindroid with

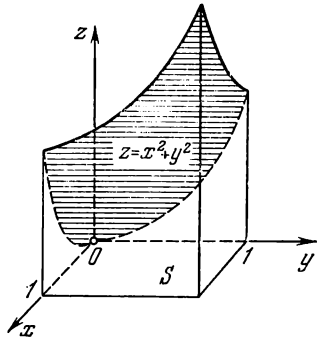


FIG. 254

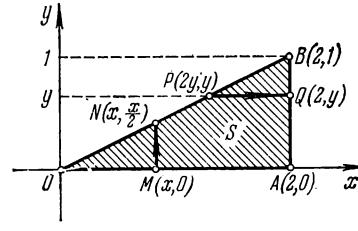


FIG 255

a square as the lower base, and bounded from above by the paraboloid of revolution $z = x^2 + y^2$ (Fig. 254).

Example 2. Evaluate the double integral

$$I = \iint\limits_S xy^2\,dx\,dy,$$

where S is a rectangle: $0 \leqslant x \leqslant 1$, $-2 \leqslant y \leqslant 3$.

Setting the limits of integration and separating the variables, we have

$$I = \int\limits_0^1 x\,dx \cdot \int\limits_{-2}^3 y^2\,dy = \frac{x^2}{2}\Big|_0^1 \cdot \frac{y^3}{3}\Big|_{-2}^3 = \frac{1}{2} \cdot \frac{27+8}{3} = \frac{35}{6} = 5\,\frac{5}{6}.$$

Example 3. Evaluate

$$I = \iint\limits_S x^2y\,dx\,dy, \tag{223-10}$$

where S is a triangle with the vertices O (0, 0), A (2, 0), and B (2, 1) (Fig. 255).

The domain S is bounded by the straight lines

$$y = 0,\quad y = \frac{x}{2},\quad x = 2$$

and is standard with respect to both y- and x-axes.

For the vertical line MN the "point of entrance" into the domain S is M $(x, 0)$, the "point of exit" being $N\left(x, \frac{x}{2}\right)$ $(0 < x < 2)$. Thus, for a fixed x the variable y, for the points of the domain S, changes from 0 to $\frac{x}{2}$. Therefore, integrating in double integral (223-10) first with respect to y for x = const, and then with respect to x, by (223-5), we shall have

$$I = \int\limits_0^2 x^2\,dx \int\limits_0^{\frac{x}{2}} y\,dy = \int\limits_0^2 x^2\,dx \cdot \frac{y^2}{2}\Big|_{y=0}^{y=\frac{x}{2}} =$$

$$= \frac{1}{8}\int\limits_0^2 x^4\,dx = \frac{1}{8} \cdot \frac{x^5}{5}\Big|_{x=0}^{x=2} = \frac{4}{5}. \tag{223-11}$$

Analogously, for the horizontal line PQ the "point of entrance" into the domain is P $(2y, y)$ and the "point of exit" is Q $(2, y)$ $(0 < y < 1)$. Consequently, with a fixed y the variable x for the points of the domain S changes from $2y$ to 2. Performing integration in integral (223-10) first with respect to x for y = const, and then

with respect to y, by (223-9), we get

$$I = \int_0^1 y\,dy \cdot \int_{2y}^{2} x^2\,dx = \int_0^1 y\,dy \cdot \frac{x^3}{3}\Big|_{x=2y}^{x=2} =$$

$$= \frac{8}{3}\int_0^1 (y - y^4)\,dy = \frac{8}{3}\left(\frac{y^2}{2} - \frac{y^5}{5}\right)\Big|_{y=0}^{y=1} = \frac{8}{3}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{4}{5}. \quad (223\text{-}12)$$

As it was to be expected, we have obtained the same result, but

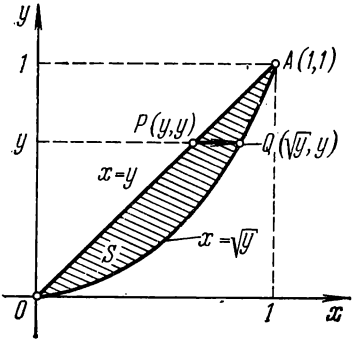


FIG. 256

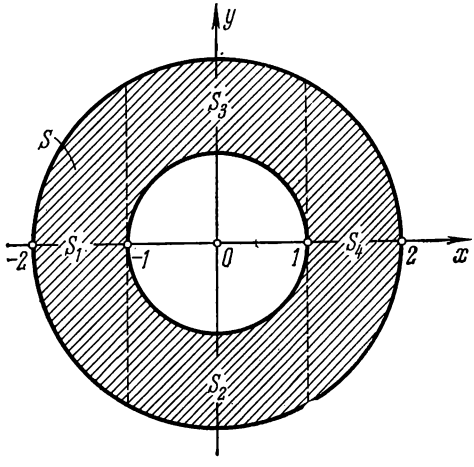


FIG. 257

the second method of computation turned out to be somewhat more complicated.

Example 4. Change the order of integration in the iterated integral

$$\int_0^1 dx \int_{x^2}^{x} f(x, y)\,dy.$$

The domain of integration S is bounded by the curves

$$y = x^2, \quad y = x$$

and

$$x = 0, \quad x = 1$$

(Fig. 256). Hence, changing the roles of the coordinate axes, we get

$$x = \sqrt{y} \geqslant 0, \quad x = y$$

and

$$y = 0, \quad y = 1.$$

Consequently,

$$\int_0^1 dx \int_{x^2}^{x} f(x, y)\, dy = \int_0^1 dy \int_y^{\sqrt{y}} f(x, y)\, dx.$$

Example 5. Set the limits of integration in the double integral

$$I = \int_S f(x, y)\, dx\, dy, \tag{223-13}$$

if the domain of integration S is an annulus bounded by the circles $x^2 + y^2 = 1$ (γ) and $x^2 + y^2 = 4$ (Γ) (Fig. 257). Here S is not a standard domain. For setting the limits of integration in integral (223-13) we split the domain S into four standard (relative to the y-axis) subdomains S_1, S_2, S_3, and S_4, as it is shown in the figure. Using the equations of the circles

$$y = \pm\sqrt{1 - x^2}\ (\gamma) \quad \text{and} \quad y = \pm\sqrt{4 - x^2}\ (\Gamma),$$

we have

$$I = \iint_{S_1} f(x, y)\, dx\, dy + \iint_{S_2} f(x, y)\, dx\, dy + \iint_{S_3} f(x, y)\, dx\, dy + \iint_{S_4} f(x, y)\, dx\, dy =$$

$$= \int_{-2}^{-1} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y)\, dy + \int_{-1}^{1} dx \int_{-\sqrt{4-x^2}}^{-\sqrt{1-x^2}} f(x, y)\, dy +$$

$$+ \int_{-1}^{1} dx \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} f(x, y)\, dy + \int_{1}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y)\, dy.$$

A similar formula will be obtained if we arrange the limits of integration in another order.

Sec. 224. Expressing a Double Integral in Polar Coordinates

Suppose in the double integral

$$\iint_S f(x, y)\, dx\, dy = \iint_S f(x, y)\, dS, \tag{224-1}$$

in usual assumptions, we desire to pass over to the polar coordinates r and φ, setting

$$x = r\cos\varphi, \quad y = r\sin\varphi. \tag{224-2}$$

Let us split the domain of integration S into elementary cells ΔS_{ij} with the aid of coordinate lines $r = r_j$ (circles) and $\varphi = \varphi_i$ (rays) (Fig. 258), and introduce the following notation:

$$\Delta r_j = r_{j+1} - r_j, \quad \Delta\varphi_i = \varphi_{i+1} - \varphi_i.$$

Since the circle is perpendicular (orthogonal) to the radii, the inner cells ΔS_{ij} may be considered, with an accuracy to infinitesimals of higher order of smallness than their areas, as rectangles with the dimensions $r_j\,\Delta\varphi_i$ and Δr_j; therefore the area of each cell will be equal to

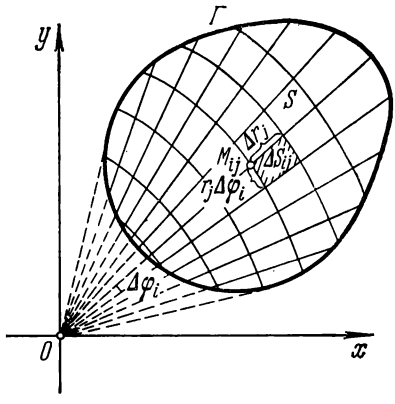


FIG. 258

$$\Delta S_{ij} \approx r_j\,\Delta\varphi_i\,\Delta r_j. \qquad (224\text{-}3)$$

As to the cells ΔS_{ij} of irregular shape adjoining the boundary Γ of the domain of integration S, they will not affect the value of the double integral (cf. Sec. 222, formula (222-8)) and, therefore, will be ignored.

For the sake of simplicity, we take the vertex of the cell ΔS_{ij} with the polar coordinates r_j and φ_i as the point $M_{ij} \in \Delta S_{ij}$. Then the Cartesian coordinates of the point M_{ij} are

$$x_{ij} = r_j \cos\varphi_i, \quad y_{ij} = r_j \sin\varphi_i$$

and, hence,

$$f(x_{ij},\ y_{ij}) = f(r_j \cos\varphi_i,\ r_j \sin\varphi_i). \qquad (224\text{-}3')$$

Double integral (224-1) represents the limit of the two-dimensional integral sum, and it is possible to show that the value of this limit is not affected by the additions to the terms of the integral sum which are infinitesimals of higher order of smallness. Therefore, taking into account (224-3) and (224-3′), we get

$$\iint_S f(x,\ y)\,dS = \lim_{d\to 0} \sum_{i,\,j} f(x_{ij},\ y_{ij})\,\Delta S_{ij} =$$

$$= \lim_{d\to 0} \sum_{i,\,j} f(r_j \cos\varphi_i,\ r_j \sin\varphi_i)\,r_j\,\Delta\varphi_i\,\Delta r_j, \qquad (224\text{-}4)$$

where d is the maximal diameter of the cells ΔS_{ij} and the sum is extended to all the cells of the above indicated form wholly contained in the domain S. On the other hand, the quantities φ_i and r_j

are numbers, and we may consider them as rectangular Cartesian coordinates of some points of the plane $O\varphi r$. Thus, sum (224-4) is an integral sum for the function

$$f(r\cos\varphi,\ r\sin\varphi)\,r,$$

corresponding to the rectangular grid with the linear elements $\Delta\varphi_i$ and Δr_j. Consequently,

$$\lim_{d\to 0}\sum f(r_j\cos\varphi_i,\ r_j\sin\varphi_i)\,r_j\,\Delta\varphi_i\,\Delta r_j = \iint\limits_S f(r\cos\varphi,\ r\sin\varphi)\,r\,d\varphi\,dr. \qquad (224\text{-}5)$$

Comparing (224-4) and (224-5), we finally obtain

$$\iint\limits_S f(x,\ y)\,dS = \iint\limits_S f(r\cos\varphi,\ r\sin\varphi)\,r\,d\varphi\,dr. \qquad (224\text{-}6)$$

The expression

$$dS = r\,d\varphi\,dr \qquad (224\text{-}7)$$

is called a *two-dimensional element of area in polar coordinates* (cf. Sec. 126, Chapter 15). Thus, *to pass to polar coordinates in double*

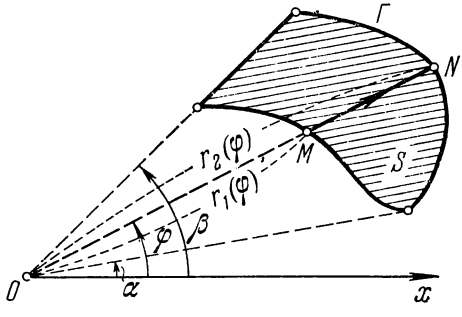


FIG. 259

integral (224-1), *it is sufficient to substitute the coordinates x and y according to formula* (224-2), *and replace the element of area dS by expression* (224-7).

To evaluate double integral (224-6), we have to replace it by an iterated one. Let the domain of integration S be defined by the inequalities

$$\alpha \leqslant \varphi \leqslant \beta, \qquad r_1(\varphi) \leqslant r \leqslant r_2(\varphi),$$

where $r_1(\varphi)$, $r_2(\varphi)$ are single-valued continuous functions on the segment $[\alpha, \beta]$ (Fig. 259). Then, by analogy with rectangular coor-

dinates (see Sec. 223), we have

$$\iint_S F(r,\ \varphi)\,d\varphi\,dr = \int_\alpha^\beta d\varphi \int_{r_1(\varphi)}^{r_2(\varphi)} F(r,\ \varphi)\,dr, \qquad (224\text{-}8)$$

where

$F(r,\ \varphi) = rf(r\cos\varphi,\ r\sin\varphi)$.

Example 1. Passing over to the polar coordinates φ and r, evaluate the double integral

$$I = \iint_S \frac{dx\,dy}{\sqrt{x^2+y^2}},$$

where S is the first quadrant of the circle of radius $R = 1$ with

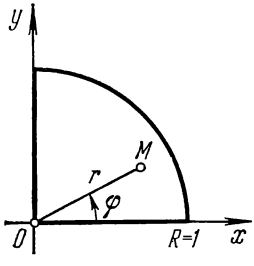


FIG. 260

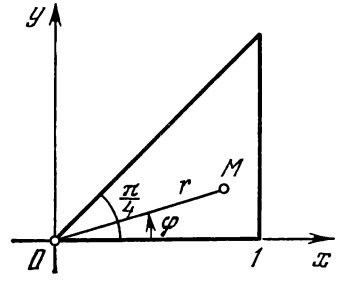


FIG. 261

centre at the point O (0, 0) (Fig. 260).

Since

$$\sqrt{x^2+y^2} = r,$$

applying formula (224-6), we get

$$I = \iint_S \frac{r\,d\varphi\,dr}{r} = \iint_S d\varphi\,dr.$$

The domain S is defined by the inequalities

$$0 \leqslant \varphi \leqslant \frac{\pi}{2},\quad 0 \leqslant r \leqslant 1.$$

Therefore, by (224-8), we have

$$I = \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 dr = \frac{\pi}{2}\cdot 1 = \frac{\pi}{2}.$$

Example 2. In the integral

$$I = \int_0^1 dx \int_0^x \sqrt{x^2+y^2}\, dy \tag{224-9}$$

pass over to polar coordinates.

Here, the domain of integration S is a triangle bounded by the straight lines $y = 0$, $y = x$, $x = 1$ (Fig. 261).

The equations of these straight lines in polar coordinates are written as follows: $\varphi = 0$, $\varphi = \frac{\pi}{4}$, $r\cos\varphi = 1$ and, hence, the domain S is defined by the inequalities

$$0 \leqslant \varphi \leqslant \frac{\pi}{4},$$

$$0 \leqslant r \leqslant \frac{1}{\cos\varphi} = \sec\varphi.$$

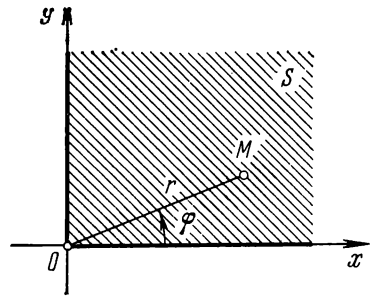


FIG. 262

Hence, by (224-6) and (224-8), and, taking into consideration that $\sqrt{x^2+y^2} = r$, we have

$$I = \iint_S r \cdot r\, d\varphi\, dr = \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\sec\varphi} r^2\, dr.$$

Sec. 225. The Euler-Poisson Integral

With the aid of polar coordinates it is easy to evaluate the *Euler-Poisson integral* which is of importance for the theory of probability

$$I = \int_0^{+\infty} e^{-x^2}\, dx. \tag{225-1}$$

Since the definite integral does not depend on the notation of the variable of integration, we may, obviously, also write

$$I = \int_0^{+\infty} e^{-y^2}\, dy. \tag{225-2}$$

Multiplying (225-1) and (225-2), and taking into consideration that the product of these single integrals may be regarded as a double integral of the product of the integrands (see Sec. 223, for-

mula (223-8)), we shall have

$$I^2 = \int\int_S e^{-(x^2+y^2)}\,dx\,dy, \tag{225-3}$$

where the domain S is defined by the inequalities

$$0 \leqslant x < +\infty, \quad 0 \leqslant y < +\infty$$

and, consequently, represents the first quadrant of the coordinate plane Oxy (Fig. 262)*.

Passing to polar coordinates in integral (225-3), we get

$$I^2 = \int\int_S e^{-r^2} r\,d\varphi\,dr = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{+\infty} re^{-r^2}\,dr =$$

$$= \varphi\Big|_0^{\frac{\pi}{2}} \cdot \left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^{+\infty} = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}.$$

Hence taking into account that the number I is positive, we find

$$I = \int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Since the function $y = e^{-x^2}$ is even, we also have

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = 2\int_0^{+\infty} e^{-x^2}\,dx = \sqrt{\pi},$$

which represents the area bounded by the x-axis and Gauss' curve $y = e^{-x^2}$ (see Sec. 92, Fig. 120).

Sec. 226. Mean-Value Theorem

Let the function $f(x, y)$ be continuous in a bounded closed domain S and $m = \min\limits_S f(x, y)$, $M = \max\limits_S f(x, y)$ the least and the greatest values of the function $f(x, y)$ in the domain S, respectively. For the two-dimensional integral sum of this function over the domain S we have the following estimates

$$mS \leqslant \sum_{i=1}^{n} f(x_i, y_i)\,\Delta S_i \leqslant MS, \tag{226-1}$$

where $S = \sum\limits_{i=1}^{n} \Delta S_i$ is the area of the domain S. Hence, proceeding

* Strictly speaking, (225-3) is an *improper* integral and needs, therefore, a special definition. But the formal operations performed lead to correct results.

to the limit as $d = \max\limits_i d(\Delta S_i) \to 0$ in inequalities (226-1) and taking into consideration the existence of the double integral, we shall obtain

$$mS \leqslant \iint\limits_S f(x, y)\, dS \leqslant MS. \tag{226-2}$$

The number

$$\mu = \frac{1}{S} \iint\limits_S f(x, y)\, dS \tag{226-3}$$

is called the *mean value* of the function $f(x, y)$ in the domain S. From inequalities (226-2) it follows that $m \leqslant \mu \leqslant M$.

Formula (226-3) can be rewritten as follows:

$$\iint\limits_S f(x, y)\, dS = \mu S \tag{226-4}$$

($m \leqslant \mu \leqslant M$). Thus, *the double integral is equal to the mean value of the integrand multiplied by the area of the domain of integration.*

The reader should not think that formula (226-4) gives a general-purpose method for evaluating double integrals. The point is that, as a rule, the mean value of a function is determined through a double integral. Therefore, here estimate (226-2) has a real sense.

Example. Estimate the integral

$$I = \iint\limits_S \sqrt{x^2 + y^2}\, dx\, dy,$$

where S is a square: $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$.

For the function $f(x, y) = \sqrt{x^2 + y^2}$ we have

$$m = \min_S f(x, y) = f(0, 0) = 0$$

and

$$M = \max_S f(x, y) = f(1, 1) = \sqrt{2}.$$

Since $S = 1$, we have

$$0 \leqslant I \leqslant \sqrt{2} = 1.41.$$

We may estimate the integral as follows

$$I \approx \frac{1}{2}(0 + 1.41) = 0.71.$$

This is a rough estimate, since the exact value of the integral is

$$I = \frac{1}{3}\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right] \approx 0.79.$$

A more exact value of the integral I can be obtained if we divide the domain of integration S into sufficiently small subdomains and apply the mean-value theorem to each of them.

Sec. 227. Geometrical Applications of the Double Integral

In Sec. 222 it was shown that a right cylindroid constructed on the base S in the coordinate plane Oxy and bounded from above by a continuous surface $z = f(x, y)$ has a volume equal to

$$V = \iint_S f(x, y)\, dx\, dy. \tag{227-1}$$

Example 1. Find the volume V of a solid bounded by the following surfaces:

$$z = x^2, \quad z = 0, \quad x = 0, \quad y = 0, \quad x + y = 1.$$

The base of the given solid is a triangle S formed by the straight lines $x = 0$, $y = 0$, and $x + y = 1$ in the xy-plane. From above

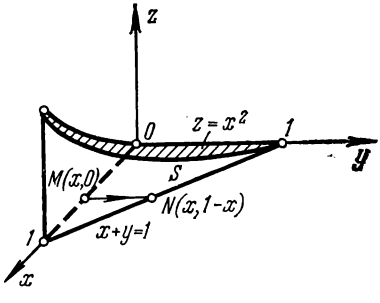


FIG. 263

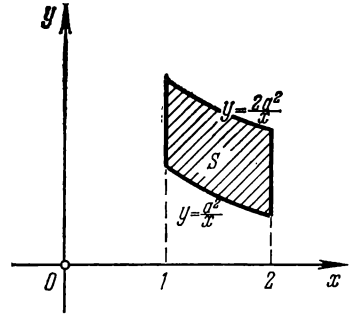


FIG. 264

the solid is bounded by the parabolic cylinder $z = x^2$ (Fig. 263). Hence, by (227-1), we get

$$V = \iint_S x^2\, dx\, dy = \int_0^1 dx \int_0^{1-x} x^2\, dy = \int_0^1 dx \cdot x^2 y \Big|_{y=0}^{y=1-x} =$$

$$= \int_0^1 x^2 (1 - x)\, dx = \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_{x=0}^{x=1} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

If in formula (227-1) we set

$f(x, y) \equiv 1$,

then we shall obtain the volume of a right cylinder with altitude $z = 1$ numerically equal to the area S of its base. Therefore, the area of the plane region S is equal to

$$S = \iint\limits_S dx\, dy. \tag{227-2}$$

Formula (227-2) can also be written in the form

$$S = \iint\limits_S dS. \tag{227-3}$$

Example 2. Find the area bounded by the hyperbolas

$$y = \frac{a^2}{x}, \quad y = \frac{2a^2}{x}, \qquad (a > 0)$$

and straight lines

$$x = 1, \quad x = 2$$

(Fig. 264).

By formula (227-2), we get that the area S is equal to

$$S = \int\limits_1^2 dx \int\limits_{\frac{a^2}{x}}^{\frac{2a^2}{x}} dy = \int\limits_1^2 \left(\frac{2a^2}{x} - \frac{a^2}{x}\right) dx = a^2 \int\limits_1^2 \frac{dx}{x} =$$

$$= a^2 \ln x \Big|_1^2 = a^2 \ln 2 \approx 0.7a^2.$$

Sec. 228. Physical Applications of the Double Integral

Let S be a material plate. If ΔS is a part of the plate S containing point M and having a mass Δm, then the ratio

$$\frac{\Delta m}{\Delta S}$$

is called the *mean surface density* of the piece ΔS, and the limit of this ratio, provided the diameter $d(\Delta S) \to 0$, is called the *surface density* $\rho(M)$ of the plate S at the point M:

$$\rho(M) = \lim_{d(\Delta S) \to 0} \frac{\Delta m}{\Delta S}.$$

Obviously, the surface density $\rho(M)$ of the plate S is a function of the point M. The concepts of the mean surface density of a plate and the surface density of the plate at a given point are quite analogous to those of the mean line density of an arc and line density of the arc at a point introduced in the preceding chapter (Sec. 217).

Let us assume that the surface density of the plate S at a current point $M(x, y)$ is equal to $\rho = \rho(x, y)$, where $\rho(x, y)$ is a known

continuous function. Consider the infinitesimal element dS of the plate containing the point M (Fig. 265). Since within the limits of this element the plate may be regarded as homogeneous with density ρ, the mass of the element dS (*elementary mass*) is equal to

$$dm = \rho\, dS. \tag{228-1}$$

Integrating expression (228-1) over the entire plate S, we find the *mass of the plate*

$$m = \iint_S \rho\, dS. \tag{228-2}$$

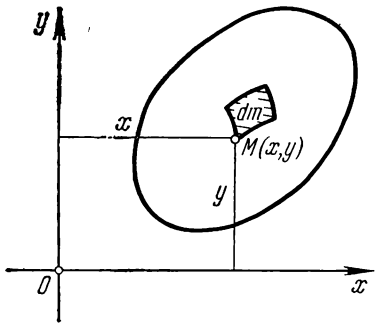


FIG. 265

Considering dm as a materia point situated at distances y and x from the x- and y-axes respectively, we get the *elementary static moments of the plate*

$$dS_x = y\, dm = y\rho\,(x,\ y)\, dS$$

and

$$dS_y = x\, dm = x\rho\,(x,\ y)\, dS.$$

Hence, integrating these expressions over the whole plate S, we find the *static moments of the plate S with respect to the coordinate axes*

$$\left.\begin{aligned} S_x &= \iint_S y\rho\,(x,\ y)\, dS, \\ S_y &= \iint_S x\rho\,(x,\ y)\, dS. \end{aligned}\right\} \tag{228-3}$$

It is proved in mechanics that the static moment of a plate about some axis coincides with the static moment of a point mass equal to the mass of the plate, concentrated at its centre of gravity, with respect to the same axis (*Varignon's theorem*). Hence, denoting by $(x_0,\ y_0)$ the coordinates of the *centre of gravity* (*centre of mass*) of the plate S, we shall have

$$S_y = mx_0, \quad S_x = my_0;$$

consequently,

$$\left.\begin{aligned} x_0 &= \frac{S_y}{m} = \frac{1}{m}\iint_S x\rho\,(x,\ y)\, dS, \\ y_0 &= \frac{S_x}{m} = \frac{1}{m}\iint_S y\rho\,(x,\ y)\, dS, \end{aligned}\right\} \tag{228-4}$$

where m is the mass (228-2) of the plate.

Analogously, for *elementary moments of inertia* of the plate S about the x- and y-axes we get the following expressions:

$$dI_x = y^2\,dm = y^2\rho\,(x,\ y)\,dS,$$

$$dI_y = x^2\,dm = x^2\rho\,(x,\ y)\,dS.$$

Hence, on integrating over the plate S, we shall have the *moments of inertia of the plate S with respect to the coordinate axes*

$$\left.\begin{aligned} I_x &= \iint\limits_S y^2\rho\,(x,\ y)\,dS, \\ I_y &= \iint\limits_S x^2\rho\,(x,\ y)\,dS. \end{aligned}\right\} \tag{228-5}$$

An *elementary polar moment of inertia* is determined by the formula

$$dI_0 = r^2\,dm = (x^2 + y^2)\,\rho\,(x,\ y)\,dS,$$

where $r^2 = x^2 + y^2$ is the square of the distance of the mass dm from the origin. Integrating this expression over the whole plate S, we obtain the *polar moment of the plate*

$$I_0 = \iint\limits_S (x^2 + y^2)\,\rho\,(x,\ y)\,dS. \tag{228-6}$$

From (228-5) and (228-6) it follows that

$$I_0 = I_x + I_y. \tag{228-7}$$

Setting $\rho\,(x,\ y) \equiv 1$ in the above formulas, we shall obtain the corresponding moments of inertia of the geometrical figure S. It should be remembered that, when computing in rectangular Cartesian coordinates, we take, as usually,

$$dS = dx\,dy,$$

and in the case of polar coordinates we have

$$dS = r\,d\varphi\,dr.$$

Example 1. Determine the coordinates of the centre of gravity of a square plate S: $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 2$, whose surface density at the point $M\,(x,\ y)$ is equal to $\rho = x + y$.

Using formula (228-2), we find the mass of the plate

$$m = \iint\limits_S \rho\,dx\,dy = \int\limits_0^2 dx \int\limits_0^2 (x + y)\,dy =$$

$$= \int\limits_0^2 dx \left(xy + \frac{y^2}{2} \right) \Bigg|_{y=0}^{y=2} = \int\limits_0^2 (2x + 2)\,dx = (x^2 + 2x) \Big|_{x=0}^{x=2} = 8.$$

Then, by (228-3), we determine the static moments of the plate S about the coordinate axes

$$S_x = \iint_S y\rho\,dx\,dy = \int_0^2 dx \int_0^2 (xy + y^2)\,dy = \int_0^2 dx \left(\frac{xy^2}{2} + \frac{y^3}{3}\right)\Big|_{y=0}^{y=2} =$$

$$= \int_0^2 \left(2x + \frac{8}{3}\right) dx = \left(x^2 + \frac{8}{3}x\right)\Big|_{x=0}^{x=2} = 4 + \frac{16}{3} = 7\frac{1}{3},$$

$$S_y = \iint_S x\rho\,dx\,dy = \int_0^2 dx \int_0^2 (x^2 + xy)\,dy = \int_0^2 dx \left(x^2 y + \frac{xy^2}{2}\right)\Big|_{y=0}^{y=2} =$$

$$= \int_0^2 (2x^2 + 2x)\,dx = \left(\frac{2x^3}{3} + x^2\right)\Big|_{x=0}^{x=2} = \frac{16}{3} + 4 = 7\frac{1}{3}.$$

The equality of the moments S_x and S_y is explained by the fact that the given figure is a symmetric one.

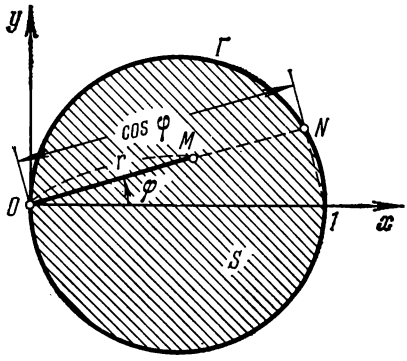


FIG. 266

By (228-4), the centre of gravity of the plate has the following coordinates

$$x_0 = \frac{S_y}{m} = \frac{28}{3 \cdot 8} = \frac{7}{6} = 1\frac{1}{6},$$

$$y_0 = \frac{S_x}{m} = 1\frac{1}{6}.$$

Example 2. Find the moment of inertia I_x of the area of the circle S: $x^2 + y^2 \leqslant x$ (Fig. 266) with respect to the x-axis.

Setting $\rho = 1$, by the first formula of (228-5), we have

$$I_x = \iint_S y^2\,dS. \qquad (228\text{-}8)$$

We shall solve the problem in polar coordinates. We have

$$x = r\cos\varphi, \quad y = r\sin\varphi$$

and

$$dS = r\,d\varphi\,dr.$$

The equation of the boundary Γ of the domain S is

$$x^2 + y^2 = x.$$

Hence, passing to polar coordinates, we get

$$r^2 = r\cos\varphi.$$

Consequently, on reducing by an unessential factor r, we have

$$r = \cos\varphi,$$

and since $r \geqslant 0$, then $-\frac{\pi}{2} \leqslant \varphi \leqslant \frac{\pi}{2}$. Thus, for each fixed $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the radius r varies within the limits $0 \leqslant r \leqslant \cos\varphi$.

Passing to polar coordinates in (228-8), we obtain

$$I_x = \iint\limits_S r^2 \sin^2\varphi \cdot r\, dr\, d\varphi = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\varphi\, d\varphi \int\limits_0^{\cos\varphi} r^3\, dr =$$

$$= \frac{1}{4} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\varphi \cos^4\varphi\, d\varphi.$$

As is known,

$$\cos^2\varphi = \frac{1}{2}(1 + \cos 2\varphi) \quad \text{and} \quad \sin^2\varphi\cos^2\varphi = \frac{1}{4}\sin^2 2\varphi,$$

therefore

$$I_x = \frac{1}{4} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}\sin^2 2\varphi \cdot \frac{1}{2}(1 + \cos 2\varphi)\, d\varphi = \frac{1}{32} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2\varphi\, d\varphi +$$

$$+ \frac{1}{32} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2\varphi \cos 2\varphi\, d\varphi =$$

$$= \frac{1}{64} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos 4\varphi)\, d\varphi + \frac{1}{64} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2\varphi\, d(\sin 2\varphi) =$$

$$= \frac{1}{64}\left(\varphi - \frac{1}{4}\sin 4\varphi\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{64} \cdot \frac{\sin^3 2\varphi}{3}\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \frac{1}{64}\left(\frac{\pi}{2} + \frac{\pi}{2} - 0\right) + 0 = \frac{\pi}{64}.$$

Sec. 229. Triple Integrals

The so-called *triple integral* is defined by analogy with the double integral. Let in the Cartesian space $Oxyz$ there be given a finite closed domain V and a bounded function $f(x, y, z)$ defined in V. Now we split the domain V into a finite number of cells ΔV_1, ΔV_2, . . ., ΔV_n and choose in each of them a point

$$M_i(x_i, y_i, z_i) \in \Delta V_i \qquad (i = 1, 2, \ldots, n)$$

(Fig. 267). The sum

$$S_n = \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta V_i, \qquad (229\text{-}1)$$

where ΔV_i is the volume of the ith cell is called the *three-dimensional integral sum*.

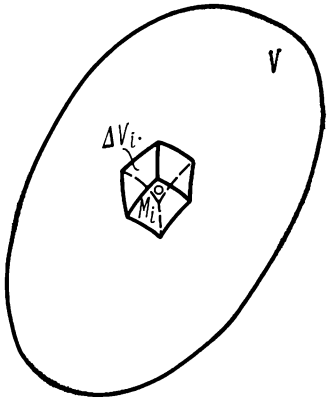


FIG. 267

Let us denote by d the greatest of the diameters of the cells ΔV_i*. Then we shall infinitely reduce the cells ΔV_i in an arbitrary way. In this case, the limit of integral sum (229-1) as $d \to 0$, provided this limit exists and is independent of the shape of the cells ΔV_i and of the choice of points M_i in them, is called the *triple integral of the function* $f(x, y, z)$ *over the domain* V and is denoted in the following way:

$$\iiint_V f(x, y, z)\,dV = \lim_{d \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta V_i. \qquad (229\text{-}2)$$

It is proved that if the integrand $f(x, y, z)$ is continuous in a closed bounded domain of integration V with a piecewise smooth boundary, then triple integral (229-2) exists.

If the domain V is filled with mass and $f(x, y, z)$ represents a continuously distributed volume density at a current point $M(x, y, z)$, then $f(x_i, y_i, z_i)\,\Delta V_i$, where $M_i(x_i, y_i, z_i) \in \Delta V_i$, with an accuracy to an infinitesimal of higher order of smallness than the maximal volume of the cells ΔV_j $(j = 1, \ldots, n)$, is the mass of the cell ΔV_i. Consequently, integral sum (229-1) is approximately equal to the mass m filling the domain V. As $d \to 0$, the limit of the sum S_n will be equal to the mass m. Hence we derive the *physical meaning of the triple integral*: if $f(x, y, z)$ is a continuous

* For the concept of diameter see Sec. 222.

density of distribution of a mass in the xyz-space, then the triple integral

$$m = \iiint\limits_V f(x, y, z)\, dV \tag{229-3}$$

represents the mass filling the domain of integration V. In particular, if density $f(x, y, z) \equiv 1$, then the mass of the domain V is numerically equal to its volume. Therefore the volume of the domain V is expressed by the triple integral

$$V = \iiint\limits_V dV. \tag{229-4}$$

If triple integral (229-2) is evaluated in the rectangular coordinates x, y, z, then the cells ΔV_{ijk} are represented by rectangular parallelepipeds with the dimensions Δx_i, Δy_j, Δz_k whose faces are parallel to the coordinate planes, i.e. we set

$$\Delta V_{ijk} = \Delta x_i\, \Delta y_j\, \Delta z_k.$$

In this case the *volume element* dV is considered to be equal to

$$dV = dx\, dy\, dz \tag{229-5}$$

(*the volume element in rectangular coordinates*) and triple integral (229-2) is written in the following form:

$$\iiint\limits_V f(x, y, z)\, dx\, dy\, dz. \tag{229-6}$$

In particular, for the volume of a solid we get the formula

$$V = \iiint\limits_V dx\, dy\, dz. \tag{229-6'}$$

In the simplest case the computation of triple integral (229-6) is reduced to *three quadratures*. Namely, let the domain of integration V be *standard* relative to the z-axis (cf. Sec. 223), i.e. be bounded from below and from above by single-valued continuous surfaces

$$z_1 = z_1(x, y)$$

$$z_2 = z_2(x, y),$$

respectively, the projection of the domain V on the xy-plane being a plane domain S (Fig. 268).

Hence it follows that with the values $(x, y) \in S$ fixed, the corresponding z-coordinates of the points contained in the domain V vary within the limits $z_1(x, y) \leqslant z \leqslant z_2(x, y)$. By analogy with

the double integral (Sec. 227), we shall have

$$\iiint\limits_V f(x, y, z)\,dx\,dy\,dz = \iint\limits_S dx\,dy \int\limits_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\,dz. \qquad (229\text{-}7)$$

If, in addition to this, the projection S is standard with respect to the y-axis and is defined by the inequalities

$$a \leqslant x \leqslant b, \quad y_1(x) \leqslant y \leqslant y_2(x),$$

where $y_1(x)$ and $y_2(x)$ are single-valued continuous functions on the

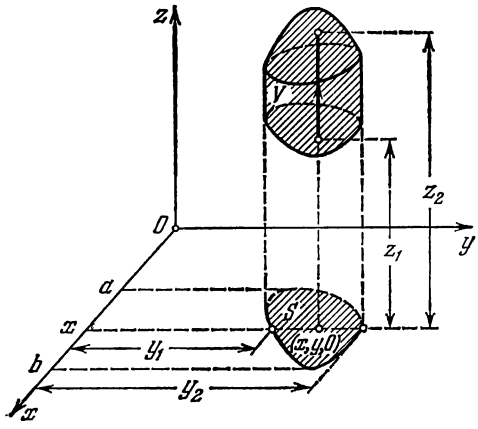


FIG. 268

segment $[a, b]$, then

$$\iint\limits_S dx\,dy \int\limits_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\,dz = \int\limits_a^b dx \int\limits_{y_1(x)}^{y_2(x)} dy \int\limits_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\,dz. \qquad (229\text{-}8)$$

From (229-7) and (229-8) we finally get

$$\iiint\limits_V f(x, y, z)\,dx\,dy\,dz = \int\limits_a^b dx \int\limits_{y_1(x)}^{y_2(x)} dy \int\limits_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\,dz. \qquad (229\text{-}9)$$

Thus, the computation of the triple integral has been reduced to three quadratures.

Note that if the domain of integration V is standard with respect to all the three coordinate axes (the x-, y-, and z-axes), then the limits of integration for triple integral (229-6) can be set in $3! = 6$ different ways.

Example. Evaluate

$$I = \iiint\limits_V xyz\,dx\,dy\,dz,$$

where V is a pyramid $OPQR$ bounded by the following planes:

$$x = 0, \quad y = 0, \quad z = 0, \quad x + y + z = 1$$

(Fig. 269).

The projection of the domain V on the coordinate plane $Oxyz$ s a triangle S bounded by the straight lines

$$x = 0, \quad y = 0, \quad x + y = 1.$$

For $(x, y) \in S$ the z-coordinates of the points $(x, y, z) \in V$ satisfy

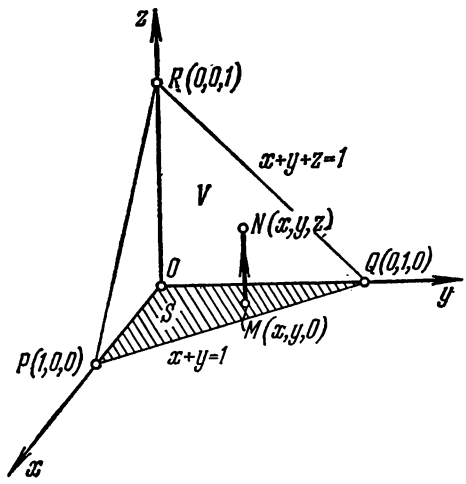


FIG. 269

the inequality $0 \leqslant z \leqslant 1 - x - y$. Therefore

$$I = \iint\limits_S xy\,dx\,dy \int\limits_0^{1-x-y} z\,dz =$$

$$= \iint\limits_S xy\,dx\,dy \cdot \frac{z^2}{2}\Big|_{z=0}^{z=1-x-y} = \frac{1}{2}\iint\limits_S xy\,(1-x-y)^2\,dx\,dy.$$

Setting the limits of integration for the triangle S, we get

$$I = \frac{1}{2}\int\limits_0^1 x\,dx \int\limits_0^{1-x} y\,[(1-x)^2 - 2(1-x)\,y + y^2]\,dy =$$

$$= \frac{1}{2}\int\limits_0^1 x\left[(1-x)^2 \cdot \frac{y^2}{2} - 2(1-x)\cdot\frac{y^3}{3} + \frac{y^4}{4}\right]\Big|_{y=0}^{y=1-x} dx =$$

$$= \frac{1}{2}\int\limits_0^1 x\,(1-x)^4\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)dx = \frac{1}{24}\int\limits_0^1 [1-(1-x)]\,(1-x)^4\,dx =$$

$$= \frac{1}{24}\left\{\int_0^1 (1-x)^4\,dx - \int_0^1 (1-x)^5\,dx\right\} =$$

$$= \frac{1}{24}\left[-\frac{(1-x)^3}{5}\Big|_{x=0}^{x=1} + \frac{(1-x)^6}{6}\Big|_{x=0}^{x=1}\right] = \frac{1}{24}\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{1}{720}.$$

The number I represents the mass of the pyramid V if its density at the current point $M(x, y, z)$ is equal to $\rho = xyz$.

Exercises

1. Evaluate the iterated integrals:

(a) $\int_0^1 dx \int_2^5 \frac{x}{y^2}\,dy$; (b) $\int_{-1}^0 dx \int_0^1 e^{x-y}\,dy$; (c) $\int_0^{2\pi} d\varphi \int_0^1 r\sin^2\varphi\,dr$.

2. Draw the domain of integration and evaluate the iterated integrals:

(a) $\int_0^1 dx \int_0^x \sqrt{x+y}\,dy$; (b) $\int_{-1}^1 dy \int_y^{y+y^2} xy\,dx$; (c) $\int_0^{2\pi} d\varphi \int_0^{\varphi} r\cos\varphi\,dr$.

3. Set the limits of integration for both orders in the double integral $\iint_S f(x, y)\,dx\,dy$ for the following domains of integration:

(a) S is a triangle with the vertices: $O(0, 0)$, $A(1, 0)$, and $B(1, 2)$;

(b) S is a trapezoid with the vertices: $O(0, 0)$, $A(2, 0)$, $B(1, 1)$, and $C(0, 1)$;

(c) S is a quadrant of a circle $x^2 + y^2 \leqslant 1$, $x \geqslant 0$, $y \geqslant 0$;

(d) S is a parabolic segment bounded by the lines $y = x^2$, $y = 4$;

(e) S is a circle $x^2 + y^2 \leqslant 2x$.

4. Change the order of integration in the following iterated integrals:

(a) $\int_0^1 dx \int_{x^2}^x f(x, y)\,dy$;

(b) $\int_0^1 dy \int_{e^y}^e f(x, y)\,dx$;

(c) $\int_0^{\pi} dx \int_0^{\sin x} f(x, y)\,dx$.

5. Pass over to polar coordinates and set the limits of integration in the following double integrals:

(a) $\iint\limits_S x^2y\,dx\,dy$, where S is a circular sector bounded by the lines $x^2+y^2=1$, $y=x$, $y=-x\,(y\geqslant 0)$;

(b) $\iint\limits_S \sqrt{x^2+y^2}\,dx\,dy$, where S is a triangle with the vertices $O(0,\ 0)$, $A(2,\ -1)$, $B(2,\ 1)$;

(c) $\iint\limits_S f(x+y)\,dx\,dy$, where S is a parabolic segment bounded by the lines $y=x$ and $y=x^2$.

6. Evaluate the following double integrals:

(a) $\iint\limits_S (x+y)\,dx\,dy$, where S is a triangle with the vertices $O(0,\ 0)$, $A(1,\ 0)$ and $B(0,\ 1)$;

(b) $\iint\limits_S xy^2\,dx\,dy$, where S is a parabolic segment bounded by the lines $y^2=x$, $x=1$;

(c) $\iint\limits_S \frac{dx\,dy}{(y+1)^2}$, where S is a parabolic segment bounded by the lines $y=1-x^2$, $y=0$;

(d) $\iint\limits_S \frac{x}{\sqrt{y}}\,dx\,dy$, where S is a hyperbolic segment bounded by the lines $xy=4$, $x+y=5$.

7. On passing to polar coordinates, evaluate the given integrals:

(a) $\iint\limits_S \sin\sqrt{x^2+y^2}\,dx\,dy$, where S is a circle $x^2+y^2\leqslant\pi^2$;

(b) $\iint\limits_S \frac{y^2}{x^2+y^2}\,dx\,dy$, where S is a triangle bounded by the straight lines $y=x$, $y=-x$, $y=1$.

8. Find the volumes of the following solids bounded by the indicated surfaces:

(a) $z=\sqrt{1-y^2}$, $z=0$, $y=x$, $x=0$;
(b) $z=2-x-y$, $z=0$, $x^2+y^2=1$, $x=0$, $y=0$;
(c) $z=x^2+y^2$, $z=0$, $y^2=x$, $x=1$;
(d) $z=\sqrt{x^2+y^2}$, $z=0$, $x^2+y^2=2x$;
(e) $z=\frac{c}{a}x$, $z=0$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(z\geqslant 0)$.

9. Find the moments of inertia of a homogeneous circular plate $x^2+y^2\leqslant R^2$ about the x- and y-axes.

10. Determine the coordinates of the centre of gravity of a homogeneous plate bounded by the curves: $ay = x^2$, $y = a$.

11. Evaluate

$$\int_0^1 dx \int_0^x dy \int_0^y xyz\, dz.$$

12. Find the volume of a solid bounded by the following surfaces: $z = y^2$, $z = 2y^2$, $xy = 1$, $xy = 4$, $x = 1$, $x = 3$.

Chapter 25

Fundamentals of the Theory of Probability

A. BASIC DEFINITIONS AND THEOREMS

Sec. 230. Random Events

In natural sciences knowledge of reality is acquired through trials (experiments) or observations; i.e. through experience (in the broad sense of the word). Generally, a trial (observation) implies the presence of a certain set of conditions. A theoretically possible result (the outcome of a trial or observation) is called the *event*, irrespective of its significance.

When constructing a theory, events are idealized, i.e. the situations unessential for a given phenomenon are ignored.

Example. Flip a coin and it will fall heads or tails. Thus, in a single tossing of a coin two events are possible: A—falling heads, and B—falling tails.

But here one more event is possible, C, when the coin falls on its edge. But for playing pitch-and-toss this circumstance is unessential and in our idealized experiment this event is neglected (the coin is tossed repeatedly!).

Definition 1. *The result of a trial which cannot be predicted is called a random event.*

In other words, an event is random in a given experiment if we cannot predict its occurrence in this trial.

For instance, falling heads in coin tossing is a random event. Of course, it is assumed that the trial is organized so that its outcome is not known beforehand.

In many cases a random event is the result of incomplete information concerning a given phenomenon. For instance, if in the experiment with the coin we knew: the pushing force, the shape of the coin, the law of air resistance and other factors determining the law of motion of the coin, we would be able to predict precisely the outcome of the trial.

Definition 2. *An event is called certain in a given trial (i.e. under realization of a certain set of conditions) if it happens inevitably in this trial.*

For instance, getting either a positive or a negative mark by a student in passing his exam is a certain event if the exam proceeds according to usual rules.

Definition 3. *An event is called impossible in a given trial if it is well known that it does not occur in this trial.*

For instance, it is impossible to take out a white ball from a box filled with coloured balls. It should be noted that the occurrence of a white ball is not excluded under other conditions. Thus, this event is impossible only under the conditions of our trial.

The branch of mathematics which deals with the study of the likelihood of an event occurring is called the *theory of probability.*

In connection with the development of new equipment and technology of special interest are the statistical rules for *mass homogeneous random events* (production quality inspection, mass production attendance, telephone exchange operation, etc.). Here we come across various versions of the basic theorem of the theory of probability, i.e. of the law of large numbers.

Let us take as an axiom that for each event A we can determine, at least theoretically, the probability of this event, i.e. the number $P(A)$ representing in a sense a "measure of certainty" of the given event and obeying the natural requirements.

It is assumed that the probability of any event satisfies the inequality

$$0 \leqslant P(A) \leqslant 1;$$

the probability of an impossible event being equal to zero, and the probability of a certain event to unity.

For practical purposes it is assumed that if the probability of an event is small, then this event is practically impossible; and vice versa, if the probability of an event is close to unity, then this event is almost certain; and well-grounded decisions are taken accordingly.

Many outstanding mathematicians (Pascal, Fermat, Laplace, Gauss, Poisson and others) contributed to the development of the theory of probability. Later, the decisive successes in this field of science belong to the Russian and Soviet mathematicians (Chebyshev, Markov, Lyapunov, Bernstein, Kolmogorov, Khinchin and others).

Probability theory is widely used in theoretical and applied sciences (in physics, geodesy, shooting theory, automatic control theory, and in many others). In particular, it serves as a theoretical basis for mathematical and applied statistics by means of which we plan and organize production.

Sec. 231. Algebra of Events

Each trial is associated with a number of events we are interested in, which can, generally speaking, occur simultaneously. For instance, let in throwing an ordinary die (a cube with numbers ranging from 1 to 6 on the faces) turning up of one pip be event A, and turning up of an odd number of pips, event B. Obviously, these events are not mutually exclusive.

Let all possible results of a trial be realized in a number of the only possible particular cases which are mutually exclusive (the so-called *elementary events* or *elementary outcomes*). Then (1) each outcome of the trial is represented by one and only one elementary event; (2) any event A related with this trial is a set of a finite or infinite number of elementary events; (3) event A takes place if and only if one of the elementary events entering into this set is realized.

Example 1. Let event A consist in turning up of an odd number of pips in a single throwing of the die.

Here for elementary events we may take the following results of the trial: (1), (2), (3), (4), (5), (6). Event A represents the set of events $\{(1), (3), (5)\}$.

Algebra of events is constructed by analogy with the set theory (see Sec. 6).

Definition 1. *The sum of two events A and B is understood to be the event*

$$A + B \equiv A \cup B,$$

which takes place if and only if at least one of the events A and B occurred.

In the general case, the *sum of several events* is understood to be an event consisting in occurrence of at least one of these events.

Example 2. Let a prize won in lottery I be event A, and a prize won in lottery II event B. Then the event $A + B$ is a prize won at least in one lottery (possibly, in two at once!).

Definition 2. *The product of two events A and B is the event*

$$AB \equiv A \cap B,$$

consisting in simultaneous occurrence of both events, A and B.

In the general case, the *product of several events* is understood to be an event consisting in simultaneous realization of all these events.

Example 3. Let events A and B mean successful passages of turns I and II in entering an institute, respectively. Then event AB represents a successful passage of both turns.

Events A and B are called incompatible in a given trial if their product is an impossible event, i.e.

$$AB = O,$$

where O is an impossible event.

In other words, two events are incompatible if the occurrence of one of them excludes the occurrence of the other, and vice versa.

Sec. 232. The Classical Definition of Probability

Let event A be some outcome of a trial and

$$E_1, E_2, \ldots, E_n \tag{232-1}$$

a finite system of all possible and the only possible pairwise incompatible elementary outcomes of this trial (*a complete system of elementary events*). Thus, event A takes place when and only when there happen some events from system (232-1) (successes or favourable outcomes, or so-called chances for event A).

Let us assume that the events of system (232-1) are *equally possible*, i.e. there is no reason to suppose that one of the events of system (232-1) prevails, in the sense of occurrence, over others. Sometimes, this can be established, using the "property of symmetry".

Definition 1. *The probability $P(A)$ of event A is understood to be the ratio of the number of equally possible elementary outcomes favourable to event A to the total number of all equally possible and the only possible elementary outcomes of a given trial.*

Thus, if m is the number of elementary outcomes, favourable to event A, and n the total number of all elementary outcomes in a given trial, and all of these outcomes are equally possible, then, by definition, we have the following formula

$$P(A) = \frac{m}{n}. \tag{232-2}$$

Since obviously,

$$0 \leqslant m \leqslant n,$$

we have

$$0 \leqslant P(A) \leqslant 1, \tag{232-3}$$

i.e. *the probability of any event is a non-negative number not exceeding unity.*

Note. From the definition of probability it follows that equally possible elementary events are *equiprobable*, i.e. they possess one and the same probability.

From the definition of probability there follow its basic properties:

1. *The probability of an impossible event is equal to zero.*

Indeed, if event A is impossible, then the number of elementary outcomes, favourable to it, $m = 0$, and we have

$$P(A) = \frac{0}{n} = 0.$$

2. *The probability of a certain event is equal to unity.*

Indeed, if event A is certain, then, obviously, $m = n$ and, consequently,

$$P(A) = \frac{n}{n} = 1.$$

Let us consider some elementary theorems on probabilities.

Definition 2. *Two events A and B are called equivalent*:

$$A = B,$$

if either of them takes place whenever the other occurs.

From the standpoint of probability theory, such events are regarded as equal.

For instance, if a box contains only white and black balls, then the occurrence of a black ball and occurrence of a non-white ball are equivalent events.

Theorem 1. *Equivalent events have equal probabilities, i.e. if $A = B$, then*

$$P(A) = P(B). \tag{232-4}$$

Indeed, each elementary outcome for event A is the same for event B and conversely. By formula (232-2), equality (232-4) holds true.

Definition 3. *Event B is said to follow from event A ($A \Rightarrow B$) if event B occurs whenever event A happened.*

For instance, for any events A and B we have: $AB \Rightarrow A$ and $AB \Rightarrow B$.

Theorem 2. *If $A \Rightarrow B$, then*

$$P(A) \leqslant P(B). \tag{232-5}$$

Indeed, let events A and B be included into a general system of equiprobable elementary outcomes, where m and m' are the numbers of favourable elementary outcomes for events A and B, respectively, and n is the total number of elementary outcomes. Since each elementary outcome for event A is also an elementary outcome for event B, we have

$$m \leqslant m'$$

and, consequently,

$$P(A)=\frac{m}{n}\leqslant\frac{m'}{n}=P(B),$$

and, thus, inequality (232-5) is proved.

Definition 4. *Event $\overline{A}$ occurring when and only when event A does not take place is called opposite to the latter.*

For instance, if when tossing a coin, event A is falling heads, then event $\overline{A}$ represents non-falling heads, i.e. falling tails.

From Definition 4 it follows that (1) event $A+\overline{A}$ is certain; (2) event $A\overline{A}$ is impossible.

Theorem 3. *The probability of an opposite event $\overline{A}$ is equal to the complement of the probability of a given event A with respect to unity,* i.e.

$$P(\overline{A})=1-P(A). \tag{232-6}$$

Indeed, let a complete system of equiprobable elementary outcomes contain n events, of which m ($m\leqslant n$) are favourable to event A. Then $n-m$ elementary outcomes are unfavourable to event A, i.e. they are favourable to event $\overline{A}$. Thus, we have

$$P(\overline{A})=\frac{n-m}{n}=1-\frac{m}{n}=1-P(A).$$

Below, we give a couple of examples on evaluating probabilities of events.

Example 1. A coin is tossed twice. What is the probability: (1) of falling heads at least once (event A); (2) of falling heads twice (event B)?

Here, equally possible elementary outcomes are: HH, HT, TH, TT; their number $n=4$.

Event A is favoured by the outcomes HH, HT, TH, whose number $m=3$. Consequently,

$$P(A)=\frac{m}{n}=\frac{3}{4}.$$

Event B is favoured by one outcome HH ($m'=1$). Therefore

$$P(B)=\frac{m'}{n}=\frac{1}{4}.$$

Example 2. A die is thrown twice. What is the probability of the sum of the pips turning up being equal to 6 (event A)?

Here, equally possible elementary outcomes are the pairs (x, y) where x and y take on the values: 1, 2, 3, 4, 5, 6, the total number of elementary outcomes being equal to $n=36$.

Event A is favoured by the pairs (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), whose number $m = 5$.

Hence,

$$P(A) = \frac{m}{n} = \frac{5}{36}.$$

Sec. 233. The Statistical Definition of Probability

The classical definition of the probability of an event assumes that (1) the number of elementary outcomes is finite; (2) these outcomes are equally possible.

But in practice we encounter trials with an infinite number of possible outcomes. Besides, there are no general methods enabling us to present the results of a trial even with a finite number of outcomes in the form of a sum of equally possible elementary outcomes.

Therefore, the application of the classical definition of probability is very limited.

We are going now to give another definition of probability which is sometimes more convenient for applications.

Let us carry out n one-type trials one of whose outcomes is a given event A.

Definition 1. *The ratio of the number of occurrences m of event A to the total number of trials n is called the relative frequency of event A.*

Thus, denoting by $W_n(A)$ the relative frequency of event A in n trials, we shall have

$$W_n(A) = \frac{m}{n}. \tag{233-1}$$

Obviously,

$$0 \leqslant W_n(A) \leqslant 1.$$

From formula (233-1) we get

$$m = W_n(A)\, n, \tag{233-2}$$

i.e. *the number of occurrences of event A is equal to its relative frequency multiplied by the number of trials.*

In one-type mass trials in many cases we come across *stability of the relative frequency of an event*, i.e. as the number of trials $n \to \infty$, the relative frequency $W_n(A)$ of event A fluctuates about some constant number p. The greater the number of trials carried out, the smaller these fluctuations are, the individual unsuccessful trials being neglected. This number p is called the *probability of event A in the statistical sense.*

Definition 2. *The probability of an event in the statistical sense is understood to be almost certain limit of its relative frequency, as the number of trials increases without bound.*

Thus, it is almost certain that the relative frequency of an event approximately coincides with its statistical probability, provided the number of trials is sufficiently large.

From this point of view, the quantity

$$\mu = np \tag{233-3}$$

represents the mean value of the number of occurrences of event A in n trials.

With broad assumption it is proved that the probabilities of an event in the classical and statistical senses coincide with each other.

Example. As a result of a number of trials it was revealed that, firing 200 shots, the rifleman hits the target 190 times on the average. What is the probability p of hitting the target by the rifleman? How many successful shots is he expected to fire if the total number of shots is equal to 1000?

Applying the statistical definition of probability, we have

$$p = \frac{190}{200} = 0.95 = 95\%.$$

Hence, the number of successful shots (of the total of 1000 shots) approximately amounts to

$$\mu = 1000 \times 0.95 = 950.$$

Sec. 234. The Theorem on Addition of Probabilities

Theorem. *The probability of a sum of two incompatible events is equal to the sum of the probabilities of these events*, i.e. if $AB = 0$, then

$$P(A + B) = P(A) + P(B). \tag{234-1}$$

Proof. Let, of the total number n of all possible and equally possible elementary outcomes of a trial, m_1 favour event A, and m_2 event B. Since the events A and B are incompatible, the occurrence of event A excludes the occurrence of event B, and conversely. Therefore, the number of successful outcomes of the event $A + B$ is exactly equal to $m_1 + m_2$. Hence, on the basis of the classical definition of probability, we get

$$P(A+B) = \frac{m_1+m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B).$$

Corollary. *The probability of a sum of a finite number of pairwise incompatible events is equal to the sum of the probabilities of these events.*

Let, for instance, events A, B, and C be pairwise incompatible, i.e. the events AB, AC, BC are impossible.

We have

$$P(A + B + C) = P[(A + B) + C] = \\ = P(A + B) + P(C) = P(A) + P(B) + P(C).$$

Note. Let now the events A and B be compatible. Then the number of favourable elementary outcomes for the event $A + B$ will be

$$m = m_1 + m_2 - m',$$

where m' is the number of outcomes favourable to the event AB. Indeed, adding the numbers of outcomes m_1 and m_2 favourable to the events A and B, we thus count the outcomes favourable to the event AB twice; consequently, when calculating the number of outcomes for the event $A + B$ the superfluous value m' should be rejected.

Therefore, in the general case, we have

$$P(A+B) = \frac{m_1+m_2-m'}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m'}{n} = \\ = P(A) + P(B) - P(AB). \quad (234\text{-}2)$$

Corollary. Since $P(AB) \geqslant 0$, from (234-2) we have

$$P(A + B) \leqslant P(A) + P(B), \quad (234\text{-}3)$$

i.e. *the probability of a sum of two events never exceeds the sum of the probabilities of these events.*

This assertion is obviously valid for several events as well.

Example. A box contains two white, three red, and five blue balls equal in size. What is the probability of the event in which a ball accidentally taken out of the box will be coloured (not white)?

Let the taking of a red ball out of the box be event A, and the drawing of a blue ball event B. Then the event $A + B$ is taking a coloured ball out of the box. Obviously, we have

$$P(A) = \frac{3}{10}, \quad P(B) = \frac{5}{10}.$$

Since the events A and B are incompatible (only one ball is taken out), by the addition theorem, we have

$$P(A + B) = P(A) + P(B) = \frac{3}{10} + \frac{5}{10} = 0.8.$$

Sec. 235. A Complete Group of Events

Definition. *The system of events*

$$A_1, A_2, \ldots, A_n \quad (235\text{-}1)$$

is called a complete group of events for a given trial if any of its outcomes is one and only one event of this group.

In other words, for a complete group of events (235-1) the following conditions are fulfilled:

(1) the event $A_1 + A_2 + \ldots + A_n$ is certain;

(2) the events A_i and A_j $(i \neq j)$ are pairwise incompatible, i.e. $A_iA_j = O$ $(i \neq j)$, where O is an impossible event.

The simplest example of a complete group of events is the pair of events A and $\overline{A}$.

Theorem. *The sum of the probabilities of events forming a complete group is equal to unity.*

Proof. For a complete group (235-1) the event $A_1 + A_2 + \ldots + A_n = D$ is certain, and the events of this group are pairwise incompatible. Hence, by the addition theorem, we have

$$P(A_1 + A_2 + \ldots + A_n) = P(A_1) + P(A_2) + \ldots + P(A_n). \qquad (235\text{-}2)$$

But

$$P(A_1 + A_2 + \ldots + A_n) = P(D) = 1,$$

therefore, from (235-2) we have

$$P(A_1) + P(A_2) + \ldots + P(A_n) = 1.$$

Sec. 236. The Theorem on Multiplication of Probabilities

Definition 1. *The probability of event A, on condition that event B has occurred, is called the conditional probability of event A and is denoted as follows:*

$$P(A/B) = P_B(A). \qquad (236\text{-}1)$$

Note. The probability of each event A in a given trial is associated with the presence of a certain complex of conditions. In defining the conditional probability we assume that this complex of conditions necessarily comprises event B. Thus, we actually have another, more burdensome, complex of conditions corresponding to a trial under new circumstances. The probability $P_B(A)$ of the occurrence of event A under these new conditions is called its *conditional probability* in contrast to the probability $P(A)$ which will be called the *unconditional probability* of event A.

Example. An urn contains seven white and three black balls. What is the probability of: (1) drawing a white ball from the urn (event A); (2) drawing a white ball from the urn after one ball is removed from it, white (event B) or black (event C)?

Here

$$P(A)=\frac{7}{10}=0.7;$$

$$P_B(A)=\frac{6}{9}=0.666\ldots;$$

$$P_C(A)=\frac{7}{9}=0.777\ldots$$

Thus, the conditional probability of an event can be both less and more than the probability of this event.

Definition 2. *Two events A and B are called independent if the probability of either of them does not depend on the occurrence or non-occurrence of the other*, i.e.

$$P(A) = P_B(A) = P_{\bar{B}}(A) \tag{236-2}$$

and

$$P(B) = P_A(B) = P_{\bar{A}}(B). \tag{236-2'}$$

Otherwise the events are called *dependent*.

Theorem 1. *The probability of a product (coincidence) of two events A and B is equal to the product of the probability of one of them by the conditional probability of the other in the assumption that the first event takes place*, i.e.

$$P(AB) = P(A)\,P_A(B). \tag{236-3}$$

Proof. Let event A be favoured by m and the event AB by k equally possible elementary outcomes of their total amount n. Then

$$P(A)=\frac{m}{n}, \quad P(AB)=\frac{k}{n}. \tag{236-4}$$

But if event A has occurred, then in this situation only those m elementary outcomes are possible which favoured event A, k of them, obivously, being favourable to event B. Thus

$$P_A(B)=\frac{k}{m}.$$

Hence, by (236-4), we have

$$P(AB)=\frac{k}{n}=\frac{m}{n}\cdot\frac{k}{m}=P(A)\,P_A(B). \tag{236-5}$$

The theorem has been proved.

Since $BA = AB$, we also have

$$P(AB) = P(BA) = P(B)\cdot P_B(A). \tag{236-6}$$

Note. Formula (236-5) remains formally true if event A is impossible.

Corollary. *For any two events A and B the following equality holds*

$$P(A)\,P_A(B) = P(B)\,P_B(A). \tag{236-7}$$

Theorem 2. *The probability of simultaneous occurrence of two independent events A and B is equal to the product of the probabilities of these events:*

$$P(AB) = P(A)\,P(B). \tag{236-8}$$

Indeed, setting $P_A(B) = P(B)$, we get (236-8) from (236-5).

Example. The probability of hitting the target by the first rifleman (event A) is equal to 0.9, and the probability of hitting the target by the second rifleman (event B) is equal to 0.8. What is the probability that the target will be hit at least by one rifleman?

Let C be the event we are interested in; the opposite event $\overline{C}$, obviously, consists in that both riflemen failed to hit the target. Thus, $\overline{C} = \overline{A}\,\overline{B}$. Since the events $\overline{A}$ and $\overline{B}$ are independent (the riflemen do not hinder each other in firing!), we have

$$P(\overline{C}) = P(\overline{A})\cdot P(\overline{B}) = [1 - P(A)]\cdot[1 - P(B)] =$$
$$= (1 - 0.9)\cdot(1 - 0.8) = 0.1\cdot 0.2 = 0.02.$$

Hence, the probability of the event in which the target will be hit at least by one rifleman is

$$P(C) = 1 - P(\overline{C}) = 1 - 0.02 = 0.98.$$

Theorem 1 allows a generalization for the case of several events. For instance, for the case of three events A, B, and C we have

$$P(ABC) = P[A\cdot(BC)] = P(A)\cdot P_A(BC) =$$
$$= P(A)\,P_A(B)\,P_{AB}(C). \tag{236-9}$$

Definition 3. *Events are called collectively independent if each of them and any product of the remaining ones (including either all the remaining events, or a part of them) are independent events.*

Collectively independent events are, obviously, pairwise independent of each other; the converse is false.

Theorem 3. *The probability of a product of a finite number of collectively independent events is equal to the product of the probabilities of these events.*

Indeed, for instance, for three collectively independent events A, B, and C from (236-9), taking into account that

$$P_A(B) = P(B), \quad P_{AB}(C) = P(C),$$

we have

$$P(ABC) = P(A)\,P(B)\,P(C)$$

and so on.

Sec. 237. The Formula of Total Probability

Suppose event A can take place as a result of the occurrence of one and only one event H_i $(i = 1, 2, \ldots, n)$ from a certain complete group of incompatible events

$$H_1, H_2, \ldots, H_n.$$

The events of this group are usually called *hypotheses.*

Theorem. *The probability of event A is equal to the sum of pairwise products of the probabilities of all the hypotheses forming a complete group by the corresponding conditional probabilities of the given event A,* i.e.

$$P(A) = \sum_{i=1}^{n} P(H_i)\,P_{H_i}(A) \tag{237-1}$$

(*the formula of total probability*), where

$$\sum_{i=1}^{n} P(H_i) = 1. \tag{237-2}$$

Proof. Since

$$A = H_1A + H_2A + \ldots + H_nA,$$

and in view of the incompatibility of the events $H_1, H_2, \ldots, H_n$ the events $H_1A, H_2A, \ldots, H_nA$ are also incompatible, by the theorem of addition and multiplication of probabilities, we have

$$P(A) = \sum_{i=1}^{n} P(H_iA) = \sum_{i=1}^{n} P(H_i)\,P_{H_i}A,$$

which is what we had to prove.

Example. A shop is supplied with the goods manufactured by three factories whose relative quotas are: I—50%, II—30%, and III—20%. The spoilage of the supplied goods is characterized by the following respective percentage: I—2%, II—3%, and III—5%. What is the probability that an article perchased accidentally at this shop will turn out to be of good quality (event A)?

The following three hypotheses are possible here: H_1, H_2, H_3 meaning that the perchased thing was made at factories I, II, and III, respectively. The system of these hypotheses is, obviously, complete,

and their probabilities are

$$P(H_1) = 0.5, \quad P(H_2) = 0.3, \quad P(H_3) = 0.2.$$

The corresponding conditional probabilities of event A are

$$P_{H_1}(A) = 1 - 0.02 = 0.98,$$

$$P_{H_2}(A) = 1 - 0.03 = 0.97,$$

$$P_{H_3}(A) = 1 - 0.05 = 0.95.$$

By the formula of total probability, we have

$$P(A) = 0.5 \cdot 0.98 + 0.3 \cdot 0.97 + 0.2 \cdot 0.95 = 0.971.$$

Sec. 238. Bayes' Formula

Consider the following problem: there is a complete group of incompatible hypotheses

$$H_1, H_2, \ldots, H_n,$$

whose probabilities $P(H_i)$ $(i = 1, 2, \ldots, n)$ are known before the trial (*a priori probabilities*). A trial is carried out and as a result of it the occurrence of event A is registered, and it is known that certain probabilities $P_{H_i}(A)$ $(i = 1, 2, \ldots, n)$ were assigned to this event by our hypotheses. What will be the probabilities of these hypotheses after the trial (*a posteriori probabilities*)?

For instance, the hypotheses denying the occurrence of event A should, obviously, be discarded. In general, the problem consists in that, having new information, we must reestimate the probabilities of our hypotheses.

In other words, we have to determine the conditional probabilities

$$P_A(H_i) \quad (i = 1, 2, \ldots, n).$$

By the multiplication theorem, we have

$$P(AH_i) = P(A) \cdot P_A(H_i) = P(H_i) \cdot P_{H_i}(A);$$

hence

$$P_A(H_i) = \frac{P(H_i) P_{H_i}(A)}{P(A)} \qquad (i = 1, 2, \ldots, n). \tag{238-1}$$

To find the probability $P(A)$, we can utilize the formula of total probability

$$P(A) = \sum_{j=1}^{n} P(H_j) P_{H_j}(A). \tag{238-2}$$

Hence we have the formula for the probabilities of the hypotheses after the trial (*Bayes' formula*)

$$P_A(H_i) = \frac{P(H_i)\, P_{H_i}(A)}{\sum_{j=1}^{n} P(H_j)\, P_{H_j}(A)} \qquad (i = 1, 2, \ldots, n). \tag{238-3}$$

Example. The probability of hitting an airplane by one shot is equal to 0.2 for the first antiaircraft-gun detachment (event A) and to 0.1 for the second (event B). Either of the guns fired one shell, and, as a result, one of them hit the plane (event C). What is the probability that the successfull shot belongs to the first detachment?

Prior to the trial the following four hypotheses are possible: $H_1 = AB$, $H_2 = A\bar{B}$, $H_3 = \bar{A}B$, and $H_4 = \bar{A}\bar{B}$. These hypotheses form a complete group of events.

With the detachments operating independently, the probabilities of these events are respectively equal to

$$P(H_1) = 0.2 \times 0.1 = 0.02,$$
$$P(H_2) = 0.2 \times 0.9 = 0.18,$$
$$P(H_3) = 0.8 \times 0.1 = 0.08,$$
$$P(H_4) = 0.8 \times 0.9 = 0.72,$$

where $P(H_1) + P(H_2) + P(H_3) + P(H_4) = 1$.

The conditional probabilities of event C for the four mentioned hypotheses will be

$$P_{H_1}(C) = 0, \quad P_{H_2}(C) = 1,$$
$$P_{H_3}(C) = 1, \quad P_{H_4}(C) = 0.$$

Hence the hypotheses H_1 and H_4 are eliminated, and the probabilities of the hypotheses H_2 and H_3 are computed by Bayes' formula

$$P_C(H_2) = \frac{0.18 \cdot 1}{0.18 \cdot 1 + 0.08 \cdot 1} \approx 0.7,$$

$$P_C(H_3) = \frac{0.08 \cdot 1}{0.18 \cdot 1 + 0.08 \cdot 1} \approx 0.3.$$

Thus, with the probability of 0.7 we can assert that the successful shot belongs to the first detachment.

B. REPEATED INDEPENDENT TRIALS

Sec. 239. Elements of Combinatorial Analysis

Consider the set of n different elements

$$a_1, a_2, \ldots, a_n.$$

An arbitrary ordered selection (possible with repetitions) of these elements

$$a_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha m} \qquad (1 \leqslant \alpha_k \leqslant n; \quad k = 1, \ldots, m)$$

will be called the *combination.*

For instance, in tenfold tossing of a coin, falling heads (H) and falling tails (T) can give a combination

HHHHTTHTTT.

Definition 1. *Permutations of n elements taken m at a time* ($m \leqslant n$) *are defined as their combinations each of which contains exactly m different elements* (*chosen from the given elements*) *and which differ either by the elements themselves, or by the order of the elements.*

Let us determine the number A_n^m of permutations of n elements taken m at a time.

Let $a_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha_m}$ be all possible permutations containing m elements. We shall construct these permutations in succession. First we are going to determine a_{α_1}, i.e. the first element of the permutation. Obviously, from the given set of n elements it can be chosen in n different ways. After the first element a_{α_1} is chosen, for the second element a_{α_2} there remains $n - 1$ ways of choice, and so on. Since each such choice yields a new permutation, we may freely combine all these choices with one another. Therefore, we have

$$A_n^m = n\,(n-1)\,(n-2) \ldots [n - (m-1)]. \tag{239-1}$$

Introducing the notation for the *factorial*

$$n! = 1 \cdot 2 \cdot 3 \ldots n,$$

we can write formula (239-1) in the following form:

$$A_n^m = \frac{n!}{(n-m)!}. \tag{239-2}$$

Definition 2. *Sets of n elements each of which contains all n elements and which differ only in the order of elements are called permutations.*

Obviously, the number of permutations of n elements is equal to

$$A_n^n = n\,(n-1)\,(n-2) \ldots [n - (n-1)] = n!. \tag{239-3}$$

It is conventionally regarded that $0! = 1$.

Definition 3. *Combinations of n elements taken m at a time are such their arrangements each of which contains exactly m given elements and which differ by at least one element.*

Let us denote by C_n^m the number of combinations of n elements taken m at a time.

Consider all permissible combinations of our elements

$$a_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha_m}.$$

Arranging their elements in $m!$ possible ways, we shall, obviously, obtain all permutations of n elements taken m at a time. Thus, we have the following formula

$$C_n^m \cdot m! = A_n^m;$$

hence

$$C_n^m = \frac{A_n^m}{m!} = \frac{n(n-1)\ldots[n-(m-1)]}{m!}. \tag{239-4}$$

Formula (239-4) can also be represented in the form

$$C_n^m = \frac{n!}{m!\,(n-m)!}. \tag{239-5}$$

The symbol C_n^m possesses the following obvious property

$$C_n^m = C_n^{n-m}, \tag{239-6}$$

which will also be true for $m = 0$ if we set $C_n^0 = 1$.

The numbers C_n^m are the coefficients in the *binomial formula*

$$(p+q)^n = p^n + C_n^1 p^{n-1} q + C_n^2 p^2 q^{n-2} + \ldots + q^n$$

and therefore they are frequently called the *binomial coefficients* (cf. Sec. 89).

Example. A batch of ten workpieces contains one non-standard piece. What is the probability of the hypothesis that all five pieces randomly selected from this batch will be standard (event A)?

Here the number of all random selections $n = C_{10}^5$, and the number of those favourable to event A is $m = C_9^5$. Thus, the sought-for probability is equal to

$$P(A) = \frac{C_9^5}{C_{10}^5} = \frac{9!}{5!4!} \cdot \frac{5!5!}{10!} = \frac{1}{2}.$$

Sec. 240. The Binomial Law of Distribution of Probabilities

Event A is called independent in a given system of trials if the probability of this event in each of them does not depend on the outcomes of other trials.

A sequence of repeated independent trials in each of which a given event A has one and the same probability $P(A) = p$ independent of the number of trial is termed the *Bernoulli scheme.*

Thus, in the Bernoulli scheme there are only two outcomes for each trial: (1) event A (success) and (2) event $\bar{A}$ (failure), with constant probabilities $P(A) = p$, and $P(\bar{A}) = q$, and, obviously, $p + q = 1$.

Consider the following problem: under the conditions of the Bernoulli scheme, determine the probability $P_n(m)$ $(0 \leqslant m \leqslant n)$ that in n trials event A, having one and the same probability $P(A) = p$ for each separate trial, will occur precisely m times.

Here, favourable sequences of trials have the form

$$A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_n},$$

where $A_{\alpha_i} = A$ or $\bar{A}$ $(i = 1, 2, \ldots, n)$, and event A occurs precisely m times, and event $\bar{A}$ exactly $(n - m)$ times. Since the trials are independent, the probability of realization of one such favourable series is equal to

$$p^m q^{n-m},$$

where $p = P(A)$, $q = P(\bar{A}) = 1 - p$. All favourable series are obtained as a result of choosing m different numbers of trials from the total of n numbers and, consequently, their number is equal to C_n^m. Hence, applying the theorem of addition of probabilities for the case of incompatible events, for the probability of the occurrence of event A exactly m times in n trials, we get the *Bernoulli formula*

$$P_n(m) = C_n^m p^m q^{n-m} = \frac{n!}{m!\,(n-m)!}\, p^m q^{n-m}. \qquad (240\text{-}1)$$

This formula is called also *binomial*, since its right-hand member represents $(m + 1)$th term of the binomial formula

$$(q + p)^n = C_n^0 q^n + C_n^1 p q^{n-1} + C_n^2 p^2 q^{n-2} + \ldots + C_n^n p^n.$$

Hence we obtain the binomial distribution of the probabilities of the number of occurrences of event A in n independent trials:

$$1 = (q + p)^n = P_n(0) + P_n(1) + P_n(2) + \ldots \ldots + P_n(n). \qquad (240\text{-}2)$$

Example. Find the probability that in a tenfold tossing of a coin heads will occur precisely five times.

Here the probability of falling heads in a single trial $p = \frac{1}{2}$, hence $q = 1 - p = \frac{1}{2}$.

By the Bernoulli formula, we have

$$P_{10}(5) = C_{10}^5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^5 = \frac{10!}{5! \cdot 5!} \cdot \left(\frac{1}{2}\right)^{10} = \frac{252}{1024} \approx 0.25.$$

Sec. 241. The Laplace Local Theorem

If the number of trials n is large, then it becomes difficult to carry out calculations using Bernoulli's formula. Laplace obtained an important approximate formula for the probability $P_n(m)$ of the

occurrence of event A exactly m times if n is a sufficiently large number.

Laplace's theorem. *Let $p = P(A)$ be the probability of event A, where $0 < p < 1$. Then the probability that under the conditions of the Bernoulli scheme event A will occur precisely m times in n trials is expressed by the Laplace approximate formula*

$$P_n(m) \approx \frac{1}{\sqrt{2\pi npq}} e^{-\frac{t^2}{2}}, \tag{241-1}$$

where

$$q = 1 - p \quad \text{and} \quad t = \frac{m - np}{\sqrt{npq}}. \tag{241-2}$$

It is proved that the relative error of formula (241-1) tends to zero as $n \to \infty$. This theorem is proved in comprehensive courses of the theory of probability.

In the statistical sense, $\mu = np$ represents the mean value of the number of occurrences m of event A in n trials; thus, $m - np$ is the deviation of the number of occurrences of event A from its mean value. As to the expression $\sigma = \sqrt{npq}$, its theoretical and probability sense will be explained later on (see Sec. 247). Considering σ as a certain scale for deviations in a sequence of n trials, the number t can be obviously imagined as the deviation of the number of occurrences of event A from its mean value measured in this scale.

Introducing the function

$$\varphi_0(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \tag{241-3}$$

(this function is tabulated), formula (241-1) can be rewritten in the form

$$P_n(m) \approx \frac{1}{\sqrt{npq}} \varphi_0(t) = \frac{1}{\sqrt{npq}} \varphi_0\left(\frac{m - np}{\sqrt{npq}}\right). \tag{241-4}$$

Since the function $\varphi_0(t)$ monotonically decreases as $|t| \to \infty$, for one and the same sequence of trials (n is fixed) greater, by absolute value, deviations $m - np$ are less probable than the smaller ones. This assertion, of course, is true for a sufficiently large number of trials, since only in this assumption the Laplace approximate formula was obtained.

Example. The probability of hitting the target by a rifleman at one shot is $p = 0.2$. What is the probability that 100 shots will ensure the necessary twenty hits?

Here $p = 0.2$, $q = 0.8$, $n = 100$, and $m = 20$.

Hence

$$\sqrt{npq} = \sqrt{100 \cdot 0 \cdot 2 \cdot 0.8} = 4$$

and, consequently,

$$t=\frac{m-np}{\sqrt{npq}}=\frac{20-100\cdot 0.2}{4}=0.$$

Taking into account that $\varphi_0(0)=\frac{1}{\sqrt{2\pi}}\approx 0.40$, from formula (241-1), we get

$$P_{100}(20)\approx 0.40\cdot\frac{1}{4}=0.10.$$

One should not be astonished that the value of the probability is small: exactly 20 hits at 100 shots is a comparatively rare event! Almost certain event here is the target hitted *about* 20 times. For instance, the probability P of the inequality $15\leqslant m\leqslant 25$ including 11 values $m=15, 16, \ldots, 24, 25$ is close to unity. This can be checked by computing this probability using the formula

$$P=\sum_{k=15}^{25}P_{100}(k).$$

Sec. 242. The Laplace Integral Theorem

Let us put the following question: what is the probability $P_n(m_1, m_2)$ that, under the conditions of the Bernoulli scheme, event A, having the probability $P(A)=p$ $(0<p<1)$, occurs not less than m_1 times, and not more than m_2 times in n trials?

By the theorem of addition of probabilities for incompatible events, we get

$$P_n(m_1, m_2)=\sum_{m=m_1}^{m_2}P_n(m). \tag{242-1}$$

Hence, approximately using the Laplace local theorem, we shall have

$$P_n(m_1, m_2)\approx\sum_{m=m_1}^{m_2}\frac{1}{\sqrt{npq}}\varphi_0(t_m), \tag{242-2}$$

where

$$t_m=\frac{m-np}{\sqrt{npq}}\qquad(m_1\leqslant m\leqslant m_2) \tag{242-3}$$

and

$$\varphi_0(t)=\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}. \tag{242-4}$$

We have

$$\Delta t_m = t_{m+1} - t_m = \frac{(m+1)-np}{\sqrt{npq}} - \frac{m-np}{\sqrt{npq}} = \frac{1}{\sqrt{npq}}$$

and, consequently,

$$P_n(m_1, m_2) \approx \sum_{m=m_1}^{m_2} \varphi_0(t_m)\,\Delta t_m. \tag{242-5}$$

Sum (242-5) is integral for the function $\varphi_0(t)$ on the segment $t_{m_1} \leqslant t \leqslant t_{m_2}$. As $n \to \infty$, i.e. as $\Delta t_m \to 0$, its limit is the corresponding definite integral. Therefore, regarding n as sufficiently large, we obtain the following approximate formula

$$P_n(m_1, m_2) \approx \int_{t_{m_1}}^{t_{m_2}} \varphi_0(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{t_{m_1}}^{t_{m_2}} e^{-\frac{t^2}{2}}\,dt, \tag{242-6}$$

where

$$t_{m_1} = \frac{m_1 - np}{\sqrt{npq}}, \qquad t_{m_2} = \frac{m_2 - np}{\sqrt{npq}}.$$

This constitutes the contents of the *Laplace integral theorem*. Let us introduce the *standard integral of probabilities* (*Laplace's function*)

$$\Phi_0(x) = \int_0^x \varphi_0(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}}\,dt, \tag{242-7}$$

which is, obviously, an antiderivative for the function $\varphi_0(x)$.

Then, by the Newton-Leibniz formula, from (242-6) we shall have

$$P_n(m_1, m_2) \approx \Phi_0(t_{m_2}) - \Phi_0(t_{m_1}). \tag{242-8}$$

Formula (242-8) approximately yields the probability that the number m of occurrences of event A at n trials satisfies the inequality $m_1 \leqslant m \leqslant m_2$, and, consequently, the random quantity t the inequality $t_{m_1} \leqslant t \leqslant t_{m_2}$. This formula is frequently written as follows:

$$P(m_1 \leqslant m \leqslant m_2) = P(t_{m_1} \leqslant t \leqslant t_{m_2}) \approx \Phi_0(t_{m_2}) - \Phi_0(t_{m_1}) \tag{242-8'}$$

(*the Laplace integral formula*).

The integral $\Phi_0(x)$ is not expressed in terms of elementary functions. It is usually computed with the aid of special tables given in full courses of the theory of probability.

Table of Values of Function $\Phi_0(x)$

x	0	0.5	1	1.5	2	2.5	3
$\Phi(x)$	0	0.192	0.341	0.433	0.477	0.494	0.499

By way of illustration, we set forth a fragment of such a table. The function $\Phi_0(x)$ (Fig. 270) possesses the following properties:
(1) $\Phi_0(0) = 0$;
(2) $\Phi_0(+\infty) = \frac{1}{2}$ (see Sec. 225, Chapter 24);
(3) the function $\Phi_0(x)$ is odd, i.e. $\Phi_0(-x) = -\Phi_0(x)$ (therefore there is no need to give the values of the function $\Phi_0(x)$ for the negative values of the argument in the above table), in particular, $\Phi_0(-\infty) = -\frac{1}{2}$;
(4) $\Phi_0(x)$ is a monotonically increasing function (this follows from the fact that $\Phi_0'(x) = \varphi_0(x) > 0$). For $x > 3$, with an accuracy to three decimal places, we may take $\Phi_0(x) = 0.500$.

Example. The probability of hitting the target with a single shot of one gun is equal to $p = 0.2$. What is the probability that a volley

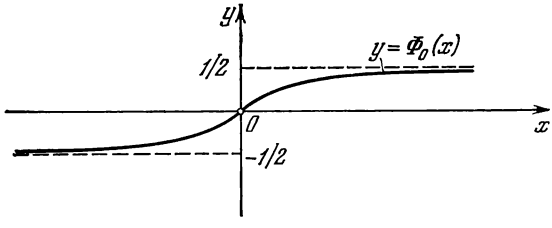


FIG. 270

fired from 100 guns will hit the target not less than 20 times?
Here $n = 100$, $20 \leqslant m \leqslant 100$. We have

$$\sqrt{npq} = \sqrt{100 \cdot 0.2 \cdot 0.8} = 4.$$

Hence

$$t_{20} = \frac{20 - 100 \cdot 0.2}{4} = 0$$

and

$$t_{100} = \frac{100 - 100 \cdot 0.2}{4} = 20.$$

By (242-8), we get

$$P(20 \leqslant m \leqslant 100) = \Phi_0(20) - \Phi_0(0) = 0.500 - 0 = 0.500.$$

Problem. Under the conditions of the Bernoulli scheme, find the probability that the deviation of the relative frequency $\frac{m}{n}$ of event A from its probability $P(A) = p$ $(0 < p < 1)$ does not exceed (by absolute value) a given number $\varepsilon > 0$.

We shall denote this probability in the following way:

$$P\left(\left|\frac{m}{n} - p\right| \leqslant \varepsilon\right).$$

From the inequality

$$\left|\frac{m}{n} - p\right| \leqslant \varepsilon \tag{242-9}$$

we obtain the equivalent inequality

$$|m - np| \leqslant n\varepsilon$$

and, hence

$$-n\varepsilon \leqslant m - np \leqslant n\varepsilon.$$

Whence, setting

$$t = \frac{m - np}{\sqrt{npq}},$$

we shall have

$$-t_1 = -\varepsilon\sqrt{\frac{n}{pq}} \leqslant t \leqslant \varepsilon\sqrt{\frac{n}{pq}} = t_1.$$

By virtue of (242-8′), taking into account that the function $\Phi_0(x)$ is odd, we find

$$P\left(\left|\frac{m}{n} - p\right| \leqslant \varepsilon\right) = P(-t_1 \leqslant t \leqslant t_1) = \Phi_0(t_1) - \Phi_0(-t_1) =$$

$$= 2\Phi_0(t_1) = 2\Phi_0\left(\varepsilon\sqrt{\frac{n}{pq}}\right).$$

Since $\Phi_0(+\infty) = \frac{1}{2}$, then $2\Phi_0\left(\varepsilon\sqrt{\frac{n}{pq}}\right) \to 1$ as $n \to \infty$.

Thus, *under the conditions of the Bernoulli scheme, no matter how small is* $\varepsilon > 0$, *with a probability arbitrarily close to unity, we may expect that with a sufficiently large number of trials n the deviation of the relative frequency of the occurrence of event A from its probability will be (by absolute value) less than* ε (the law of large numbers in the Bernoulli form).

Sec. 243. Poisson's Theorem

Let there be carried a series of n independent trials ($n = 1, 2, 3, \ldots$), and the probability of the occurrence of a given event A in this series $P(A) = p_n > 0$ depends on its number n and tends to zero as $n \to \infty$ (a sequence of rare events). Let us assume that for each series the mean value of the number of occurrences of event A is constant, i.e.

$$np_n = \mu = const; \tag{243-1}$$

hence

$$p_n = \frac{\mu}{n}. \tag{243-2}$$

By virtue of the binomial formula (Sec. 240) for the probability of the occurrence of event A in the nth series exactly m times, we have the expression

$$P_n(m) = C_n^m p_n^m (1 - p_n)^{n-m} = C_n^m \left(\frac{\mu}{n}\right)^m \left(1 - \frac{\mu}{n}\right)^{n-m}. \tag{243-3}$$

Let m be fixed and $n \to \infty$. Then

$$C_n^m \left(\frac{\mu}{n}\right)^m = \frac{n(n-1)(n-2)\ldots[n-(m-1)]}{m!\, n^m} \mu^m =$$

$$= \frac{\mu^m}{m!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{m-1}{n}\right) \to \frac{\mu^m}{m!}.$$

Besides (see Sec. 55, Chapter 7), taking into account that

$$\lim_{\alpha \to 0} (1 + \alpha)^{\frac{1}{\alpha}} = e$$

and, consequently,

$$\lim_{\alpha \to 0} (1 - \alpha)^{\frac{1}{\alpha}} = e^{-1},$$

we have

$$\left(1 - \frac{\mu}{n}\right)^{n-m} = \left[\left(1 - \frac{\mu}{n}\right)^{\frac{n}{\mu}}\right]^{\mu} \left(1 - \frac{\mu}{n}\right)^{-m} \to e^{-\mu} \cdot 1 = e^{-\mu},$$

if $n \to \infty$.

Thus, passing to the limit in formula (243-3) as $n \to \infty$, we get

$$\lim_{n \to \infty} P_n(m) = \lim_{n \to \infty} C_n^m \left(\frac{\mu}{n}\right)^m \cdot \lim_{n \to \infty} \left(1 - \frac{\mu}{n}\right)^{n-m} = \frac{\mu^m}{m!} e^{-\mu}. \tag{243-4}$$

If n is large, then the probability $P_n(m)$ differs only slightly from its limit (243-4). Hence for large n's for the required probability

$P_n(m)$ we have the approximate *Poisson formula*

$$P_n(m) \approx \frac{\mu^m}{m!} e^{-\mu}, \tag{243-5}$$

where $\mu = np_n$ (*Poisson's theorem*).

In general, the Poisson formula should be applied in the cases when the number of trials is large, the probability of an event $p_n = p$ is small, and $\mu = np$ is neither small, nor large.

Poisson's formula is used in the theory of mass-scale service*.

Example. In manufacturing some mass-scale products the probability of the occurrence of one non-standard article is equal to 0.01. What is the probability that in a batch of 100 articles of this production 2 articles will be non-standard?

Here the probability $p = 0.01$ is small, the number $n = 100$ is large, and

$$\mu = np = 100 \cdot 0.01 = 1.$$

Using Poisson's law for the sought-for probability, we obtain the following value:

$$P_{100}(2) \approx \frac{\mu^2}{2!} e^{-\mu} = \frac{1}{2} e^{-1} = 0.184.$$

C. RANDOM VARIABLES AND THEIR NUMERICAL CHARACTERISTICS

Sec. 244. A Random Discrete Variable and Its Distribution Law

A variable is called *random* if it attains its values depending on the outcomes of some trial, and it has a unique value for each elementary outcome. A random variable is said to be *discrete* (in a narrow sense) if the set of all its possible values is finite.

Geometrically, the set of all possible values of a discrete random variable represents a finite system of points on the number axis.

Let X be a discrete random variable whose possible and the only possible values are the numbers

$$x_1, x_2, \ldots, x_n.$$

Let us denote by

$$p_i = P(X = x_i) \qquad (i = 1, 2, \ldots, n)$$

the probabilities of these values (i.e. p_i is the probability of the event consisting in that X takes on the value x_i).

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* As the queueing theory is usually termed in the Soviet literature.

The events $X = x_i$ $(i = 1, 2, \ldots, n)$, obviously, form a complete group of events, therefore

$$p_1 + p_2 + \ldots + p_n = 1. \qquad (244\text{-}1)$$

Definition. *The correspondence between all possible values of a discrete random variable and their probabilities is called the law of distribution of the given random variable.*

In the simplest cases it is convenient to specify the law of distribution of a discrete random variable X in a tabular way:

X	x_1	x_2	$\ldots$	x_n
p	p_1	p_2	$\ldots$	p_n

The first row of this table contains all possible values of the random variable, and the second their probabilities.

It should be noted that, if it is expedient, the table of the values of the discrete random variable X can always be formally supplemented with a collection of any numbers regarding them as the values of X with the probabilities equal to zero.

Example 1. Raffled in a money lottery are the following prizes: one 1000-rouble prize, ten 100-rouble prizes, and a hundred of 1-rouble prizes, the total number of tickets being equal to 10 000. Find the law of distribution of a random prize X for the holder of one lottery ticket.

Here, the possible values for X are

$$x_1 = 1000, \quad x_2 = 100, \quad x_3 = 1, \quad x_4 = 0.$$

Their respective probabilities will be

$$p_1 = 0.0001, \quad p_2 = 0.001, \quad p_3 = 0.01$$

and

$$p_4 = 1 - (p_1 + p_2 + p_3) \approx 0.9889.$$

The distribution law for the prize X can be represented by a table:

X	1000	100	1	0
p	0.0001	0.001	0.01	0.9889

Example 2. The number of the occurrences of event A in n independent trials may be considered as a random variable X with the

values $m = 0, 1, 2, \ldots, n$. The law of distribution of this variable is given by the binomial formula

$$p_m = P(X = m) = C_n^m p^m q^{n-m},$$

where $p = P(A)$, $q = P(\bar{A}) = 1 - p$ (binomial distribution).

In particular, if p is small and n is large, and $pn = \mu$ is a bounded quantity enclosed between two fixed positive numbers, then there holds true the Poisson approximate distribution

$$p_m = P(X = m) \approx \frac{\mu^m}{m!} e^{-\mu} \qquad (m = 0, 1, 2, \ldots, n).$$

Sec. 245. Mathematical Expectation

Definition. *Mathematical expectation of a discrete random variable is the sum of pairwise products of all its possible values by their probabilities.*

If $x_1, x_2, \ldots, x_n$ is a complete collection of all the values of a discrete random variable X and $p_1, p_2, \ldots, p_n$ are their corresponding probabilities, then denoting mathematical expectation by M, we shall have

$$M(X) = \sum_{i=1}^{n} x_i p_i, \tag{245-1}$$

where

$$\sum_{i=1}^{n} p_i = 1. \tag{245-2}$$

Obviously, the mathematical expectation of a random variable will remain unchanged if the table of its values is supplemented with a finite number of any numbers, considering that the probabilities of these numbers are equal to zero.

The mathematical expectation $M(X)$ of a random variable is a constant and, therefore, represents a numerical characteristic of the random variable X.

Example. Find the mathematical expectation of the prize X in Example 1 from the preceding section.

Using the table given there, we have

$$M(X) = 1000 \cdot 0.0001 + 100 \cdot 0.001 + 1 \cdot 0.01 + 0 \cdot 0.9889 = \\ = 0.21 \text{ rouble} = 21 \text{ kopecks}.$$

As it is not difficult to grasp, $M(X) = 21$ kop. is a "fair" price of one ticket.

Note 1. Individual terms $x_i p_i$ ($i = 1, 2, \ldots, n$) of sum (245-1) represent the mathematical expectations of random variables X_i ($i = 1, 2, \ldots, n$) whose possible values are x_i and 0 with the probabilities p_i and $1 - p_i$, respectively.

Note 2. Let $\underline{x} = \min\limits_i x_i$ and $\bar{x} = \max\limits_i x_i$ be the least and the greatest possible values of the random variable X, respectively. We have

$$\underline{x} = \sum_{i=1}^{n} \underline{x} p_i \leqslant \sum_{i=1}^{n} x_i p_i \leqslant \sum_{i=1}^{n} \bar{x} p_i = \bar{x}.$$

Thus

$$\underline{x} \leqslant M(X) \leqslant \bar{x}. \tag{245-3}$$

Thus, the mathematical expectation of a random variable is its certain mean value.

Note 3. The mathematical expectation of the number of occurrences of event A in one trial coincides with the probability of this event $P(A) = p$.

Indeed, let X be the number of the occurrences of event A in the given trial. The random variable X can attain two values: $x_1 = 1$ (event A has occurred) with the probability $p_1 = p$ and $x_2 = 0$ (event A has not occurred) with the probability $p_2 = 1 - p = q$.

Therefore

$$M(X) = 1 \cdot p + 0 \cdot q = p.$$

Sec. 246. Basic Properties of Mathematical Expectation

We are going to indicate the most important properties of mathematical expectation. The relevant proofs will be carried out for discrete random variables. But the appropriate theorems hold true for continuous random variables as well, therefore, when formulating these theorems we shall not mention that the random variables under consideration are discrete.

We have to explain the meaning of the arithmetical operations $X + Y$, $X - Y$, XY, etc., where X and Y are discrete random variables. It is not difficult to give the corresponding definitions.

For instance, the sum $X + Y$ is understood to be a random variable Z whose values are the permissible sums $z_{ij} = x_i + y_j$, where x_i and y_j, are all possible values of the random variables X and Y, respectively, the corresponding probabilities being equal

$$\pi_{ij} = P(Z = z_{ij}) = P(X = x_i)\, P(Y = y_j / X = x_i).$$

If some of the combinations $x_k + y_l$ is impossible, we conventionally set $\pi_{kl} = 0$; this will not affect the mathematical expectation of the sum.

The rest of the expressions are determined analogously.

We also distinguish between independent and dependent random variables. Two random variables are considered to be *independent* if the possible values and the distribution law of either of them are the same at any choice of the permissible values of the other. Otherwise they are called *dependent*. Several random variables are called *mutually independent* if the possible values and distribution laws of any of them do not depend on what possible values are attained by the remaining random variables.

Theorem 1. *The mathematical expectation of a constant is equal to this constant*, i.e. if C is a constant, then

$$M(C) = C. \tag{246-1}$$

Proof. The constant C may be regarded as a discrete random variable attaining only one possible value C with the probability $p = 1$. Therefore

$$M(C) = C \cdot 1 = C.$$

Theorem 2. *The mathematical expectation of a sum of two (or several) random variables is equal to the sum of the mathematical expectations of these variables*, i.e. if X and Y are random variables, then

$$M(X + Y) = M(X) + M(Y) \tag{246-2}$$

and so on.

Proof. (1) Let the random variable X take on the values x_i with the probabilities p_i ($i = 1, 2, \ldots, n$), and the random variable Y attain the values y_j with the probabilities p'_j ($j = 1, 2, \ldots, m$). Then the possible values of the random variable $X + Y$ will be the sums $x_i + y_i$ whose probabilities are equal to

$$\pi_{ij} = P(X=x_i)\, P(Y=y_j/X=x_i) = P(Y=y_j)\, P(X=x_i/Y=y_j). \tag{246-3}$$

As it was mentioned above, all combinations (i, j) ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$) may be regarded as permissible, and if the sum $x_i + y_j$ is impossible, then we put $\pi_{ij} = 0$.

We have

$$M(X+Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i + y_j)\, \pi_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i \pi_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{m} y_j \pi_{ij}. \tag{246-4}$$

Taking advantage of the obvious properties of a sum: (1) it is independent of the order of terms, and (2) a factor which does not depend on the summation index can be taken outside the summation sign, from (246-4) we obtain

$$M(X+Y)=\sum_{i=1}^{n} x_i \sum_{j=1}^{m} \pi_{ij}+\sum_{j=1}^{m} y_j \sum_{i=1}^{n} \pi_{ij}. \tag{246-5}$$

The sum $\sum_{j=1}^{m} \pi_{ij}$ represents the probability of the event consisting in that the random variable X attains the value x_i, provided the random variable Y takes on one of its possible values (which is certain). This complex event is, obviously, equivalent to that X attains the value x_i, and therefore

$$\sum_{j=1}^{m} \pi_{ij}=P(X=x_i)=p_i.$$

Analogously,

$$\sum_{i=1}^{n} \pi_{ij}=P(Y=y_j)=p'_j.$$

Then, from formula (246-5) we get

$$M(X+Y)=\sum_{i=1}^{n} x_i p_i+\sum_{j=1}^{m} y_j p'_j=M(x)+M(Y),$$

which was to be proved.

(2) For several random variables, for instance, for three (X, Y, and Z), we have

$$M(X+Y+Z)=M[(X+Y)+Z]=$$
$$=M(X+Y)+M(Z)=M(X)+M(Y)+M(Z),$$

and so on.

Corollary. If C is a constant, then

$$M(X+C)=M(X)+C.$$

Theorem 3. *The mathematical expectation of a product of two independent random variables is equal to the product of their mathematical expectations*, i.e.

$$M(XY)=M(X)\,M(Y), \tag{246-6}$$

where X and Y are independent random variables.

Proof. Let (x_i, p_i) $(i=1, 2, \ldots, n)$ and (y_j, p'_j) $(j=1, 2, \ldots \ldots, m)$ be the laws of distribution of the random variables X and Y, respectively. Since X and Y are independent, a complete set

of values of the random variable XY consists of all products of the form x_iy_j ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$), the probabilities of these values, by the multiplication theorem for independent events, are equal to $p_ip'_j$.

We have

$$M(XY) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_iy_jp_ip'_j = \sum_{i=1}^{n} x_ip_i \sum_{j=1}^{m} y_jp'_j =$$

$$= \sum_{i=1}^{n} x_ip_iM(Y) = M(X)\,M(Y), \quad (246\text{-}7)$$

which was to be proved.

Corollary 1. *The mathematical expectation of a product of several mutually independent random variables is equal to the product of the mathematical expectations of these variables.*

Indeed, for instance, for three mutually independent random variables X, Y, Z we have

$$M(XYZ) = M[(XY)Z] = M(XY)\,M(Z) = M(X)\,M(Y)M(Z),$$

and so on.

Corollary 2. *A constant factor may be taken outside the sign of mathematical expectation.*

If C is a constant and X is any random variable, then, taking into account that C and X are independent, on the basis of Theorem 1, we get

$$M(CX) = M(C)\,M(X) = CM(X).$$

Corollary 3. *The mathematical expectation of a difference of any two variables X and Y is equal to the difference of the mathematical expectations of these variables*, i.e.

$$M(X - Y) = M(X) - M(Y).$$

Indeed, using the theorem about the sum of mathematical expectations and Corollary 2, we obtain

$$M(X - Y) = M[X + (-Y)] = M(X) + M(-Y) =$$
$$= M(X) + (-1)\,M(Y) = M(X) - M(Y).$$

Sec. 247. Variance

Let X be a random variable and $M(X)$ its mathematical expectation (mean value). The random variable $X - M(X)$ is then called the *deviation*.

Theorem 1. *For any random variable X the mathematical expectation of its deviation is equal to zero*, i.e.

$$M[X - M(X)] = 0.$$

Proof. Indeed, taking into consideration that $M(X)$ is a constant, we have

$$M[X - M(X)] = M(X) - M[M(X)] = M(X) - M(X) = 0.$$

Definition. *The variance (also called dispersion) of a random variable is defined as the mathematical expectation of the square of the deviation of this variable from its mathematical expectation.*

Hence, denoting the variance by D, for a random variable X we shall have

$$D(X) = M\{[X - M(X)]^2\}. \tag{247-1}$$

It is obvious that the variance of a random variable is constant, i.e. it is a numerical characteristic of this variable.

If a random variable X has the distribution law (x_i, p_i) $(i = 1, 2, \ldots, n)$, then, denoting, for brevity, $M(X) = \mu$, from (247-1) we shall have

$$D(X) = \sum_{i=1}^{n} (x_i - \mu)^2 p_i. \tag{247-2}$$

The square root of the variance $D(X)$ is called the *root mean square deviation* σ (or the standard) of this variable:

$$\sigma(X) = \sqrt{D(X)}. \tag{247-3}$$

Example. Let the law of distribution of a random variable be given by the following table:

X	4	10	20
p	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Determine the mathematical expectation $M(X)$, variance $D(X)$, and root mean square deviation $\sigma(X)$.

We have

$$M(X) = 4 \cdot \frac{1}{4} + 10 \cdot \frac{1}{2} + 20 \cdot \frac{1}{4} = 11;$$

hence

$$D(X) = (4-11)^2 \cdot \frac{1}{4} + (10-11)^2 \cdot \frac{1}{2} + (20-11)^2 \cdot \frac{1}{4} = 33$$

and

$$\sigma(X) = \sqrt{D(X)} = \sqrt{33} \approx 5.75.$$

The variance $D(X)$ serves as a measure of the dispersion or scattering of the values of the discrete random variable X. Indeed, let

$D(X)$ be small. Then, from (247-2) we obtain that all the terms $(x_i - \mu)^2 \cdot p_i$ $(i = 1, 2, \ldots, n)$ are also small. Hence it follows that if we pay no attention to the values having a small probability (such values are practically impossible), then all the remaining values of x_i deviate only slightly from μ. Thus, at small variance $D(X)$ it is almost certain that the values of a random variable *are concentrated* about its mathematical expectation (with the exception of, perhaps, a comparatively small number of individual values). In particular, if $D(X) = 0$, then, obviously, $X = \mu$ and the random variable represents a point on the number axis. If $D(X)$ is great, then the concentration of the values of the random variable X about some centre is excluded.

Theorem 2. *The variance of a random variable is equal to the difference between the mathematical expectation of the square of this variable and the square of its mathematical expectation*, i.e.

$$D(X) = M(X^2) - [M(X)]^2. \tag{247-4}$$

Proof. Using the basic theorems on the mathematical expectations of random variables, we have

$$D(X) = M\{[X - M(X)]^2\} = M\{X^2 - 2XM(X) + [M(X)]^2\} =$$
$$= M(X^2) - 2M(X) \cdot M(X) + [M(X)]^2 = M(X^2) - [M(X)]^2.$$

Theorem 3. *The variance of a constant is equal to zero.*

Indeed, if C is a constant, then $M(C) = C$ and, consequently,

$$D(C) = M[(C - C)^2] = M(0) = 0.$$

This result is obvious, since a constant quantity is represented by one point on the Ox number axis and has no dispersion.

Theorem 4. *The variance of a sum of two independent random variables X and Y is equal to the sum of the variances of these variables*, i.e.

$$D(X + Y) = D(X) + D(Y). \tag{247-5}$$

Proof. Since

$$M(X + Y) = M(X) + M(Y), \tag{247-6}$$

we have

$$D(X + Y) = M\{[(X + Y) - M(X + Y)]^2\} =$$
$$= M\{[(X - M(X)) + (Y - M(Y))]^2\} =$$
$$= M\{[X - M(X)]^2\} + M\{[Y - M(Y)]^2\} +$$
$$+ 2M\{[X - M(X)] \cdot [Y - M(Y)]\} =$$
$$= D(X) + D(Y) + 2K(X, Y),$$

where

$$K(X, Y) = M\{[X - M(X)] \cdot [Y - M(Y)]\}$$

is the so-called *correlation moment* of the variables X and Y. If the random variables X and Y are independent, then the random variables $X - M(X)$ and $Y - M(Y)$, differing from X and Y by constants, are, obviously, also independent. Therefore, by virtue of Theorem 3 from the preceding section and Theorem 1 of the present section, we have

$$K(X, Y) = M[X - M(X)] \cdot M[Y - M(Y)] = 0$$

and, consequently, formula (247-6) is valid.

Corollary 1. *The variance of a sum of several mutually independent random variables is equal to the sum of the variances of these variables.*

Corollary 2. *If C is a constant, then*

$$D(X + C) = D(X).$$

Thus, the random variables X and $X + C$ have an equal measure of dispersion.

Theorem 5. *A constant factor may be taken outside the variance sign in the squared form*, i.e.

$$D(CX) = C^2 D(X).$$

Proof. If C is a constant factor, then, by Theorem 2, we have

$$D(CX) = M(C^2X^2) - [M(CX)] = \\ = C^2 M(X^2) - C^2[M(X)]^2 = C^2 D(X).$$

Thus, the dispersion of the variable CX is C^2 times the dispersion of the variable X.

Corollary. *The variance of a difference of two independent random variables is equal to the sum of the variances of these variables*, i.e. if the random variables X and Y are independent, then

$$D(X - Y) = D(X) + D(Y).$$

Indeed, on the basis of Theorems 4 and 5, we have

$$D(X - Y) = D[X + (-Y)] = D(X) + D(-Y) = \\ = D(X) + (-1)^2 D(Y) = D(X) + D(Y).$$

The mathematical expectation and variance of a random variable are its principal numerical characteristics.

Problem. Determine the mathematical expectation and variance for the number X of the occurrence of event A in n independent trials in each of which the probability of the event $P(A) = p$ is constant.

The random variable X takes on the values $0, 1, 2, \ldots, n$ and is distributed according to the binomial law

$$p_m = P(X = m) = C_n^m p^m q^{n-m} \qquad (m = 0, 1, 2, \ldots, n), \tag{247-7}$$

where $q = P(\overline{A}) = 1 - p$.

The variable X may be regarded as a sum of independent random variables

$$X = X_1 + X_2 + \ldots + X_n,$$

where X_i $(i = 1, 2, \ldots, n)$ is the number of the occurrences of event A in the ith trial. The random variable X_i attains only two values: 1 if event A occurred in the ith trial, and 0 if event A did not occur in the ith trial. The probabilities of these values are $P(A) = p$ and $P(\bar{A}) = q$. Hence

$$M(X_i) = 1 \cdot p + 0 \cdot q = p \qquad (i = 1, 2, \ldots, n)$$

(see also Sec. 245). Hence, taking advantage of the theorem about the mathematical expectation of a sum, we shall have

$$M(X) = M(X_1) + M(X_2) + \ldots + M(X_n) = np. \qquad (247\text{-}8)$$

Thus, the mathematical expectation of the number of the occurrences of event A under the conditions of the Bernoulli scheme coincides with the "average number" of the occurrences of this event in a given sequence of trials.

For the variance of the random variable X_i we obtain

$$D(X_i) = (1 - p)^2 p + (0 - p)^2 q = q^2 p + p^2 q = pq(q+p) = pq.$$

Hence, by the property of the variance of a sum of independent random variables (Theorem 4), we shall have

$$D(X) = D(X_1) + D(X_2) + \ldots + D(X_n) = npq. \qquad (247\text{-}9)$$

Therefore, the root mean square deviation (the standard)

$$\sigma(X) = \sqrt{D(X)} = \sqrt{npq}. \qquad (247\text{-}10)$$

Formulas (247-8) and (247-9) yield the mathematical expectation and variance for the binomial distribution law.

Note. Now the meaning of the random variable

$$t = \frac{m - np}{\sqrt{npq}}$$

becomes clear in Laplace's approximate formulas (Secs. 241-242), namely, t represents the deviation of the number of occurrences of event A from its mathematical expectation measured in standards (so-called *standardized deviation*).

Consider n discrete pairwise independent random variables $X_1, X_2, \ldots, X_n$ whose variances $D(X_i)$ $(i = 1, 2, \ldots, n)$ are uniformly bounded

$$0 \leqslant D(X_i) \leqslant K \qquad (i = 1, 2, \ldots, n).$$

These variables, perhaps, have a considerable scattering, but their arithmetic mean

$$\hat{X}_n = \frac{1}{n}(X_1 + X_2 + \ldots + X_n)$$

behaves sufficiently "compactly".

Namely, under the above mentioned conditions there takes place the remarkable Chebyshev theorem: *For any positive* $\varepsilon > 0$ *the probability of the inequality*

$$|\hat{X}_n - M(\hat{X}_n)| < \varepsilon,$$

where

$$M(\hat{X}_n) = \frac{1}{n}\sum_{i=1}^{n} M(X_i),$$

is arbitrarily close to unity if the number of random variables n *is sufficiently large*, i.e.

$$\lim_{n\to\infty} P(|\hat{X}_n - M(\hat{X}_n)| < \varepsilon) = 1$$

(the law of large numbers in Chebyshev's form).

Chebyshev's theorem finds its application in error theory, statistics, etc.

Sec. 248. Continuous Random Variables. Distribution Functions

A random variable X will be called *continuous* if all its possible values wholly fill a certain finite or infinite interval $\langle a, b\rangle$ of the number axis. It is assumed that in each trial the random variable X takes on one and only one value $x \in \langle a, b\rangle$. It should be noted that discrete and continuous random variables do not exhaust all types of random variables.

The diversity of random variables is extremely great. The number of assumed values may be finite, countable and uncountable; the values may be distributed discretely or fill the intervals continuously, or not fill the intervals, but be spread out, everywhere dense. In order to specify the probabilities of the values of random variables that are so diversified, and to be able to specify them in one and the same fashion, we introduce into the theory of probability the concept of the *distribution function* of a random variable:

$$\Phi(x) = P(-\infty < X < x), \tag{248-1}$$

which is called an integral distribution law*.

* Here probability is understood in axiomatic sense (see B. Gnedenko, *The Theory of Probability*, Chapter 1, Sec. 8, Mir Publishers, Moscow, 1978).

If we consider the values of the random variable X as points of the number axis Ox, then $\Phi(x)$ represents the probability of the event consisting in that the observed value of the random variable X belongs to the interval $(-\infty, x)$, i.e. is situated to the left of the point x. This interval depends on its right-hand end-point x, and, therefore, the probability is naturally a function of x defined over the entire axis $-\infty < x < +\infty$.

Note that the distribution function has sense also for discrete random variables.

The distribution function $\Phi(x)$ possesses the following properties.

I. The function $\Phi(x)$ is a non-decreasing function of the argument x, i.e. if $x < x'$, then $\Phi(x) \leqslant \Phi(x')$.

Indeed, if $x' > x$, then from the event $X \in (-\infty, x)$ there follows the event $X \in (-\infty, x')$. But then the probability $\Phi(x')$ of the second event is not less than the probability $\Phi(x)$ of the first one (see Sec. 232, Theorem 2).

II. Since $\Phi(x)$ is a probability, the following inequality holds:

$$0 \leqslant \Phi(x) \leqslant 1.$$

III.

$$\Phi(-\infty) = 0, \qquad \Phi(+\infty) = 1.$$

Indeed, the event $X \in (-\infty, -\infty)$ is, obviously, impossible, while the event $X \in (-\infty, +\infty)$ is certain.

Knowing the function of distribution $\Phi(x)$, we can for any interval $[a, b)$ determine $P(a \leqslant X < b)$, i.e. the probability of entering of the random variable into this interval (here, the left-hand end point a of the interval is conventionally included, and the right-hand end point b is not included in this interval).

Indeed, let A denote the event $X \in (-\infty, a)$, B the event $X \in (-\infty, b)$, and C the event $X \in [a, b)$.

Then we, obviously, have

$$B = A + C.$$

Since the events A and C are incompatible, by the addition theorem we get

$$P(B) = P(A) + P(C),$$

hence

$$P(C) = P(B) - P(A),$$

i.e.

$$P(a \leqslant X < b) = \Phi(b) - \Phi(a), \tag{248-2}$$

where $\Phi(b) - \Phi(a) \geqslant 0$ by virtue of Property I.

Thus, *the probability that a random variable X will attain a value belonging to the interval* $[a, b)$ *is equal to the increment of its distribution function on this interval.*

Henceforth, the random function X will be called continuous only if its distribution function $\Phi(x)$ is continuous on the axis $(-\infty, +\infty)$.

Theorem. *The probability (prior to the trial) that a continuous random variable X will take on a preassigned, strictly defined value a is equal to zero.*

Indeed, by formula (248-2), we have

$$P(a \leqslant X < x) = \Phi(x) - \Phi(a). \tag{248-3}$$

Suppose $x \to a$; then in the limit the interval $[a, x)$ will contain the only point a. Besides, by virtue of the continuity of the function $\Phi(x)$ at point a, we have

$$\lim_{x \to a} \Phi(x) = \Phi(a).$$

Passing to the limit as $x \to a$ in (248-3), we get

$$P(a) = \lim_{x \to a} \Phi(x) - \Phi(a) = \Phi(a) - \Phi(a) = 0.$$

Thus, with a continuous distribution function the probability of "hitting a point" is equal to zero.

Corollary. *For a continuous random variable X the following equalities are valid:*

$$P(a \leqslant X \leqslant b) = \Phi(b) - \Phi(a). \tag{248-2'}$$

and

$$P(a < X < b) = \Phi(b) - \Phi(a), \tag{248-2''}$$

where $\Phi(x) = P(a \leqslant X < x)$ *is its distribution function.*

Indeed

$$P(a \leqslant X \leqslant b) = P(a \leqslant X < b) + P(X = b) =$$
$$= [\Phi(b) - \Phi(a)] + 0 = \Phi(b) - \Phi(a).$$

The second equality is proved in a similar way.

Note. In the general case impossible events and events with zero probability may turn out to be non-equivalent.

Let us assume now that for the continuous random variable X its distribution function $\Phi(x)$ has a continuous derivative

$$\Phi'(x) = \varphi(x). \tag{248-4}$$

The function $\varphi(x)$ is called the *probability density* (for a given distribution) or the *differential law of distribution of a random variable* X.

The term 'probability density' has the following meaning. Let $[x, x + dx]$ be an infinitely small interval; then, by (248-2'), we have

$$P(x \leqslant X \leqslant x + dx) = \Phi(x + dx) - \Phi(x).$$

Replacing the infinitesimal increment of the function $\Phi(x + dx) - \Phi(x)$ by its differential $\Phi'(x)\,dx = \varphi(x)\,dx$, we obtain the approximate equality

$$P(x \leqslant X \leqslant x + dx) = \varphi(x)\,dx. \tag{248-5}$$

Thus, the probability density represents the ratio of the probability of getting of a point into an infinitely small interval to the length of this interval.

Since the probability density $\varphi(x)$ is the derivative of a non-decreasing function $\Phi(x)$, it is non-negative: $\varphi(x) \geqslant 0$. In contrast to probability, probability density can attain arbitrarily large values.

Since $\Phi(x)$ is an antiderivative for $\varphi(x)$, by the Newton-Leibniz formula, we have

$$\int_a^b \varphi(x)\,dx = \Phi(b) - \Phi(a).$$

Hence, by virtue of (248-2'), we get

$$P(a \leqslant X \leqslant b) = \int_a^b \varphi(x)\,dx. \tag{248-6}$$

Geometrically, this probability represents the area S of a curvilinear trapezoid bounded by the graph of the probability density

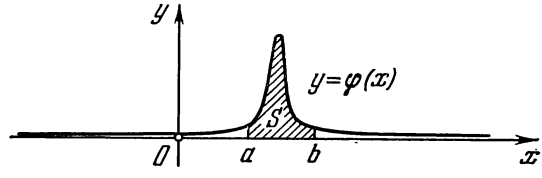

FIG. 271

$y = \varphi(x)$, x-axis, and two ordinates: $x = a$ and $x = b$ (Fig. 271).

Setting $a = -\infty$ and $b = +\infty$, we obtain a certain event $X \in (-\infty, +\infty)$ whose probability is equal to unity. Consequently,

$$\int_{-\infty}^{+\infty} \varphi(x)\,dx = 1. \tag{248-7}$$

Putting in formula (248-6) $a = -\infty$, $b = x$ and denoting, for clearness, the integration variable x by another letter, for instance, by t (that is valid for the definite integral), we get the distribution function

$$\Phi(x) = P(-\infty < X < x) = \int_{-\infty}^{x} \varphi(t)\, dt. \tag{248-8}$$

Sec. 249. Numerical Characteristics of a Continuous Random Variable

We shall consider the infinitesimal interval $[x, x + dx]$ as a "bold-faced point" x of the axis Ox. Then the probability that a random variable X takes on the value coinciding with this "bold-faced point" x is equal to $\varphi(x)\, dx$ and the mathematical expectation of this event is

$$dM = x\varphi(x)\, dx.$$

Regarding the straight line $-\infty < x < +\infty$ as an infinite set of such "bold-faced points", by analogy with definition of the mathematical expectation of a discrete random variable, we obtain a natural definition of the mathematical expectation of a continuous random variable (but here summation is substituted by integration).

Definition. *The mathematical expectation of a continuous random variable X is understood to be the number*

$$M(X) = \int_{-\infty}^{+\infty} x\varphi(x)\, dx \tag{249-1}$$

(*of course, this definition has sense only for such random variables X for which integral* (249-1) *converges*).

For the variance of a continuous random variable X we shall retain the previous definition

$$D(X) = M\{[X - M(X)]^2\}.$$

From (249-1) it follows that

$$D(X) = \int_{-\infty}^{+\infty} [x - M(X)]^2 \varphi(x)\, dx \tag{249-2}$$

(of course, in the assumption that integral (249-2) converges). We may also make use of the formula

$$D(X) = M(X^2) - [M(X)]^2 = \int_{-\infty}^{+\infty} x^2\varphi(x)\, dx - \left[\int_{-\infty}^{+\infty} x\varphi(x)\, dx\right]^2. \tag{249-3}$$

We can prove that the basic properties of the mathematical expectation and variance of discrete random variables are valid for continuous random variables as well.

Let now all possible values of a continuous random variable X entirely fill a closed interval $[a, b]$. Then $\varphi(x) = 0$ for $-\infty < x < a$ and for $b < x < +\infty$, and consequently,

$$M(X) = \int_{-\infty}^{+\infty} x\varphi(x)\,dx = \int_{-\infty}^{a} x\varphi(x)\,dx + \int_{a}^{b} x\varphi(x)\,dx + \int_{b}^{+\infty} x\varphi(x)\,dx = 0 + \int_{a}^{b} x\varphi(x)\,dx + 0 = \int_{a}^{b} \varphi(x)\,dx.$$

Analogously,

$$D(X) = \int_{a}^{b} [x - M(X)]^2 \varphi(x)\,dx,$$

where

$$\int_{a}^{b} \varphi(x)\,dx = 1.$$

Sec. 250. Uniform Distribution

A continuous random variable X all possible values of which fill a finite interval $\langle a, b\rangle$ is said to be *uniformly distributed* if its probability density $\varphi(x)$ is constant on this interval.

In other words, for a uniformly distributed random variable all its possible values are equally possible.

Let, for instance, $X \in [a, b]$. Since in this case $\varphi(x) = \text{const}$ for $x \in [a, b]$ and $\varphi(x) = 0$ for $x \notin [a, b]$, we have

$$\int_{a}^{b} \varphi(x)\,dx = \varphi(x) \int_{a}^{b} dx = (b-a)\,\varphi(x) = 1;$$

hence

$$\varphi(x) = \frac{1}{b-a} \quad \text{for} \quad a \leqslant x \leqslant b.$$

Let $[\alpha, \beta] \in [a, b]$ (Fig. 272). Then

$$p = P(\alpha \leqslant X \leqslant \beta) = \int_{\alpha}^{\beta} \varphi(x)\,dx = \frac{\beta - \alpha}{b-a},$$

i.e.

$$p=\frac{l}{L}, \tag{250-1}$$

where L is the length (a linear measure) of the whole closed interval $[a, b]$ and l the length of the partial interval $[\alpha, \beta]$.

The values of the random variable X, i.e. the points x of the interval $[a, b]$ may be considered as all possible elementary outcomes of

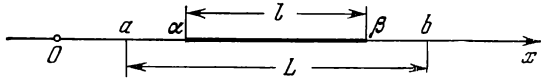

FIG. 272

a certain trial. Let event A consist in that the result of the trial belongs to the interval $[\alpha, \beta] \subset [a, b]$. Then the points of the interval $[\alpha, \beta]$ are successful elementary outcomes of event A.

According to formula (250-1), we have the geometrical definition of probability: *the probability of event A is understood to be the ratio of the measure l of the set of elementary outcomes favourable to event A to the measure L of the set of all possible elementary outcomes in the assumption that they are equally possible*:

$$P(A)=\frac{l}{L}\leqslant 1.$$

This definition naturally extends the classical definition of probability to the case of an infinite number of elementary outcomes.

An analogous definition may also be introduced when elementary outcomes of a trial represent points in the plane or in space.

Example 1. During an hour $0 \leqslant t \leqslant 1$ (t time in hours) a bus-stop is approached by one and only one bus. What is the probability

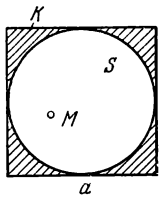

FIG. 273

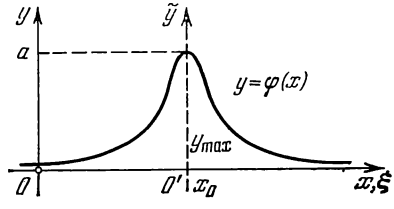

FIG. 274

that a passenger who came to this bus-stop at the moment of time $t = 0$ would have to wait for the bus not more than 10 minutes?

Here the set of all elementary outcomes form a closed interval $[0, 1]$ whose temporal length $L = 1$, and the set of successful elementary outcomes form the interval $\left[0, \frac{1}{6}\right]$ of temporary length $l = \frac{1}{6}$.

Therefore the sought-for probability is

$$p = \frac{l}{L} = \frac{1}{6}.$$

Example 2. A material point M is thrown accidentally into a square K with the side a and a circle S inscribed in it (Fig. 273). What is the probability of this point getting into the circle S?

Here the area of the square is $K = a^2$, and the area of the circle $S = \pi \left(\frac{a}{2}\right)^2 = \frac{\pi}{4} a^2$.

It is only natural to take for the sought-for probability the ratio

$$p = \frac{S}{K} = \frac{\pi}{4} \approx 0.785.$$

This probability, and, hence, the number π can, obviously, be determined experimentally.

Sec. 251. Normal Distribution

The distribution of probabilities of a random variable X is called *normal* if the probability density obeys Gauss' law

$$\varphi(x) = ae^{-b(x-x_0)^2}, \tag{251-1}$$

where a, b, and x_0 are certain constants, and $a > 0$, $b > 0$. In this case the graph of the probability density represents a displaced Gauss, curve symmetric about the straight line $x = x_0$ and with the maximal ordinate $y_{\max} = a$ (Fig. 274).

For the sake of convenience, we centre this curve by introducing new coordinates: $\xi = x - x_0$ and $\eta = y$. Then Gauss' law will take the form

$$\widetilde{\varphi}(\xi) = ae^{-b\xi^2} \tag{251-2}$$

and will represent the differential law of distribution of the random variable $\widetilde{X} = X - x_0$.

The constants a and b in (251-2) are not arbitrary, since for the probability density $\widetilde{\varphi}(\xi)$ the following condition must be fulfilled

$$\int_{-\infty}^{+\infty} \widetilde{\varphi}(\xi)\, d\xi = a \int_{-\infty}^{+\infty} e^{-b\xi^2}\, d\xi = 1. \tag{251-3}$$

Substituting the variable

$$b\xi^2 = t^2, \quad \xi = \frac{t}{\sqrt{b}}, \quad d\xi = \frac{dt}{\sqrt{b}},$$

we shall have

$$\int_{-\infty}^{+\infty} e^{-b\xi^2}\,d\xi = \int_{-\infty}^{+\infty} e^{-t^2}\frac{dt}{\sqrt{b}} = \sqrt{\frac{\pi}{b}} \tag{251-4}$$

(see Sec. 225, Chapter 24). Hence, by (251-3), we find

$$a\sqrt{\frac{\pi}{b}} = 1, \tag{251-5}$$

i.e.

$$a = \sqrt{\frac{b}{\pi}}. \tag{251-6}$$

Thus,

$$\widetilde{\varphi}(\xi) = \sqrt{\frac{b}{\pi}}\,e^{-b\xi^2}. \tag{251-7}$$

For the mathematical expectation of a random variable we shall have

$$M(\widetilde{X}) = \int_{-\infty}^{+\infty} \xi\widetilde{\varphi}(\xi)\,d\xi = \sqrt{\frac{b}{\pi}}\int_{-\infty}^{+\infty} \xi e^{-b\xi^2}\,d\xi = 0$$

(since the integrand is odd). Hence

$$M(X) = M(\widetilde{X} + x_0) = M(\widetilde{X}) + M(x_0) = 0 + x_0 = x_0. \tag{251-8}$$

Thus, with normal distribution of the random variable X, its mathematical expectation x_0 coincides with the point of intersection of the axis of symmetry of the graph of the corresponding Gauss' curve and the x-axis (*centre of dispersion*).

For the variance of the random variable $\widetilde{X}$ we obtain

$$D(\widetilde{X}) = M(\widetilde{X}^2) - [M(\widetilde{X})]^2 = M(\widetilde{X}^2) = \int_{-\infty}^{+\infty} \xi^2\widetilde{\varphi}(\xi)\,d\xi = \sqrt{\frac{b}{\pi}}\int_{-\infty}^{+\infty} \xi^2 e^{-b\xi^2}\,d\xi. \tag{251-9}$$

Setting $u = \xi$ and $dv = \xi e^{-b\xi^2}\,d\xi = d\left(-\frac{e^{-b\xi^2}}{2b}\right)$ and integrating by parts, taking into account (251-4), we shall have

$$\int_{-\infty}^{+\infty} \xi^2 e^{-b\xi^2}\,d\xi = -\xi\cdot\frac{e^{-b\xi^2}}{2b}\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \frac{e^{-b\xi^2}}{2b}\,d\xi = \frac{1}{2b}\sqrt{\frac{\pi}{b}}.$$

Thus, from (251-9) we get

$$D(\widetilde{X}) = \sqrt{\frac{b}{\pi}} \cdot \frac{\sqrt{\pi}}{2b\sqrt{b}} = \frac{1}{2b};$$

and consequently,

$$D(X) = D(\widetilde{X} + x_0) = D(\widetilde{X}) + D(x_0) = \frac{1}{2b} + 0 = \frac{1}{2b}. \qquad (251\text{-}10)$$

Hence, for the root mean square deviation of the variable X we obtain

$$\sigma(X) = \sqrt{D(X)} = \frac{1}{\sqrt{2b}}.$$

Introducing the notation $\sigma = \sigma(X)$, we shall have

$$b = \frac{1}{2\sigma^2}, \qquad a = \sqrt{\frac{b}{\pi}} = \frac{1}{\sigma\sqrt{2\pi}}.$$

Substituting these values into formula (251-1), we get the *standard form of the normal law of distribution* of a random variable X *in the differential form*:

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-x_0)^2}{2\sigma^2}}, \qquad (251\text{-}11)$$

where $x_0 = M(X)$ and $\sigma = \sqrt{D(X)}$.

Thus, the normal distribution law depends only on two parameters: *mathematical expectation* and *root mean square deviation*.

The normal distribution law of a random variable in the integral form has the form

$$\Phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{(z-x_0)^2}{2\sigma^2}} dz. \qquad (251\text{-}12)$$

Formulas (251-11) and (251-12) are simplified by introducing the *standardized deviation*

$$t = \frac{x - x_0}{\sigma}; \qquad (251\text{-}13)$$

then

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2}} \equiv \frac{1}{\sigma}\varphi_0(t)$$

(see Sec. 241). Setting in integral (251-12) $\tau = \frac{z-x_0}{\sigma}$, $d\tau = \frac{dz}{\sigma}$, we get

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-x_0}{\sigma}} e^{-\frac{\tau^2}{2}} d\tau =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{\tau^2}{2}} d\tau + \frac{1}{\sqrt{2\pi}} \int_{0}^{t} e^{-\frac{\tau^2}{2}} d\tau = \frac{1}{2} + \Phi_0(t),$$

where t is determined by the formula (251-13) and $\Phi_0(t)$ is the standard integral of probabilities (see Sec. 242).

Hence we get that for a random variable X, obeying the normal law, its probability of getting onto the interval $[a, b]$ is

$$P(a \leqslant x \leqslant b) = \Phi(b) - \Phi(a) =$$

$$= \left[\frac{1}{2} + \Phi_0\left(\frac{b-x_0}{\sigma}\right)\right] - \left[\frac{1}{2} + \Phi_0\left(\frac{a-x_0}{\sigma}\right)\right] =$$

$$= \Phi_0\left(\frac{b-x_0}{\sigma}\right) - \Phi_0\left(\frac{a-x_0}{\sigma}\right). \quad (251\text{-}14)$$

In particular, the probability that the deviation of the variable X from its mathematical expectation x_0 will be less than α by absolute value is equal to

$$P(|X - x_0| < \alpha) = P(x_0 - \alpha < X < x_0 + \alpha) = 2\Phi_0\left(\frac{\alpha}{\sigma}\right).$$

Setting $\alpha = 3\sigma$, we get

$$P(|x - x_0| < 3\sigma) = 2\Phi_0(3) = 0.9973,$$

i.e. such deviation is almost certain (*three-sigma rule*).

The normal law of distribution of probabilities is widely applied in error theory, shooting theory, physics, etc.

Exercises

1. Let A be a random event and C a certain event. What do the events $A + A$, AA, $A + C$, AC mean?

2. The die is thrown once. What are the probabilities of the following events: A—one pip turning up; B—an odd number of pips turning up; C—at least three pips turning up?

3. An urn contains two white and eight black balls. How many white balls should be added into the urn so that the probability of drawing one white ball out of it would be not less than 0.99?

4. From an urn containing five white and three black balls four balls are drawn at random. What are the probabilities that: (a)

equal numbers of white and black balls will be among them; (b) the number of white balls will exceed the number of black ones?

5. Of a total of thirty examination cards a student learnt twenty-four cards. What is the probability (in per cent) of his successful answer at the examination: (1) with a card drawn singly, and (2) with a card drawn repeatedly (the firstly drawn card is not returned!)?

6. Three riflemen are shooting at a target simultaneously. The probability of hitting the target by the first rifleman is equal to 0.9, by the second rifleman to 0.8, and by the third to 0.6. What is the probability that: (a) the target will be hit at least by one rifleman; (b) the target will be hit at least twice; (c) none of the riflemen will hit the target?

7. Prove that if events A and B are independent of each other, then the events $\overline{A}$ and $\overline{B}$ are also mutually independent.

8. The probability of hitting the target at a single shot is equal to $p = \frac{1}{3}$. Find the distribution of probabilities for the number of hits m with the number of shots $n = 5$.

9. The probability of hitting the target with one shot is equal to $p = 0.2$. At what number of shots will the target be hit at least once with the probability of not less than 0.999?

10. Repeated measurements of a certain quantity yielded the following results (containing random errors): $x_1 = 1.2$ (two times); $x_2 = 1.3$ (five times); $x_3 = 1.4$ (three times). Assuming that the measurements are equally accurate, find the mathematical expectation (the mean value) and variance of the result of measurement. What is the mean square error of the result of measurement?

11. Discrete random variables X and Y are independent. Prove that the variables $\widetilde{X} = X + C$ and Y (C constant) are also independent.

12. All values of a random variable X belong to the interval (0, 2), and the probability density $\varphi(x) = \frac{1}{4}$ for $0 < x \leqslant 1$ and $\varphi(x) = \frac{3}{4}$ for $1 < x < 2$. Find the distribution function $\Phi(x)$, mathematical expectation $M(X)$ and variance $D(X)$.

13. A random variable X is uniformly distributed on a centred closed interval $[-l, l]$. Find the probability density $\varphi(x)$, mathematical expectation $M(X)$, variance $D(X)$, and standard $\sigma(X)$.

14. Show that the Gauss curve

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

has points of inflection at $x = \pm\sigma$.

15. In experimental shooting it was revealed that the deviation Δ of the point of strike from the target obeys the normal law with the

mathematical expectation $M = 0$ and variance $D = 4$ m. What is the probability that $|\Delta| < 1$ m?

16. In packing a dry substance a package is considered to be standard if its mass differs from the preset mass of 1 kg by not more than 20 g (in either side). It is verified that in a careful packing process the errors in filling the packages obey the normal law with the mathematical expectation $M = 0$ and mean square deviation $\sigma = 10$ g. A certain batch of this substance totalling to 10 000 packages contains 9000 standard packages. Does it correspond to the given normal law?

Chapter 26

The Concept of Linear Programming

Sec. 252. An n-Dimensional Vector Space

In Chapter 18 we saw that each vector $\mathbf{x}$ of a three-dimensional space can be specified by three coordinates: $\mathbf{x} = \{x_1, x_2, x_3\}$. Generalizing this property, we come to the notion of an n-dimensional vector.

Definition. *An ordered system of n numbers* $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ *is called the n-dimensional point or n-dimensional vector.*

The numbers $x_1, x_2, \ldots, x_n$ are called the *coordinates* of the point (vector) $\mathbf{x}$; we shall assume that they are real. A vector with zero coordinates $\mathbf{0} = \{0, 0, \ldots, 0\}$ is called the *null-vector*. In geometrical representation, vector $\mathbf{x}$ may be considered as the radius vector of a corresponding point.

The basic operations on the n-dimensional vectors $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ and $\mathbf{y} = \{y_1, y_2, \ldots, y_n\}$ are defined by analogy with three-dimensional vectors.

I. *Equality of vectors.* Two vectors are equal if and only if their corresponding coordinates are equal, i.e.

$$\mathbf{x} = \mathbf{y} \Leftrightarrow x_i = y_i \qquad (i = 1, 2, \ldots, n).$$

II. *Sum of vectors.* By definition, we set

$$\mathbf{x} + \mathbf{y} = \{x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n\}.$$

In particular, for the two oppositely directed vectors $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ and $-\mathbf{x} = \{-x_1, -x_2, \ldots, -x_n\}$ we have

$$\mathbf{x} + (-\mathbf{x}) = \mathbf{0},$$

where $\mathbf{0}$ is the null-vector.

III. *Difference of vectors*:

$$\mathbf{x} - \mathbf{y} = \{x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\}.$$

The following relation is valid

$$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y}).$$

IV. *Multiplication by a scalar.* If k is a scalar, then

$$k\mathbf{x} = \{kx_1, kx_2, \ldots, kx_n\}.$$

V. *Scalar product*:

$$(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

The *absolute value* (*the norm*) of vector $\mathbf{x}$ is understood to be the number

$$\mathbf{x} = |\mathbf{x}| = \sqrt{(\mathbf{x} \cdot \mathbf{x})} = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{\frac{1}{2}} \geqslant 0.$$

The *distance* between the points $\mathbf{x}$ and $\mathbf{y}$ is determined by the formula

$$\rho(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \left\{ \sum_{i=1}^{n} (x_i - y_i)^2 \right\}^{\frac{1}{2}}.$$

In particular, $|\mathbf{x}|$ is the distance of point $\mathbf{x}$ from the origin.
Setting

$$(\mathbf{x}, \mathbf{y}) = xy \cos \varphi,$$

we obtain the *angle* between the vectors

$$\varphi = \arccos \frac{(\mathbf{x}, \mathbf{y})}{xy}.$$

In particular, if $\mathbf{x} \cdot \mathbf{y} = 0$, then $\varphi = \frac{\pi}{2}$ (orthogonal vectors).

For operations I to V the usual properties are valid (see Chapter 18). In particular, if

$$\mathbf{x} = \sum_{k=1}^{m} c_k \mathbf{x}^k$$

is a linear combination of the vectors $\mathbf{x}^1, \ldots, \mathbf{x}^m$ (with vectors we use superscripts) and $c_1, c_2, \ldots, c_m$ are scalars, then

$$\left(\sum_{k=1}^{m} c_k \mathbf{x}^k, \mathbf{y} \right) = \sum_{k=1}^{m} c_k (\mathbf{x}^k, \mathbf{y}).$$

The collection of all n-dimensional vectors (points) for which operations I to V are defined is called the *n-dimensional Euclidean space* E^n.

Definition. *Vectors* $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^m$ *of the space* E^n *are called linearly dependent if there exist scalars* $c_1, c_2, \ldots, c_m$, *not all equal to*

zero ($|c_1| + |c_2| + \ldots + |c_m| \neq 0$), *such that*

$$\sum_{k=1}^{m} c_k \mathbf{x}^k = 0.$$

In a particular case, for $m = 2$ these vectors are called *collinear*, and for $m = 3$ they are called *coplanar* (cf. Secs. 149 and 150, Chapter 18). Otherwise, these vectors are termed *linearly independent.*

We can prove that in space E^n there exist at the most n linearly independent vectors. Thus, the dimension dim E^n of the vector space E^n may be defined as the maximum number of linearly independent vectors enclosed in this space.

Let $\mathbf{x} \in E^n$ and $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n$ be linearly independent vectors from E^n. Since the vectors $\mathbf{x}, \mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n$ are linearly dependent, we have

$$c\mathbf{x} + \sum_{k=1}^{n} c_k \mathbf{x}^k = 0,$$

where, obviously, $c \neq 0$.

Hence we obtain

$$\mathbf{x} = \sum_{k=1}^{n} \xi_k \mathbf{x}^k \left(\xi_k = -\frac{c_k}{c}\right).$$

Thus, each vector $\mathbf{x} \in E^n$ can be decomposed (in the only possible way!) along the vectors $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n$. In other words, these vectors form the *basis* of the space E^n. The numbers ξ_k ($k = 1, 2, \ldots, n$) are called the *coordinates* of the vector $\mathbf{x}$ in the given basis $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n$. In particular, the coordinates of the given vector $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ are its coordinates in the basis of unit vectors

$$\mathbf{e}^1 = \{1, 0, 0, \ldots, 0\},$$

$$\mathbf{e}^2 = \{0, 1, 0, \ldots, 0\},$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\mathbf{e}^n = \{0, 0, 0, \ldots, 1\}.$$

Sec. 253. Sets in n-Dimensional Space

A collection of points $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ of the n-dimensional space E^n is called the *set* of this space.

The set of points $\mathbf{x}$ such that

$$0 < \rho(\mathbf{x}, \mathbf{x}_0) < \varepsilon$$

will be called the *neighbourhood* $U_{\mathbf{x}_0}$ of the point $\mathbf{x}_0$ (more precisely, *ε-neighbourhood*). The point $\mathbf{x}_0$ does not enter into its neighbourhood, i.e. the neighbourhood is punctured.

A set G is said to be *open* if each point $\mathbf{x} \in G$ of this set belongs to it together with its neighbourhood.

A set F is said to be *closed* if it is the complement of some open set G, i.e. $F = E^n \setminus G$.

Let $E \subset E^n$. A point ξ not necessarily belonging to E is called *boundary* for the set E if ξ is an interior point neither for the set E, nor for its complement $E^C = E^n \setminus E$. The totality of all boundary points of the set E is called its *boundary*: $\Gamma(E) = \Gamma$. Obviously, $\Gamma(E^C) = \Gamma(E)$.

Definition 1. *The set of points* $\mathbf{x} = \{x_1, x_2, \ldots, x_n\} \in E^n$ *whose coordinates satisfy the linear equation*

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \tag{253-1}$$

($a_1, a_2, \ldots, a_n, b$ *are constant*), *where the coefficients* $a_1, a_2, \ldots, a_n$ *are not all equal to zero, is called the hyperplane of the space* E^n.

In the case of a three-dimensional space this set is an ordinary plane.

Introducing the *normal vector* of hyperplane (253-1)

$$\mathbf{x} = \{a_1, a_2, \ldots, a_n\},$$

its equation can be written in the form

$$\mathbf{a} \cdot \mathbf{x} = b. \tag{253-2}$$

Hyperplane (253-2) separates the space E^n into two closed half-spaces

$$\mathbf{a} \cdot \mathbf{x} \leqslant b \ (E^-), \qquad \mathbf{a} \cdot \mathbf{x} \geqslant b \ (E^+)^*$$

If the constant term $b \neq 0$, then the half-spaces E^- and E^+ are distinguished in the following way: (1) if $b > 0$, then E^- contains the origin $\mathbf{0}$, and E^+ does not contain $\mathbf{0}$; (2) for $b < 0$, it is vice versa.

Example. The straight line $x_1 + x_2 = 1$ divides the plane Ox_1x_2 into two half-planes: $x_1 + x_2 \leqslant 1$ and $x_1 + x_2 \geqslant 1$ (Fig. 275).

Definition 2. *The collection of points*

$$\mathbf{z} = \mathbf{x} + \beta(\mathbf{y} - \mathbf{x}), \tag{253-3}$$

where $0 \leqslant \beta \leqslant 1$ (Fig. 276), *is called the closed interval* $[\mathbf{x}, \mathbf{y}]$ *with the end-points* $\mathbf{x}$ *and* $\mathbf{y}$.

* Here, the points of the hyperplane $\mathbf{a} \cdot \mathbf{x} = b$ are traditionally, attached to both half-spaces E^- and E^+. This hyperplane is their common boundary.

Formula (253-3) can be written in the following symmetric form

$$\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}, \tag{253-3'}$$

where $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha + \beta = 1$. For $\alpha = 0$, $\beta = 1$ and $\alpha = 1$, $\beta = 0$ we obtain the end-points of the interval; for $0 < \alpha < 1$,

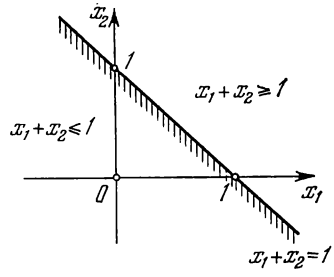

FIG. 275

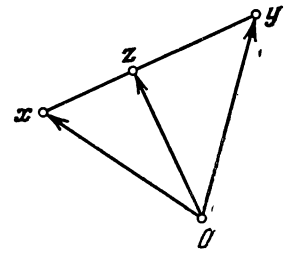

FIG. 276

$0 < \beta < 1$ its interior points. Note that point $\mathbf{x}$ may be regarded as an interval $[\mathbf{x}, \mathbf{x}]$ with coinciding ends.

This definition generalizes the concept of a closed interval for two- and three-dimensional spaces.

Definition 3. *The set $A \subset E^n$ is called convex if for any two points $\mathbf{x} \in A$ and $\mathbf{y} \in A$ the segment $[\mathbf{x}, \mathbf{y}]$ joining them, also belongs to the set A: $[\mathbf{x}, \mathbf{y}] \subset A$.*

The point, interval, circle, sphere, the whole space E^n, etc. are examples of convex sets. The empty set $\varnothing$ is, conventionally, considered to be convex.

Lemma. *Any half-plane in E^n is a convex set.*

Indeed, let the hyperplane

$$\mathbf{a} \cdot \mathbf{x} = b, \tag{253-4}$$

be the boundary of the half-plane E, and the half-plane itself is defined by the inequality

$$\mathbf{a} \cdot \mathbf{x} \leqslant b. \tag{253-5}$$

Consider arbitrary points $\mathbf{y}$ and $\mathbf{z}$ belonging to the set E, i.e.

$$\mathbf{a} \cdot \mathbf{y} \leqslant b, \quad \mathbf{a} \cdot \mathbf{z} \leqslant b, \tag{253-6}$$

and let $\mathbf{t} = \alpha\mathbf{y} + \beta\mathbf{z}$ ($\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha + \beta = 1$) be an arbitrary point of the interval $[\mathbf{y}, \mathbf{z}]$.

Using the properties of scalar product, by inequalities (253-6), we have

$$\mathbf{a} \cdot \mathbf{t} = \mathbf{a} \cdot (\alpha\mathbf{y} + \beta\mathbf{z}) = \alpha(\mathbf{a} \cdot \mathbf{y}) + \beta(\mathbf{a} \cdot \mathbf{z}) \leqslant \alpha \cdot \mathbf{b} + \beta \cdot \mathbf{b} = (\alpha + \beta)\mathbf{b} = \mathbf{b},$$

i.e.

$$\mathbf{a} \cdot \mathbf{t} \leqslant b. \tag{253-7}$$

Thus, $\mathbf{t} \in E$ and, consequently, the half-plane E is convex.

Theorem. *An intersection of any number of convex sets is a convex set.*

We shall carry out the proof for the case of two convex sets; for an arbitrary (finite or infinite) number of convex sets reasoning is analogous.

Let A and B be convex sets and $C = A \cap B$ their intersection. If C is empty, then, by definition, it is convex. Let C be not empty and $\mathbf{x}$ and $\mathbf{y}$ any two points (perhaps, coinciding ones) belonging to C. From the concept of intersection it follows that $\mathbf{x}, \mathbf{y} \in A$ and also $\mathbf{x}, \mathbf{y} \in B$. Consider the line segment $[\mathbf{x}, \mathbf{y}]$. Since the sets A and B are convex, we have $[\mathbf{x}, \mathbf{y}] \subset A$ and $[\mathbf{x}, \mathbf{y}] \subset B$, and, hence, $[\mathbf{x}, \mathbf{y}] \subset C$. Thus, the set C is also convex.

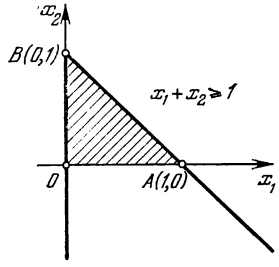

FIG. 277

Corollary. *An intersection of any number of half-spaces is a convex set (possibly, empty).*

Note. Consider the following system of linear inequalities

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leqslant b_i \quad (i = 1, 2, \ldots, n). \tag{253-8}$$

Any set of numbers $x_1, x_2, \ldots, x_n$ satisfying all inequalities (253-8) is called the *solution* of this system of inequalities. The numbers $x_1, x_2, \ldots, x_n$ satisfying only one of the inequalities of system (253-8) fill a half-plane in the space E^n. The solutions belong to the intersection of all these half-spaces. It follows from the theorem that the set of all the solutions of system (253-8) represents a convex polyhedron. These are examples of convex polyhedrons in a three-dimensional space: the pyramid, parallelepiped, prism, spherical segment bounded by two parallel planes, etc.

Example. (a) The solutions of the inequalities

$$x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_1 + x_2 \leqslant 1$$

in the space E^2 fill a triangle with the vertices O (0, 0), A (1, 0), B (0, 1) (Fig. 277);

(b) the set of the solutions of the system

$$x_1 \geqslant 0, \quad x_1 + x_2 \geqslant 1$$

in the space E^2 is an angle with the point B (0, 1) as vertex, and the ray BA and semi-axis Bx_2 as sides (Fig. 277).

Definition. *A point of a convex set which is an interior point for none of the non-zero intervals entirely belonging to this set is called its terminal point.*

For instance, for an interval its terminal points are its end-points, for a triangle its vertices, and so on.

It is intuitively clear that a convex polyhedron, i.e. an intersection of a finite number of half-spaces, has a finite number of terminal points (its vertices).

Sec. 254. The Problem of Linear Programming

Nowadays mathematical methods are widely applied in planning of national economy, organization of industry control, elaboration of military operations, etc. From the general point of view, the problems of control and planning are usually reduced to a choice of a certain system of numerical parameters or a function (characteristics of a plan) ensuring the most effective achievement of the preplanned aim (optimum plan) with the limited possible resources taken into account. To estimate the effectiveness of a plan, we introduce the so-called objective function (i.e. plan quality index) expressed in terms of the plan characteristics and attaining the extremum value (i.e. the least or the greatest value) for an optimal plan.

For a large number of practically interesting problems the objective function is expressed linearly in terms of plan characteristics, the permissible values of the parameters also obeying linear equalities or inequalities. Finding the absolute extremum of the objective function is called *linear programming* (the term 'linear planning' would be more felicitous).

Mathematically, the problem of linear programming is formulated in the following way: it is required to find the absolute extremum (the least or the greatest value depending on the sense of the problem) of the linear function

$$S = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{254-1}$$

$$(c_1^2 + c_2^2 + \ldots + c_n^2 \neq 0)$$

(the objective function) provided the variables $x_1, x_2, \ldots, x_n$ are restricted with limitations in the form of linear equalities or inequalities:

$$x_1 \geqslant 0,\ x_2 \geqslant 0,\ \ldots,\ x_k \geqslant 0^* \qquad (k \leqslant n) \tag{254-2}$$

$$\sum_{j=1}^{n} a_{ij}x_j \leqslant b_i^{**} \qquad (i = 1, 2, \ldots, m). \tag{254-3}$$

* The inequality of the opposite sense $x_p \leqslant 0$ is reduced to inequalities of type (254-2) by introducing a new coordinate $x'_p = -x_p$ $(1 \leqslant p \leqslant k)$.

** The inequalities of the opposite sense are reduced to the inequalities of the form (254-3) by termwise multiplication by -1.

Definition 1. *The maximum totality* Ω *of values* $x_1, x_2, \ldots, x_n$ *satisfying all inequalities* (254-2) *and* (254-3) *is called the domain of permissible values of the problem of linear programming* (*or simply, the permissible domain*).

The permissible domain Ω is the domain of definition of the function S.

From the theorem of the preceding section it follows that the *permissible domain of the problem of linear programming represents a convex polyhedron* (which is, possibly, an empty set if the system of inequalities (254-1), (254-3) is incompatible).

Definition 2. *The collection of values* $x_1, x_2, \ldots, x_n$ *from the permissible domain* Ω *for which objective function* (254-1) *takes on, according to the sense of the problem, either the smallest or the largest value is called the solution of the problem of linear programming* (*or the optimal plan*). *If there exists at least one solution, then the problem of linear programming is said to be solvable.*

Theoretically, three cases are possible: (A) Conditions (254-2), (254-3) are contradictory, and, consequently, the permissible set Ω is empty. The problem (254-1), (254-2), (254-3) has no solution. (B) The permissible domain Ω is unbounded; the problem (254-1), (254-2), (254-3) may have a solution, or may have no solution. (C) The system of conditions (254-2), (254-3) is compatible and the permissible domain Ω is bounded. By virtue of Weierstrass' theorem (Sec. 179, Chapter 20) the problem of linear programming (254-1), (254-2), (254-3) is solvable. We shall confine ourselves to considering only case (C).

According to the general theory (see Secs. 178-9, Chapter 20) objective function (254-1) reaches its absolute extremum either at a critical point or on the boundary Γ of the permissible domain Ω. Since its partial derivatives

$$\frac{\partial S}{\partial x_i} = c_i \qquad (i = 1, 2, \ldots, n)$$

simultaneously vanish nowhere within the domain Ω, the objective function S reaches its absolute extremum on the boundary Γ of the permissible domain Ω. This result can be made more precise.

Theorem. *A linear objective function can reach its strict absolute extremum only at terminal points of the permissible domain.*

Proof. Indeed, let

$$S(\mathbf{x}) = \sum_{i=1}^{n} c_i x_i \equiv \mathbf{c} \cdot \mathbf{x}$$

be an objective function, $\mathbf{c} = \{c_1, c_2, \ldots, c_n\}$, $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ and Ω its permissible domain. Suppose, for instance, that $S(\xi) = M$, where $\xi = \{\xi_1, \xi_2, \ldots, \xi_n\} \in \Omega$, is a strict maximum

of the objective function S (**x**) and ξ is a non-terminal point of a convex domain Ω. Then there exists a non-zero interval $[\mathbf{y}, \mathbf{z}] \subset \Omega$ for which ξ is an interior point, i.e.

$$\xi = \alpha \mathbf{y} + \beta \mathbf{z}. \tag{254-4}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta = 1$, and, obviously, $\xi \neq \mathbf{y}$ and $\xi \neq \mathbf{z}$. Since $S(\xi) = M$ is a strict maximum of the function $S(\mathbf{x})$, choosing a sufficiently small interval $[\mathbf{y}, \mathbf{z}]$ (which, obviously, can be done), we shall have

$$S(\mathbf{y}) < M, \quad S(\mathbf{z}) < M. \tag{254-5}$$

From (254-4), taking into account that $S(\mathbf{x})$ is a linear homogeneous function, we get

$$S(\xi) = \mathbf{c} \cdot (\alpha \mathbf{y} + \beta \mathbf{z}) = \alpha (\mathbf{c} \cdot \mathbf{y}) + \beta (\mathbf{c} \cdot \mathbf{z}) = \alpha S(\mathbf{y}) + \beta S(\mathbf{z}) < \alpha M + \beta M = M,$$

and, thus, we have come to a contradiction. The theorem is proved.

Note. In the general case, we can prove that an absolute extremum of a linear objective function $S(\mathbf{x})$ (provided it exists) is always realized at terminal points of a convex permissible domain Ω, i.e. at the vertices of the polyhedron Ω (it is possible, of course, that an absolute extremum of the objective function is reached at other points as well; for instance, the function $S(\mathbf{x})$ may have its greatest value at all points of one of the faces of the polyhedron Ω, and so on).

Thus, the solution of the problem of linear programming is reduced to finding a finite number of values of the objective function and comparing them with one another.

Example. Find the least and the greatest values of the function

$$S = x_1 + 2x_2$$

in the domain Ω: $0 \leqslant x_1 \leqslant 20$, $x_1 + 2x_2 \geqslant 20$, $x_1 - x_2 \leqslant -30$.

The domain Ω represents a quadrilateral with the vertices A (20, 0), B (20, 50), C (0, 30), D (0, 10) (Fig. 278). We have

$$S(A) = 20, \quad S(B) = 120,$$

$$S(C) = 60, \quad S(D) = 20.$$

Hence, the least value of the function S in the domain Ω is equal to $m = 20$ (reached at the vertices A and D), its greatest value being $M = 120$ (reached at the vertex B).

This result becomes geometrically obvious if we construct level lines of the function S, i.e. the straight lines

$$x_1 + 2x_2 = \text{const}.$$

But such an easy solution of the problem is possible only in simplest cases. The method used here, even with comparatively small

number of variables n and inequalities m, involves a considerable amount of work and requires the use of high-speed computers. Therefore, new methods were originated aimed at ordered consideration of the vertices of a permissible domain (simplex method, potential method, etc.)*.

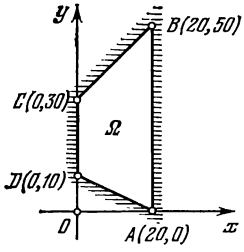

FIG. 278

We should like to conclude this section with two concrete problems to be solved by the method of linear programming.

I. Organization of supply. Let a consuming centre ensure its necessary supply of some homogeneous product (for instance, potatoes) from n points.

We denote by x_i the amount of the product (say, in tons) perchased by the centre in ith point, and by c_i the price of one ton of potatoes in this point (including transport costs), $i = 1, 2, \ldots, n$. The constants c_i are different since one point produces cheaper goods, another more expensive. Besides, the points are situated at different distances from the centre and, consequently, transport costs are different. Then, the expenses of the centre, under the given purchase plan, amount to

$$S = \sum_{i=1}^{n} c_i x_i. \tag{254-6}$$

If a is consumers' demand for the goods in question, then the following condition must be ensured

$$\sum_{i=1}^{n} x_i = a. \tag{254-7}$$

Besides, the volume of production in the ith point is limited by a certain quantity b_i. It is also possible that the capacity of the transport means which can be used for delivering the goods to the centre is restricted by the quantity b_i'. In this case the following restrictions are, naturally, put on x_i:

$$0 \leqslant x_i \leqslant \beta_i, \tag{254-8}$$

where $\beta_i = \min(b_i, b_i')$ $(i = 1, 2, \ldots, n)$.

With a rational perchase plan the expenses of the centre must be minimal. Thus, we come to the problem of linear programming: to find the least value of expenditure function (254-6) under conditions (254-7) and (254-8). In the present case the problem has a simple solution.

* For more detail see special literature.

II. Transport problem. Let the points $A_1, A_2, \ldots, A_m$ produce a certain homogeneous product which is consumed by the centres situated at points $B_1, B_2, \ldots, B_n$.

Suppose the transport expenses due to the delivery of a unit of goods from point A_i to point B_j amount to c_{ij} money units, and the amount of the goods (for instance, in tons) delivered from A_i to B_j is equal to x_{ij} ($i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$). Then the total transport expenses will amount to

$$S = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}. \tag{254-9}$$

The volume of production in point A_i is limited by a certain quantity a_i ($i = 1, 2, \ldots, m$) which depends on the production capacity of the manufacturing enterprise. Assuming that the whole amount of the manufactured goods is delivered to the consumer (i.e. to the consuming points), we have the following conditions

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad (i = 1, 2, \ldots, m). \tag{254-10}$$

Besides, the demand for the product in the point B_j is dictated by producer's interests and amounts to a certain quantity b_j ($j = 1, 2, \ldots, n$). Therefore

$$\sum_{i=1}^{n} x_{ij} = b_j \qquad (j = 1, 2, \ldots, n). \tag{254-11}$$

It is required to work out such a delivery plan which would ensure minimum transport expenses S on the condition that the product manufactured at points A_i is entirely consumed (conditions (254-10)) and the need of the consuming points is completely met (conditions (254-11)). This is a typical problem of linear programming.

It is possible to prove (see *Elements of Linear Algebra and Linear Programming* by F. I. Karpelovich and L. E. Sadovskii, "Nauka", 1967 (in Russian)) that for a transport problem to be solvable it is necessary and sufficient to fulfil the following condition

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j, \tag{254-12}$$

i.e. the total volume of production must be equal to the total volume of consumption.

APPENDICES

A. MOST IMPORTANT CONSTANTS

$\pi = 3.14159,$ $\qquad \pi^{-1} = 0.31831,$

$\pi^2 = 9.86960,$ $\qquad \pi^{-2} = 0.10132,$

$e = 2.71828,$ $\qquad e^{-1} = 0.36788,$

$e^2 = 7.38906,$ $\qquad e^{-2} = 0.13534,$

$M = \log e = 0.43429,$ $\qquad M^{-1} = \ln 10 = 2.30259.$

B. LIST OF FORMULAS (CLASSIFIED AND EXPLAINED)

I. Plane Analytic Geometry

1. *Translation* of a coordinate system

$$x' = x - a, \quad y' = y - b,$$

where O' (a, b) is the new origin, (x, y) are the old coordinates of a point, and $[x', y']$ its new coordinates.

2. *Rotation of the coordinate axes* (with the origin fixed)

$$x = x' \cos \alpha - y' \sin \alpha,$$

$$y = x' \sin \alpha + y' \cos \alpha,$$

where (x, y) are the old coordinates of a point, $[x', y']$ its new coordinates, α the angle of rotation of the axes.

3. *The distance between two points*: (x_1, y_1) and (x_2, y_2)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

4. The coordinates of the point dividing a line segment with the end-points (x_1, y_1) and (x_2, y_2) *in a given ratio* l:

$$x = \frac{x_1 + l x_2}{1 + l}, \qquad y = \frac{y_1 + l y_2}{1 + l}.$$

For $l = 1$ we have the coordinates of the *mid-point*:

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}.$$

5. The *area of a triangle* with the vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3):

$$S = \pm \frac{1}{2} [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)].$$

6. The *slope-form* equation of the straight line

$$y = kx + b,$$

where $k = \tan \varphi$ (the slope) is the inclination of the line to the x-axis and b is the length of the segment intercepted by the line on the y-axis.

7. The *tangent* of the angle between the straight lines with the slopes k and k'

$$\tan \theta = \frac{k' - k}{1 + k'k}.$$

The *condition of parallelism* of straight lines: $k' = k$; the *condition of perpendicularity* of straight lines: $k' = -\frac{1}{k}$.

8. The equation of a straight line *through a given point* (x_1, y_1):

$$y - y_1 = k(x - x_1),$$

where k is the slope of the line.

9. The equation of a straight line *through two points* (x_1, y_1) and $(x_2, y_2$

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}.$$

10. The equation of a straight line *intercepting segments* a and b on the coordinate axes (the intercept form of the equation of a straight line)

$$\frac{x}{a} + \frac{y}{b} = 1.$$

11. The *general equation* of the straight line

$$Ax + By + C = 0 \qquad (A^2 + B^2 \neq 0).$$

12. The *distance of the point* (x_1, y_1) *from the straight line* $Ax + By + C = 0$:

$$\delta = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

13. The *equation of a circle* of radius R with centre (x_0, y_0):

$$(x - x_0)^2 + (y - y_0)^2 = R^2.$$

14. The *canonical equation of the ellipse* with the semiaxes a and b:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

The *foci of ellipse*: $F(c, 0)$ and $F'(-c, 0)$, where $c^2 = a^2 - b^2$.

15. The *focal radii* of a point (x, y) of ellipse (1):

$$r = a - \varepsilon x, \qquad r' = a + \varepsilon x,$$

where $\varepsilon = \frac{c}{a} < 1$ is the *eccentricity of the ellipse*.

16. The *canonical equation of the hyperbola* with the semiaxes a and b:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \tag{2}$$

The *foci of hyperbola*: $F(c, 0)$ and $F(-c, 0)$, where $c^2 = a^2 + b^2$.

17. The *focal radii* of a point (x, y) of hyperbola (2):

$$r = \pm(\varepsilon x - a), \quad r' = \pm(\varepsilon x + a),$$

where $\varepsilon = c/a > 1$ is the *eccentricity of the hyperbola*.

18. The *asymptotes of hyperbola* (2):

$$y = \pm\frac{b}{a}x.$$

19. The *graph of inverse proportionality*

$$xy = c \ (c \neq 0)$$

is an equilateral hyperbola with the asymptotes $x = 0$ and $y = 0$.

20. The *canonical equation of the parabola* with the parameter p:

$$y^2 = 2px.$$

The *focus of parabola*: $F\left(\frac{p}{2}, 0\right)$; the *equation of the directrix*: $x = -\frac{p}{2}$; the *focal radius* of point (x, y) of the parabola: $r = x + \frac{p}{2}$.

21. The *graph of the quadratic trinomial*

$$y = Ax^2 + Bx + C$$

is a vertical parabola with the vertex $O'\left(-\frac{B}{2A}, -\frac{B^2-4AC}{4A}\right)$.

22. The *polar coordinates* of a point with the rectangular coordinates x and y:

$$\rho = \sqrt{x^2+y^2}, \quad \tan\varphi = \frac{y}{x}.$$

The *rectangular coordinates* of a point with the polar coordinates ρ and φ:

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi.$$

23. The *parametric equations of a circle* of radius R with the origin as centre:

$$x = R\cos t, \quad y = R\sin t$$

(t parameter)

24. The *parametric equations of an ellipse with the semiaxes* a and b:

$$x = a\cos t, \quad y = b\sin t.$$

25. The *parametric equations of the cycloid*:

$$x = a(t - \sin t), \quad y = a(1 - \cos t).$$

II. Differential Calculus—Functions of One Variable

1. *Basic limit theorems*:

(a) $\lim\limits_{x\to a}[f(x)+g(x)-h(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x) - \lim\limits_{x\to a} h(x)$.

(b) $\lim\limits_{x\to a}[f(x)\,g(x)] = \lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a} g(x)$;

in particular

$$\lim_{x\to a}[cf(x)]=c\lim_{x\to a}f(x).$$

(c) $\lim\limits_{x\to a}[f(x)/g(x)]=\lim\limits_{x\to a}f(x):\lim\limits_{x\to a}g(x)\left(\lim\limits_{x\to a}g(x)\neq 0\right)$.

2. *Remarkable limits*:

(a) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$;

(b) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=\lim\limits_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e=2.71828\ldots$

3. *Relation between common and natural logarithms*:

$$\log x = M \ln x,$$

where $M = \log e = 0.43429\ldots$

4. The *increment of function* $y = f(x)$, corresponding to the increment Δx of the argument x:

$$\Delta y = f(x+\Delta x) - f(x).$$

5. The *condition of continuity* of function $y = f(x)$:

$$\lim_{\Delta x\to 0}\Delta y=0.$$

The *basic property* of a continuous function:

$$\lim_{x\to x_1}f(x)=f\left(\lim_{x\to x_1}x\right).$$

6. The *derivative*

$$y'=\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}.$$

Geometrically, $y' = f'(x)$ is the slope of the tangent to the graph of the function $y = f(x)$ at a point with the abscissa x.

For the rules for finding derivatives and the list of formulas for derivatives see Sec. 82 (Chapter 10).

7. The *Lagrange theorem* on finite increments of differentiable functions

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(\xi),$$

where $\xi \in (x_1, x_2)$.

8. The function $y = f(x)$ *increases* if $f'(x) > 0$, and *decreases* if $f'(x) < 0$.

9. L'Hospital's rule for the indeterminate forms $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$:

$$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)}=\lim_{x\to a}\frac{\varphi'(x)}{\psi'(x)},$$

if the right-hand limit exists.

10. *Taylor's local formula*:

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+o\left[(x-x_0)^n\right],$$

where $f^{(n)}(x)$ exists in a certain complete neighbourhood of x_0.

11. (a) The *necessary condition for extremum of a function* $f(x)$ at a point x_0: $f'(x_0) = 0$ or does not exist.

(b) The *sufficient conditions for extremum of a function* $f(x)$ at a point x_0:

(1) $f'(x_0) = 0$, $f'(x_0 - h_1)\, f'(x_0 + h_2) < 0$ for any sufficiently small $h_1 > 0$ and $h_2 > 0$, or

(2) $f'(x_0) = 0$, $f''(x_0) \neq 0$.

12. (a) The graph of function $y = f(x)$ is *concave upwards* if $f''(x) > 0$, and is *concave downwards* if $f''(x) < 0$.

(b) The *necessary condition for a point of deflection* of the graph of function $y = f(x)$ at $x = x_0$: $f''(x_0) = 0$ or $f''(x_0)$ does not exist.

The *sufficient condition for a point of deflection* at $x = x_0$:

$$f''(x_0) = 0, \qquad f''(x_0 - h_1)\, f''(x_0 + h_2) < 0$$

for any sufficiently small $h_1 > 0$ and $h_2 > 0$.

13. If the function $f(x)$ is continuous on $[\alpha, \beta]$ and $f(\alpha)\, f(\beta) < 0$, then the root ξ of the equation $f(x) = 0$ can be found approximately, applying the formulas:

$$\text{(a)}\ \xi_1 = \alpha - \frac{f(\alpha)}{f(\beta) - f(\alpha)}(\beta - \alpha)$$

(the *method of chords*);

$$\text{(b)}\ \xi_1' = \alpha - \frac{f(\alpha)}{f'(\alpha)},$$

where $f'(\alpha) \neq 0$, $f(\alpha)\, f''(\alpha) > 0$ (the *method of tangents*).

14. The *differential of an independent variable* x: $dx = \Delta x$. The *differential of the function* $y = f(x)$: $dy = y'\, dx$. The relation between the increment Δy of the function and its differential dy:

$$\Delta y = dy + \alpha\, \Delta x,$$

where $\alpha \to 0$ as $\Delta x \to 0$.

For the basic properties of differentials and their formulas see Sec. 101, Chapter 12.

15. The *small increment* of a differentiable function

$$f(x + \Delta x) - f(x) \approx f'(x)\, \Delta x.$$

16. The *second-order differential* of the function $y = f(x)$, where x is an independent variable ($d^2x = 0$):

$$d^2y = y''\, dx^2.$$

III. Integral Calculus

1. If

$$dy = f(x)\, dx,$$

then

$$y = \int f(x)\, dx$$

(*indefinite integral*).

2. *The basic properties of indefinite integrals*:

$$\text{(a)}\ d\int f(x)\, dx = f(x)\, dx, \quad \left[\int f(x)\, dx\right]' = f(x);$$

(b) $\int dF(x) = F(x) + C$;

(c) $\int Af(x)\,dx = A\int f(x)\,dx \ (A \neq 0)$;

(d) $\int [f(x) + g(x) - h(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx - \int h(x)\,dx$.

For the table of simple indefinite integrals see Sec. 105, Chapter 13.

3. *Principal methods of integration.*

(a) *Expansion method*

$$\int f(x)\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx,$$

where $f(x) = f_1(x) + f_2(x)$.

(b) *Substitution method*: if $x = \varphi(t)$, then

$$\int f(x)\,dx = \int f(\varphi(t))\,\varphi'(t)\,dt.$$

(c) *The method of integration by parts*:

$$\int u\,dv = uv - \int v\,du.$$

4. The *Newton-Leibniz formula*: if $f(x)$ is continuous and $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

5. The *definite integral* as the limit of the integral sum:

$$\int_a^b f(x)\,dx = \lim_{\max|\Delta x_i|\to 0} \sum_{i=0}^{n-1} f(\bar{x}_i)\,\Delta x_i, \text{ where } \bar{x}_i \in [x_i, x_{i+1}]$$

and

$$\Delta x_i = x_{i+1} - x_i.$$

6. The *basic properties of the definite integrals* (the below functions are continuous):

(a) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$;

(b) $\int_a^a f(x)\,dx = 0$;

(c) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$;

(d) $\int_a^c f(x)\,dx+\int_c^b f(x)\,dx=\int_a^b f(x)\,dx;$

(e) $\int_a^b Af(x)\,dx=A\int_a^b f(x)\,dx;$

(f) $\int_a^b [f(x)+g(x)-h(x)]\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx-\int_a^b h(x)\,dx;$

(g) $\frac{d}{dx}\int_a^x f(t)\,dt=f(x),\quad \frac{d}{dx}\int_x^b f(t)\,dt=-f(x).$

7. The *mean value theorem*: if $f(x)$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx=(b-a)\,f(c),$$

where $a<c<b$.

8. *The formula for integrating by parts in the definite integral*:

$$\int_a^b u(x)\,v'(x)\,dx=u(x)\,v(x)\Big|_a^b-\int_a^b v(x)\,u'(x)\,dx.$$

9. *The formula for substituting the variable in the definite integral*:

$$\int_a^b f(x)\,dx=\int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt,$$

where $a=\varphi(\alpha)$ and $b=\varphi(\beta)$.

10. *The trapezoid formula*:

$$\int_a^b y\,dx=h\left(\frac{1}{2}y_0+y_1+\ldots+y_{n-1}+\frac{1}{2}y_n\right),$$

where $h=\frac{b-a}{n}$, $x_0=a$ and $x_n=b$, $y=f(x)$, $y_i=f(x_0+ih)$ $(i=0, 1, 2, \ldots, n)$.

11. *Simpson's formula*:

$$\int_a^b y\,dx=\frac{h}{3}\left[y(a)+4y\left(\frac{a+b}{2}\right)+y(b)\right],$$

where $h=\frac{1}{2}(b-a)$.

12. *The improper integral*:

$$\int_a^{+\infty} f(x)\,dx=\lim_{b\to+\infty}\int_a^b f(x)\,dx.$$

13. *The area of a curvilinear trapezoid* bounded by a continuous line $y = f(x)$ ($f(x) \geqslant 0$), the x-axis, and two vertical lines $x = a$ and $x = b$ ($a < b$):

$$S = \int_a^b y\,dx.$$

14. The *area of a sector* bounded by a continuous line $\rho = f(\varphi)$ (ρ and φ polar coordinates) and two rays $\varphi = \alpha$ and $\varphi = \beta$ ($\alpha < \beta$):

$$S = \frac{1}{2}\int_\alpha^\beta \rho^2\,d\varphi.$$

15. The *arc length* of a smooth curve $y = f(x)$ *in the rectangular coordinates* x and y from the point $x = a$ to the point $x = b$ ($a < b$):

$$l = \int_a^b \sqrt{1+y'^2}\,dx.$$

16. The *arc length* of a smooth curve $\rho = f(\varphi)$ *in the polar coordinates* ρ and φ from the point $\varphi = \alpha$ to the point $\varphi = \beta$ ($\alpha < \beta$):

$$l = \int_\alpha^\beta \sqrt{\rho^2+\rho'^2}\,d\varphi.$$

17. The *arc length* of a smooth curve $x = \varphi(t)$, $y = \psi(t)$, *represented parametrically* ($t_0 < T$):

$$l = \int_{t_0}^T \sqrt{x'^2+y'^2}\,dt.$$

18. The *volume of a solid* with the cross-section $S(x)$ known:

$$V = \int_a^b S(x)\,dx.$$

19. The *volume of a solid of revolution*:
(a) about the x-axis:

$$V_x = \pi\int_a^b y^2\,dx \qquad (a < b);$$

(b) about the y-axis:

$$V_y = \pi\int_c^d x^2\,dy \qquad (c < d).$$

20. The *work performed by a variable force* $F = F(x)$ on $[a, b]$:

$$A = \int_a^b F(x)\,dx.$$

IV. Complex Numbers, Determinants, and Systems of Simultaneous Equations

1. The *complex number*: $z = x + iy$, where $x = \operatorname{Re} z$, $y = \operatorname{Im} z$ are real numbers and $i^2 = -1$. The *modulus of a complex number*

$$|z| = \sqrt{x^2+y^2} \geqslant 0.$$

The equality of complex numbers:

$$z_1 = z_2 \Leftrightarrow \operatorname{Re} z_1 = \operatorname{Re} z_2 \quad \text{and} \quad \operatorname{Im} z_1 = \operatorname{Im} z_2.$$

2. *The conjugate number for the complex number* $z = x + iy$:

$$\bar{z} = x - iy.$$

3. *Arithmetical operations on the complex numbers* $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

(a) $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$;

(b) $z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$;

(c) $$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{|z_2|^2} = \frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2} \qquad (z_2 \neq 0).$$

In particular,

$$\operatorname{Re} z = \frac{1}{2}(z+\bar{z}), \quad \operatorname{Im} z = \frac{1}{2i}(z-\bar{z}), \quad |z|^2 = z\bar{z}.$$

4. The *trigonometric form of a complex number*:

$$z = r(\cos\varphi + i\sin\varphi),$$

where $r = |z|$ and $\varphi = \operatorname{Arg} z$.

5. *Theorems on the modulus and argument*:

(a) $|z_1 + z_2| \leqslant |z_1| + |z_2|$;

(b) $|z_1z_2| = |z_1|\,|z_2|$, $\operatorname{Arg} z_1z_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$;

(c) $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad \operatorname{Arg}\frac{z_1}{z_2} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2 \quad (z_2 \neq 0);$$

(d) $|z^n| = |z|^n$, $\operatorname{Arg} z^n = n \operatorname{Arg} z$ (n integer).

6. *The root of a complex number*:

$$\sqrt[n]{z} = \sqrt[n]{|z|}\left(\cos\frac{\arg z + 2k\pi}{n} + i\sin\frac{\arg z + 2k\pi}{n}\right) \quad (k = 0, 1, 2, \ldots, n-1).$$

7. *The exponential form of a complex number*:

$$z = re^{i\varphi},$$

where $r = |z|$ and $\varphi = \operatorname{Arg} z$.

8. *The second-order determinant*:

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

9. *The solution of the system*:

$$\left.\begin{aligned} a_1x + b_1y &= c_1, \\ a_2x + b_2y &= c_2 \end{aligned}\right\}$$

is given by the formulas

$$x=\frac{D_x}{D}, \quad y=\frac{D_y}{D} \qquad \text{(Cramer's rule)},$$

where

$$D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \quad D_x=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad D_y=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

10. *The solutions of the homogeneous system*:

$$\left.\begin{aligned} a_1x+b_1y+c_1z=0, \\ a_2x+b_2y+c_2z=0 \end{aligned}\right\}$$

are given by the formulas

$$x=D_1t, \quad y=-D_2t, \quad z=D_3t, \qquad (-\infty<t<+\infty),$$

where

$$D_1=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad D_2=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \quad D_3=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

are the minors of the matrix

$$\left\| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix} \right\|.$$

11. *The third-order determinant*:

$$D=\left\| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} \right\| = a_1A_1+b_1B_1+c_1C_1,$$

where

$$A_1=\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \quad B_1=-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \quad C_1=\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

are cofactors of the corresponding elements of the determinant.

12. *The solution of the system*

$$\left.\begin{aligned} a_1x+b_1y+c_1z=d_1, \\ a_2x+b_2y+c_2z=d_2, \\ a_3x+b_3y+c_3z=d_3 \end{aligned}\right\}$$

is given by *Cramer's formulas*

$$x=\frac{D_x}{D}, \quad y=\frac{D_y}{D}, \quad z=\frac{D_z}{D},$$

where

$$D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0, \quad D_x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \qquad D_z=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}.$$

13. The solutions of the homogeneous system

$$\left.\begin{aligned} a_1x+b_1y+c_1z=0,\\ a_2x+b_2y+c_2z=0,\\ a_3x+b_3y+c_3z=0, \end{aligned}\right\}$$

if

$$D=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3 \end{vmatrix}=0, \qquad \delta=\begin{vmatrix} a_1 & b_1\\ a_2 & b_2 \end{vmatrix}\neq 0$$

are found from the subsystem (see item 10)

$$\left.\begin{aligned} a_1x+b_1y+c_1z=0,\\ a_2x+b_2y+c_2z=0. \end{aligned}\right\}$$

V. Elements of Vector Algebra

1. The sum of the vectors **a**, **b**, **c** is the vector $\mathbf{s}=\mathbf{a}+\mathbf{b}+\mathbf{c}$, which is the closing line (the resultant) of a vector line with the segments **a**, **b**, **c**.

2. The difference between the vectors **a** and **b** is the vector $\mathbf{d}=\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$, where $-\mathbf{b}$ is a negative vector opposite to the vector **b**.

3. The *product of a vector* **a** *by a scalar* k is a vector $\mathbf{b}=k\mathbf{a}$ such that $b=|k|a$, where $a=|\mathbf{a}|$ and $b=|\mathbf{b}|$, the direction of the vector **b** coinciding with that of the vector **a** if $k>0$, and opposite to it if $k<0$.

4. The vectors **a** and **b** are *collinear* if $\mathbf{b}=k\mathbf{a}$ (k scalar).

Vectors **a**, **b**, **c** are *coplanar* if $\mathbf{c}=k\mathbf{a}+l\mathbf{b}$ (k, l scalars).

5. The *scalar product of the vectors* **a** and **b** is a scalar

$$\mathbf{ab}=ab\cos\varphi,$$

where $\varphi=\angle(\mathbf{a},\mathbf{b})$.

The vectors **a** and **b** are *orthogonal* if $\mathbf{ab}=0$.

If $\mathbf{a}=\{a_x, a_y, a_z\}$ and $\mathbf{b}=\{b_x, b_y, b_z\}$, then

$$\mathbf{ab}=a_xb_x+a_yb_y+a_zb_z.$$

6. The *vector product of vectors* **a** and **b** is a vector:

$$\mathbf{c}=\mathbf{a}\times\mathbf{b},$$

where $\mathbf{c}\perp\mathbf{a}$, $\mathbf{c}\perp\mathbf{b}$ and $c=ab\sin\varphi$ ($\varphi=\angle(a,b)$); **a**, **b**, **c** being a right-hand triple.

If $\mathbf{a}=\{a_x, a_y, a_z\}$ and $\mathbf{b}=\{b_x, b_y, b_z\}$, then

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z \end{vmatrix},$$

where **i**, **j**, **k** are unit vectors directed along the corresponding coordinate axes.

7. *Mixed product*

$$\mathbf{abc}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

represents the volume of a parallelepiped constructed on the vectors **a**, **b**, **c**.

If $\mathbf{a}=\{a_x, a_y, a_z\}$, $\mathbf{b}=\{b_x, b_y, b_z\}$, $\mathbf{c}=\{c_x, c_y, c_z\}$, then

$$\mathbf{abc}=\begin{vmatrix} a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z \end{vmatrix}.$$

VI. Solid Analytic Geometry

1. *Cartesian rectangular coordinates of point* $M(x, y, z)$ in the xyz-plane are

$$x = r_x, \quad y = r_y, \quad z = r_z,$$

where $\mathbf{r} = \overrightarrow{OM}$ is the radius vector of the point M.

2. The *length and direction of the vector* $\mathbf{a} = \{a_x, a_y, a_z\}$ are determined by the formulas

$$a = |\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2},$$

$$\cos\alpha = \frac{a_x}{a}, \quad \cos\beta = \frac{a_y}{a}, \quad \cos\gamma = \frac{a_z}{a}$$

$$(\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1),$$

where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are *direction cosines* of the vector $\mathbf{a}$.

3. The *distance between two points* $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$:

$$d = |\overrightarrow{M_1M_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

4. The *equation of a plane* (with a normal vector $\mathbf{N} = \{A, B, C\} \neq 0$) passing through the point $M_0(x_0, y_0, z_0)$ is

$$\mathbf{N} \cdot (\mathbf{r} - \mathbf{r}_0) = 0, \tag{1}$$

where $\mathbf{r}$ is the radius vector of a current point of the plane $M(x, y, z)$ and $\mathbf{r}_0$ is the radius vector of the point M_0.

In coordinates, equation (1) has the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

or

$$Ax + By + Cz + D = 0, \tag{2}$$

where $D = -Ax_0 - By_0 - Cz_0$ (*the general equation of the plane*).

5. The *distance of the point* $M_1(x_1, y_1, z_1)$ *from plane* (2) is equal to

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

6. *The vector equation of a straight line in space*:

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{s}t, \tag{3}$$

where $\mathbf{r} = \{x, y, z\}$ is a current radius vector of the line, $\mathbf{r}_0 = \{x_0, y_0, z_0\}$ the radius vector of a fixed point of the straight line, $\mathbf{s} = \{m, n, p\} \neq 0$ the direction vector of the straight line, and t the parameter $(-\infty < t < +\infty)$.

The equation of straight line (3) in the coordinate form:

$$\frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p}.$$

7. A *straight line as the intersection of two planes* is given by the following equations:

$$\left.\begin{aligned} Ax + By + Cz + D = 0, \\ A'x + B'y + C'z + D' = 0. \end{aligned}\right\} \tag{4}$$

The direction vector of straight line (4) is $\mathbf{s} = \mathbf{N} \times \mathbf{N}'$, where $\mathbf{N} = \{A, B, C\}$, $\mathbf{N}' = \{A', B', C'\}$.

8. The *equation of a sphere* of radius R with (x_0, y_0, z_0) as centre:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2.$$

9. The *equation of a three-axes ellipsoid* with semi-axes a, b, and c:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

10. The equation of the paraboloid of revolution about the z-axis:

$$x^2 + y^2 = 2pz.$$

VII. Differential Calculus—Functions of Several Variables

1. The *condition of continuity of a function* $z = f(x, y)$:

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta z = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} [f(x+\Delta x,\ y+\Delta y)-f(x,\ y)]=0$$

or

$$\lim_{\substack{x_1\to x\\ y_1\to y}} f(x_1,\ y_1)=f(x,\ y).$$

The continuity of the function $u = f(x, y, z)$ is determined in a similar way.

2. *Partial derivatives of the function* $z = f(x, y)$ with respect to the variables x and y:

$$\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,\ y)-f(x,\ y)}{\Delta x},$$

$$\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,\ y+\Delta y)-f(x,\ y)}{\Delta y}.$$

3. The *total differential of the function* $z = f(x, y)$ of independent variables x and y:

$$dz=\frac{\partial z}{\partial x}\,dx+\frac{\partial z}{\partial y}\,dy,$$

where $dx = \Delta x$, and $dy = \Delta y$.

If $u = f(x, y, z)$, then

$$du=\frac{\partial u}{\partial x}\,dx+\frac{\partial u}{\partial y}\,dy+\frac{\partial u}{\partial z}\,dz.$$

4. *The small increment of a differentiable function*

$$\Delta z \approx \frac{\partial z}{\partial x}\,\Delta x+\frac{\partial z}{\partial y}\,\Delta y.$$

5. The (*directional*) *derivative of the function* $u = f(x, y)$ *in the direction* $\mathbf{l} = \{\cos\alpha,\ \cos\beta\}$ is equal to

$$\frac{\partial u}{\partial l}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta.$$

Analogously if $u = f(x, y, z)$ and $\mathbf{l} = \{\cos\alpha,\ \cos\beta,\ \cos\gamma\}$, then

$$\frac{\partial u}{\partial l}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma.$$

6. The *points of extremum* (or more precisely, the points of possible extremum) of a differentiable function $u = f(x, y, z)$ are determined from the equations:

$$f'_x(x, y, z) = 0, \quad f'_y(x, y, z) = 0, \quad f'_z(x, y, z) = 0.$$

7. The *gradient of a scalar field* $u = f(x, y, z)$ is the vector

$$\operatorname{grad} u = \left\{ \frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial u}{\partial z} \right\}.$$

Hence

$$|\operatorname{grad} u| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2}.$$

8. If $P(x, y)\,dx + Q(x, y)\,dy$ is a total differential in a domain G, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad ((x, y) \in G)$$

(*the test for a total differential*)

VIII. Series

1. *The basic definition*:

$$\sum_{n=1}^{\infty} u_n = \lim_{N \to \infty} \sum_{n=1}^{N} u_n.$$

2. The *necessary test for convergence of a series*: if the series $\sum_{n=1}^{\infty} u_n$ converges, then

$$\lim_{n \to \infty} u_n = 0.$$

3. The *geometric progression*:

$$\sum_{n=1}^{\infty} aq^{n-1} = \frac{a}{1-q}, \qquad \text{if} \quad |q| < 1.$$

4. The *harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

(diverges).

5. *D'Alembert's test.* Let for the series $\sum_{n=1}^{\infty} u_n$ $(u_n > 0)$ there exist a limit

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l.$$

Then: (a) if $l < 1$, the series converges; (b) if $l > 1$, then the series diverges and $u_n \not\to 0$.

6. *Absolute convergence.* If the series $\sum_{n=1}^{\infty} |u_n|$ converges, then $\sum_{n=1}^{\infty} u_n$ also converges (*absolutely*).

7. *Leibniz' test.* If $v_1 \geqslant v_2 \geqslant v_3 \geqslant \ldots \geqslant 0$ and $v_n \to 0$ as $n \to \infty$, then the alternating series

$$v_1 - v_2 + v_3 - v_4 + \ldots$$

converges.

8. The *radius of convergence* of the power series

$$a_0 + a_1x + a_2x^2 + \dots$$

is determined by the formula

$$l = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

if the latter has sense.

9. *Maclaurin's series*:

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \dots + \frac{f^{(n)}(0)}{n!}\,x^n + \dots$$

10. *Basic functions expanded into power series*:

(a) $\dfrac{1}{1-x} = 1 + x + x^2 + \dots + x^n + \dots \quad (|x| < 1);$

(b) $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dots + (-1)^{n-1}\dfrac{x^n}{n} + \dots \quad (-1 < x \leqslant 1);$

(c) $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dots + (-1)^{n-1}\dfrac{x^{2n-1}}{2n-1} + \dots \quad (|x| \leqslant 1);$

(d) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dots + \dfrac{x^n}{n!} + \dots \quad (|x| < +\infty);$

(e) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots + (-1)^{n-1}\dfrac{x^{2n-1}}{(2n-1)!} + \dots \quad (|x| < +\infty);$

(f) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots + (-1)^n\dfrac{x^{2n}}{(2n)!} + \dots \quad (|x| < +\infty);$

(g) $(1+x)^m = 1 + mx + \dfrac{m(m-1)}{1\cdot 2}\,x^2 + \dots + \dfrac{m(m-1)\dots[m-(n-1)]}{1\cdot 2\dots n}\,x^n + \dots \quad (|x| < 1).$

11. *Taylor's series*:

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{f''(a)}{2!}\,(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}\,(x-a)^n + \dots$$

12. *The series in complex domain*:

$$\sum_{n=1}^{\infty} (u_n + iv_n) = \sum_{n=1}^{\infty} u_n + i \sum_{n=1}^{\infty} v_n.$$

13. *Absolute convergence of series with complex terms.* If the series

$$\sum_{n=1}^{\infty} |u_n + iv_n| = \sum_{n=1}^{\infty} \sqrt{u_n^2 + v_n^2}$$

converges, then the series $\sum_{n=1}^{\infty}(u_n+iv_n)$ also converges (absolutely).

14. *Euler's formulas*:

$$e^{ix}=\cos x+i\sin x, \qquad e^{-ix}=\cos x-i\sin x.$$

15. The *Fourier trigonometric series* of a piecewise smooth function $f(x)$ with the period $2l$ has the form

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right), \tag{1}$$

where

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\,dx \qquad (n=0,\ 1,\ 2,\ \ldots),$$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\,dx \qquad (n=1,\ 2,\ \ldots)$$

(the Fourier coefficients of the function $f(x)$).

For the function $f(x)$ with the period 2π we get

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

where

$$\left.\begin{matrix}a_n\\ b_n\end{matrix}\right\}=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\left\{\begin{matrix}\cos nx\\ \sin nx\end{matrix}\right.dx \qquad (n=0,\ 1,\ 2,\ \ldots).$$

At the points of discontinuity of the function $f(x)$ the sum of series (1) is equal to

$$S(x)=\frac{1}{2}[f(x-0)+f(x+0)].$$

16. If a $2l$-periodic function $f(x)$ is *even*, then

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l},$$

where

$$a_n=\frac{2}{l}\int_{0}^{l}f(x)\cos\frac{n\pi x}{l}\,dx \qquad (n=0,\ 1,\ 2,\ \ldots).$$

If it is *odd*, then

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l},$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx \qquad (n = 1, 2, \ldots).$$

IX. Differential Equations

1. The *differential equation with variables separable*

$$X(x)\, Y(y)\, dx + X_1(x)\, Y_1(y)\, dy = 0$$

has the *general integral*

$$\int \frac{X(x)}{X_1(x)} dx + \int \frac{Y_1(y)}{Y(y)} dy = C. \qquad (1)$$

Special solutions not entering into integral (1) are determined from the equations

$$X_1(x) = 0 \quad \text{and} \quad Y(y) = 0.$$

2. The first-order *homogeneous differential equation*

$$P(x, y)\, dx + Q(x, y)\, dy = 0,$$

where $P(x, y)$ and $Q(x, y)$ are homogeneous continuous functions is solved with the aid of the substitution

$$y = ux$$

(u new function).

3. The first-order *linear differential equation*

$$a(x)\, y' + b(x)\, y + c(x) = 0$$

can be solved with the aid of the substitution

$$y = uv.$$

4. *Integrable cases* of a second-order differential equation:

(a) if

$$y'' = f(x),$$

then the *general solution*

$$y = \int dx \int f(x)\, dx + C_1 x + C_2;$$

(b) if

$$y'' = f(y),$$

then the *general integral*

$$\int \frac{dy}{\sqrt{2 \int f(y)\, dy + C_1}} = \pm (x + C_2);$$

(c) if

$$y'' = f(y'),$$

then the *general integral* of the equation can be found from the relation

$$\int \frac{dp}{f(p)} = x + C_1,$$

where $y' = p$.

5. The *cases of reduction of the order* for a second-order differential equation: (a) if:

$$y'' = f(x, y'),$$

then, putting $y' = p$, we get

$$\frac{dp}{dx} = f(x, p);$$

(b) if

$$y'' = f(y, y'),$$

then, setting $y' = p$, we shall have

$$p\frac{dp}{dy} = f(y, p).$$

6. The *general solution of a* second-order *linear homogeneous differential equation*

$$y'' + p(x)\, y' + q(x)\, y = 0$$

has the form

$$y = C_1 y_1 + C_2 y_2,$$

where y_1 and y_2 are linear independent particular solutions.

7. The *general solution of a* second-order *linear non-homogeneous differential equation*

$$y'' + p(x)\, y' + q(x)\, y = f(x)$$

has the form

$$y = \bar{y} + z,$$

where $\bar{y}$ is the general solution of the corresponding homogeneous equation and z the particular solution of the given non-homogeneous equation.

8.

Table 1

General Solutions of the Homogeneous Equation $y'' + py' + qy = 0$ (p and q constants) Depending on the Roots of the Characteristic Equation

$$k^2 + pk + q = 0$$

No.	The roots k_1 and k_2 of the characteristic equation are	General solution
I	real and distinct	$y = C_1 e^{k_1 x} + C_2 e^{k_2 x}$
II	equal: $k_1 = k_2$	$y = (C_1 + C_2 x) e^{k_1 x}$
III	complex: $k_1 = \alpha + i\beta$ and $k_2 = \alpha - i\beta$	$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

9.

Table 2

Particular Solutions of the Non-homogeneous Equation $y''+py'+qy=f(x)$ (p and q constants) Depending on the Right Member $f(x)$

No.	Right member $f(x)$	Cases	Particular solution
I	$f(x)=ae^{mx}$ (a, m constants)	1) $m^2+pm+q\neq 0$, 2) $m^2+pm+q=0$	$z=Ae^{mx}$, $z=Axe^{mx}$ or $z=Ax^2e^{mx}$
II	$f(x)=M\cos\omega x+N\sin\omega x$ (M, N, ω constants; $\omega\neq 0$)	1) $p^2+(q-\omega^2)^2\neq 0$, 2) $p=0$, $q=\omega^2$	$z=A\cos\omega x+B\sin\omega x$, $z=x(A\cos\omega x+B\sin\omega x)$
III	$f(x)=ax^2+bx+c$ (a, b, c constants)	1) $q\neq 0$, 2) $q=0$, $p\neq 0$	$z=Ax^2+Bx+C$, $z=x(Ax^2+Bx+C)$

A, B, C constant coefficients.

X. Line Integrals

1. The *line integral of the first kind* of a continuous function $f(x, y)$ taken over a piecewise smooth curve K: $x=x(t)$, $y=y(t)$ $(t\in[\alpha,\beta])$ is equal to

$$\int_K f(x, y)\,ds=\int_\alpha^\beta f(x(t), y(t))\sqrt{x'^2(t)+y'^2(t)}\,|\,dt\,|. \tag{1}$$

If the curve K is given by the equation $y=y(x)$ $(a\leqslant x\leqslant b)$, then

$$\int_K f(x, y)\,ds=\int_a^b f(x, y(x))\sqrt{1+y'^2(x)}\,dx.$$

The line integral of the first kind for a space curve K is determined in an analogous manner.

If $f(x, y)$ is the line density of K, then integral (1) represents the mass of the line K.

2. The *line integral of the second kind* of a pair of continuous functions $X(x, y)$, $Y(x, y)$ taken along a piecewise smooth path K: $x=x(t)$, $y=y(t)$

($t \in [\alpha, \beta]$) is determined by the formula

$$\int_K X(x, y)\,dx + Y(x, y)\,dy = \int_\alpha^\beta [X(x(t), y(t))\,x'(t) + Y(x(t), y(t))\,y'(t)]\,dt. \quad (2)$$

If the path K is given by the equation $y = y(x)$ ($x \in [a, b]$), then

$$\int_K X(x, y)\,dx + Y(x, y)\,dy = \int_a^b [X(x, y(x)) + Y(x, y(x))\,y'(x)]\,dx.$$

The line integral of the second kind for a space curve K is determined in a similar way.

Physically, integral (2) represents the work performed by a variable force $\mathbf{F} = \{X(x, y), Y(x, y)\}$ along the path K.

3. If the condition

$$X(x, y)\,dx + Y(x, y)\,dy = dU(x, y)$$

is fulfilled, then integral (2) is path-independent and

$$\int_K X(x, y)\,dx + Y(x, y)\,dy = U(x, y)\Big|_{(x_1, y_1)}^{(x_2, y_2)} = U(x_2, y_2) - U(x_1, y_1), \quad (3)$$

where (x_1, y_1) is the initial point and (x_2, y_2) the terminal point of the path.

The physical meaning of integral (3): the work performed by a force having the potential $U(x, y)$.

XI. Double and Triple Integrals

1. The *double integral* of the function $f(x, y)$ over the domain S is defined as the number

$$\iint_S f(x, y)\,dS = \lim_{d \to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta S_i, \quad (1)$$

where $(x_i, y_i) \in \Delta S_i$ ($i = 1, 2, \ldots, n$) and d is the maximum diameter of the cells ΔS_i.

If $f(x, y) \geqslant 0$, then, geometrically, integral (1) represents the volume of a right cylindroid constructed on the base S and bounded from above by the surface $z = f(x, y)$.

2. If the domain of integration S is standard with respect to the y-axis and is defined by the inequalities $a \leqslant x \leqslant b$, $y_1(x) \leqslant y \leqslant y_2(x)$, where $y_1(x)$, $y_2(x)$ are continuous functions, then the *double integral in rectangular Cartesian coordinates* of a continuous function $f(x, y)$ is expressed by the formula

$$\iint_S f(x, y)\,dx\,dy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x, y)\,dy.$$

3. The *double integral in the polar coordinates* φ and r, where

$$x = r\cos\varphi, \quad y = r\sin\varphi,$$

has the form

$$\iint\limits_S f(x, y)\,dS = \iint\limits_S f(r\cos\varphi, r\sin\varphi)\,r\,d\varphi\,dr.$$

If the domain of integration S is defined by the inequalities $\alpha \leqslant \varphi \leqslant \beta$, $r_1(\varphi) \leqslant r \leqslant r_2(\varphi)$, then

$$\iint\limits_S f(x, y)\,dS = \int\limits_\alpha^\beta d\varphi \int\limits_{r_1(\varphi)}^{r_2(\varphi)} rf(r\cos\varphi, r\sin\varphi)\,dr.$$

4. If $\rho = f(x, y)$ is the surface density of the plate S, then its *mass* is

$$m = \iint\limits_S \rho(x, y)\,dS = \iint\limits_S \rho\,dx\,dy \tag{2}$$

(*the physical meaning of the double integral*). In particular, for $\rho = 1$ we obtain the formula for the *area of the plate*

$$S = \iint\limits_S dS = \iint\limits_S dx\,dy.$$

5. The *statical moments of the plate* S about the x- and y-axes are expressed by the respective integrals

$$S_x = \iint\limits_S \rho y\,dS, \qquad S_y = \iint\limits_S \rho x\,dS,$$

where $\rho = f(x, y)$ is the surface density of the plate S.

6. The *coordinates of the centre of gravity of the plate* S are determined by the formulas

$$x_0 = \frac{S_y}{m}, \qquad y_0 = \frac{S_x}{m}, \tag{3}$$

where m is the mass of the plate (see (2)).

For a homogeneous plate we may put $\rho = 1$ in formulas (2) and (3).

7. The *moments of inertia of the plate* S about the x- and y-axes are expressed by the respective integrals

$$I_x = \iint\limits_S \rho y^2\,dS, \qquad I_y = \iint\limits_S \rho x^2\,dS,$$

where $\rho = \rho(x, y)$ is the surface density of the plate.

8. The *triple integral* of the function $f(x, y, z)$ over the domain V is the number

$$\iiint\limits_V f(x, y, z)\,dV = \lim_{d\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta V_i, \tag{4}$$

where $(x_i, y_i, z_i) \in \Delta V_i$ $(i = 1, 2, \ldots, n)$ and d is the maximum diameter of the cells ΔV_i.

If $f(x, y, z)$ is the density at the point (x, y, z), then this integral represents the mass filling the volume V.

9. The *volume* V of a solid is equal to

$$V = \int\limits_V \int\int dV = \int\limits_V \int\int dx\, dy\, dz.$$

10. If a domain of integration V is defined by the inequalities $a \leqslant x \leqslant b$, $y_1(x) \leqslant y \leqslant y_2(x)$, $z_1(x, y) \leqslant z \leqslant z_2(x, y)$, where $y_i(x)$, $z_i(x, y)$ $(i = 1, 2)$ are continuous functions, then the *triple integral in rectangular coordinates* of a function $f(x, y, z)$ can be computed by the formula

$$\int\limits_V \int\int f(x, y, z)\, dx\, dy\, dz = \int\limits_a^b dx \int\limits_{y_1(x)}^{y_2(x)} dy \int\limits_{z_1(x, y)}^{z_2(x, y)} f(x, y, z)\, dz.$$

XII. Probability Theory

1. The *sum* of two events A and B

$$A + B = A \cup B$$

is understood to be an event which occurs if and only if at least one of the events A and B is realized.

2. The *product* of two events A and B

$$AB = A \cap B$$

is understood to be an event which takes place if and only if both event A and event B are realized.

3. The *probability* (in the classical sense) of *event* A

$$P(A) = \frac{m}{n} \qquad (0 \leqslant P(A) \leqslant 1)$$

is the ratio of the number m of equally possible elementary outcomes favourable to event A to the number n of all only possible and equally possible elementary outcomes.

4. *The probability of an opposite event*:

$$P(\overline{A}) = 1 - P(A).$$

5. The *addition theorem* for two incompatible events A and B:

$$P(A + B) = P(A) + P(B).$$

In the general case

$$P(A + B) = P(A) + P(B) - P(AB).$$

6. The *multiplication theorem*:

$$P(AB) = P(A)\, P_A(B),$$

where $P_A(B)$ is the corresponding conditional probability of event B.

If events A and B are independent, then

$$P(AB) = P(A) \cdot P(B).$$

7. *The formula for total probability*:

$$P(A) = \sum_{i=1}^{n} P(H_i)\, P_{H_i}(A),$$

where $H_1, H_2, \ldots, H_n$ is a complete group of hypotheses;

$$A=\sum_{i=1}^{n} H_i A, \quad H_i H_j = O \quad \text{for} \quad i \neq j, \quad \sum_{i=1}^{n} P(H_i)=1.$$

8. Bayes' formula:

$$P_A(H_i)=\frac{P(H_i)\,P_{H_i}(A)}{\sum_{j=1}^{n} P(H_j)\,P_{H_j}(A)}$$

($i = 1, 2, \ldots, n$), where $H_1, H_2, \ldots, H_n$ is a complete group of hypotheses.

9. *The basic formulas of combinatorial analysis*:

(a) the *number of distributions* of n elements m at a time:

$$A_n^m = n(n-1)\ldots[n-(m-1)] = \frac{n!}{(n-m)!};$$

(b) the *number of permutations* of n elements: $A_n^n = n!$;

(c) the *number of combinations* of n elements m at a time

$$C_n^m = \frac{n(n-1)\ldots[n-(m-1)]}{m!} = \frac{n!}{m!\,(n-m)!}.$$

10. *The binomial formula*:

$$(q+p)^n = q^n + C_n^1 q^{n-1} p + C_n^2 q^{n-2} p^2 + \ldots + p^n.$$

11. The *binomial distribution law*: under the conditions of the Bernoulli scheme the probability of the occurrence of event A in n trials exactly m times ($0 \leqslant m \leqslant n$) is equal to

$$P_n(m) = C_n^m p^m q^{n-m} = \frac{n!}{m!\,(n-m)!}\, p^m q^{n-m},$$

where $P(A) = p$, $P(\bar{A}) = q = 1 - p$ in a single trial.

12. The *Laplace local formula*:

$$P_n(m) \approx \frac{1}{\sqrt{2\pi npq}}\, e^{-\frac{t^2}{2}}$$

(see item 11), where $0 < p < 1$, $q = 1 - p$, $t = (npq)^{-\frac{1}{2}}(m - np)$.

13. The *Laplace integral formula*:

$$P(m_1 \leqslant m \leqslant m_2) \approx \Phi_0(t_{m_2}) - \Phi_0(t_{m_1}),$$

where

$$t_m = (npq)^{-\frac{1}{2}}(m-np), \qquad \Phi_0(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}}\, dt.$$

14. *Poisson's formula*:

$$P_n(m) \approx \frac{\mu^m}{m!}\, e^{-\mu},$$

where $\mu = np$, the probability p being small.

15. *Mathematical expectation* of a discrete random variable $X=\{x_1, x_2, \ldots, x_n\}$, where $p_i=P(X=x_i)\,(i=1, 2, \ldots, n)$, $\sum_{i=1}^{n} p_i=1$, is

$$M(X)=\sum_{i=1}^{n} x_i p_i.$$

Its basic properties:

(1) $M(C)=C$;

(2) $M(X+Y)=M(X)+M(Y)$; (4) $M(CX)=CM(X)$;

(3) $M(X-Y)=M(X)-M(Y)$; (5) $M(XY)=M(X)\,M(Y)$ (X and Y independent).

16. The *variance* of a discrete random variable X:

$$D=M\{[X-M(X)]^2\}=M(X^2)-[M(X)]^2.$$

Its basic properties:

(1) $D(C)=0$;

(2) $D(X\pm Y)=D(X)+D(Y)$ (X and Y independent);

(3) $D(CX)=C^2D(X)$.

17. For the binomial law of distributions of the number of occurrences X of event A in n trials we have:

(1) $P(X=m)=C_n^m p^m q^{n-m}$ $(m=0, 1, 2, \ldots, n)$,

where $p=P(A)$, $q=P(\overline{A})$;

(2) $M(X)=np$; (3) $D(X)=npq$.

18. For a continuous random variable X we have

(a) *distribution function*

$$\Phi(x)=P(-\infty<X<x)=\int_{-\infty}^{x}\varphi(t)\,dt$$

($-\infty<x<+\infty$), where $\varphi(x)$ is probability density;

(b) *mathematical expectation*

$$M(X)=\int_{-\infty}^{+\infty} x\varphi(x)\,dx;$$

(c) *variance*

$$D(X)=\int_{-\infty}^{+\infty}[x-M(X)]^2\varphi(x)\,dx.$$

19. For the *normal law of distribution* of a random variable X the probability density has the form

$$\varphi(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-x_0)^2}{2\sigma^2}}$$

where $x_0=M(x)$, $\sigma=\sqrt{D(X)}$.

In this case

$$P(a\leqslant x\leqslant b)=\Phi_0\left(\frac{b-x_0}{\sigma}\right)-\Phi_0\left(\frac{a-x_0}{\sigma}\right),$$

where $\Phi_0(x)$ (see item 13) is the standard integral of probabilities.

Answers

Chapter 1

2. (0, 0), (10, 0), $(5, 5\sqrt{3})$. 3. (a) $M(1, -2)$; (b) $M(-1, 2)$; (c) $M(2, 1)$; (d) $M(-2, -1)$. 4. $A_1(8, 4)$, $B_1(11, 3)$. 5. $x_1=6$, $y_1=0$; $x_2=0$, $y_2=0$. 6. $C\left(2\frac{4}{5}, 6\frac{1}{5}\right)$. 7. $B(4, 5)$. 8. $\frac{10}{3}\sqrt{2}$. 9. $(4, 3)$. 10. $N\left(1\frac{1}{3}, 1\right)$. 11. $x_i = x_0+ih$, $y_i=y_0+ik$ $(i=1, 2, \ldots, n-1)$, where $h=\frac{x-x_0}{n}$ and $k=\frac{y-y_0}{n}$. 12. 12. 14. $y_1=10$, $y_2=-5$. 15. $\frac{1}{2}$ ha.

Chapter 2

1. $y=2x$ and $y=-2x$. 2. $5x+y+2=0$. 3. A circle: $x^2+y^2=4$. 4. (a) A collection of coordinate axes; (b) the origin (0, 0); (c) a pair of straight lines parallel to the y-axis and lying on both of its sides at a distance equal to unity; (d) a pair of straight lines parallel to the x-axis and lying at distances equal to 1 and 2 from it; (e) a collection of the bisector of the first and third quadrants and y-axis. 6. Points A and B lie on the curve; points C, D, and E do not lie on the curve. 7. (0, 2), (−1, 0), (2, 0). 8. (2, 2) and (−2, −2).

Chapter 3

2. $y=0$, $y=2\sqrt{3}$, $y=\sqrt{3}(x+5)$, $y=-\sqrt{3}(x-5)$. 3. $x-3y+9=0$, $3x+y-13=0$. 4. $12x+8y-15=0$. 5. $x+2y=0$. 6. $7x-2y-1=0$, $\sqrt{53}$. 7. $7x+8y-11=0$, $\frac{71}{\sqrt{113}}$. 8. $6x+5y-56=0$. 9. $x+y-7=0$. 10. $y=0.1x-5$; 25m, 150m. 11. (a) $x=305$, $y=215$; (b) no point of intersection, the lines are parallel; (c) $x=t$, $y=2t$, where t is arbitrary, the lines coincide. 12. $y=x$. 13. $(-4, 0)$. 14. $x=\frac{ak_2}{k_2-k_1}$, $y=\frac{ak_1k_2}{k_2-k_1}$, where $k=\tan\alpha$ and $k_2=\tan\beta$. 15. $\left(1\frac{1}{3}, 1\right)$. 16. 13. 17. 2, 1, 0. 18. 6, 4. 19. 7. 20. $8x-6y-15=0$, $8x-6y+25=0$. 21. $y=\frac{3}{4}x$ and $y=\frac{7}{24}x$.

Chapter 4

1. (a) $C(-4,\ 0)$, $R=5$; (b) $x^2+y^2=2x$; (c) $x^2+y^2-x-y+\frac{1}{4}=0$; (d) $x^2+y^2-4x-8y+7=0$. 2. $5x-7x+13=0$; $\frac{1}{2}\sqrt{74}$. 3. $x-y-1=0$. 4. $a=2\sqrt{2}$, $b=2$, $c=2$, $F(2,\ 0)$, $F'(-2,\ 0)$, $\varepsilon=\frac{1}{\sqrt{2}}$. 5. (a) $\frac{x^2}{25}+\frac{y^2}{9}=1$; (b) $\frac{x^2}{16}+\frac{y^2}{9}=1$; (c) $\frac{x^2}{100}+\frac{y^2}{36}=1$. 6. 4. 7. (a) $a=2$, $b=\frac{3}{2}$, $c=\frac{5}{2}$, $F\left(2\frac{1}{2},\ 0\right)$, $F'\left(-2\frac{1}{2},\ 0\right)$, $\varepsilon=1.25$; (b) $\frac{x^2}{16}-\frac{y^2}{9}=1$. 8. $\frac{150}{\sqrt{481}}$. 9. $\frac{5}{3}$, $\frac{5}{4}$. 10. $F_1F_2=5\sqrt{2}$. 11. $\pm x\pm y=10$. 12. $\frac{5}{3}$. 13. $y^2=20x$. 14. $(0,\ 1)$; $y=-1$. 15. 7.5 cm from the vertex. 16. 24. 17. $y=1+\frac{x^2}{4}$. 18. $y_1=2x_1^2$, $x_1=x-2$, $y_1=y+3$; $O_1(2,\ -3)$. 19. $y_1=-x_1^2$, $x_1=x-2$, $y_1=y-1$; $O_1(2,\ 1)$. 20. $y_1^2=x_1$, $x_1=x-\frac{7}{4}$, $y_1=y-\frac{1}{2}$; $O_1\left(\frac{7}{4},\ \frac{1}{2}\right)$. 21. $h_1=9$m.

Chapter 5

2. $A(5,\ 0)$, $B(3\sqrt{2},\ 3\sqrt{2})$, $C(0,\ 2)$, $D(-2\sqrt{2},\ -2\sqrt{2})$. 4. (a) $\rho=\sec\varphi$; (b) $\rho=-2\operatorname{cosec}\varphi$; (c) $\varphi=\frac{\pi}{4}$ and $\varphi=\frac{5\pi}{4}$; (d) $\varphi=\arctan 2$ and $\varphi=\pi+\arctan 2$; (e) $\rho=\sec\left(\varphi-\frac{\pi}{4}\right)$; (f) $\rho=5$. 5. $\rho=\frac{p}{\cos(\varphi-\alpha)}$. 6. $x^2+y^2=ax$; a circle with centre at point $C\left(\frac{a}{2},\ 0\right)$ of radius $R=\frac{|a|}{2}$. 7. $y^2=2x+1$ (parabola). 8. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (ellipse). 9. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (hyperbola). 10. $y^2=16x$ (parabola). 11. 505, $(-305,\ -405)$.

Chapter 6

1. $y=bx\left(1-\frac{x}{2h}\right)$ $(0\leqslant x\leqslant h)$. 2. (a) $x\geqslant 2$; (b) $|x|\geqslant 1$; (c) $0<x<1$; (d) $x>-1$. 3. $f(0)=2$, $f(1)=0$, $f(2)=0$, $f(3)=2$, $f(-x)=x^2+3x+2$, $f\left(\frac{1}{x}\right)=\frac{1-3x+2x^2}{x^2}$, $f(x+1)=x^2-x$. 4. -1.25. 5. $y=2x-10$. 6. $f(x)=x^2+2x-3$. 7. $\varphi[\psi(x)]=2^{2x}$, $\psi[\varphi(x)]=2^{x^2}$. 8. $f[f(x)]=1-\frac{1}{x}$; $f\{f[f(x)]\}=x$. 9. (a) $x=\frac{y-3}{2}$; (b) $x=\sqrt[3]{1-y^3}$, $x=2$ Arcsin y. 11. $x_1=-1.88$; $x_2=0.35$; $x_3=1.53$. 12. 4.49. 13. 1.8796, 0.6928.

Chapter 7

1. (a) $-3<x<-1$; (b) $-\infty<x\leqslant 0$, $6\leqslant x<+\infty$; (c) $\frac{1}{2}<x<1$, $1<x<1\frac{1}{2}$; (d) $-2\leqslant x\leqslant -1$, $1\leqslant x\leqslant 2$. 2. $\approx 0.4\%$. 3. Two. 4. (a) 0; (b) 1; (c) $-\frac{1}{2}$. 5. (a) $\frac{1}{6}$; (b) $\frac{1}{12}$. 6. 0. 7. 5. 8. $\frac{1}{2}$. 9. $\cos a$. 10. 2. 11. $\lim\limits_{a\to 0} x_1 = -\frac{c}{b}$; $\lim\limits_{a\to 0} x_2=\infty$. 13. 1. 14. 0. 15. e^{-1}. 16. e^3. 17. e^{-2}. 18. e^{-1}.

Chapter 8

1. $\Delta x = 90$, $\Delta y = 1$. 5. -1, -2, -3. 6. $\frac{n\pi}{2}+\frac{\pi}{8}$ $(n=0, \pm 1, \pm 2, \ldots)$. 7. $0, \pm 1, \pm 2, \ldots$ 8. 3. 9. $\frac{1}{4}$. 10. 2. 11. (a) $x=0$, a point of discontinuity of the first kind; (b) $x=1$, a point of discontinuity of the first kind; (c) $x=0$, a point of discontinuity of the second kind.

Chapter 10

1. (a) $y'=6x-1$; (b) $f'(x)=-2/x^3$, $f'(1)=-2$, $f'(-10)=0.002$. 2. $y'=\frac{1}{\sqrt{x}}+\frac{2}{\sqrt[3]{x}}+\frac{1}{3x\sqrt[3]{x}}$, $y'=x-\frac{4}{x^3}$. 4. $y'=\frac{a}{a+b}$. 5. $y'=6x^2-2x$. 6. $y'=(3x+1)/2\sqrt{x}$. 7. $y'=x^2\cos x+2x\sin x$. 8. $y'=-2x/(1+x^2)^2$. 9. $y'=\frac{2(1+x^2)}{(1-x^2)^2}$. 10. $y'=\frac{2}{(\sin x+\cos x)^2}$. 11. $y'=\sin 2x$. 12. $y'=2x\cos x^2$. 13. $y'=-\frac{3}{4}\sin x\cos\frac{x}{2}$. 14. $y'=-\frac{3}{2}x^2\sin\frac{x^3}{2}$. 15. $y'=\frac{x}{\sqrt{1+x^2}}$. 16. $y'=\frac{1}{x\ln x}$. 17. $y'=\frac{2\ln x}{x}$. 18. $y'=\frac{2}{x}$. 19. $y'=\frac{1}{\sin x}$. 20. $y'=nx^{n-1}+n^x\ln n$. 21. $y'=-2xe^{-x^2}$. 22. $y'=\frac{x^2}{2}e^{\frac{x}{2}}$. 23. $y'=\frac{1}{\sqrt{4-x^2}}$. 24. $y'=\frac{1}{|x|\sqrt{x^2-1}}$, if $|x|>1$. 25. $y'=-\frac{1}{x^2+1}$. 26. $y'=-\frac{16\cos 2x}{\sin^3 2x}$. 27. $y'=\frac{x^2}{1+x^2}$. 28. $y'=\frac{1}{\sqrt{1+x^2}}$. 29. $y'=\sqrt{1-x^2}$. 30. $y'=\frac{1}{1-x^4}$. 31. $y'=\frac{2x-y}{x-2y}$. 32. $y'=\frac{x^2-y}{x-y^2}$. 33. (a) $y'=\frac{y}{y-1}$; (b) $y'=-3$. 34. (a) $y'_x=-\frac{b}{a}\cot t$; (b) $y'_x=\frac{1+\alpha t}{\omega}e^{\alpha t}$; (c) $y'_x=-1$. 35. (a) $y''=(4x^2-2)e^{-x^2}$; (b) $y''=-4\sin 2x$. 36. $y''=-1/x^2$. 37. $y''=(x^2-4x+2)e^{-x}$. 38. $f(0)=1$, $f'(0)=0$, $f''(0)=-9$. 39. (a) $y=12x-16$; (b) $y=\pi-x$; (c) $x-4y+8=0$. 40. $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$. 41. $y=\frac{\sqrt{3}}{5}(20-x)$. 42. (a) $y=-x$; (b) $4x+y-36=0$; (c) $4x+3y=0$. 43. 40π m²/s. 44. ≈ 0.8 m/s. 45. $v=\alpha+\beta t$, $j=\beta$. 46. $v=1-\cos t$, $j=\sin t$.

Chapter 11

3. $(-\infty, 2)$, $(2, +\infty)$. 4. $(-\infty, -1)$, $(-1, 1)$, $(1, +\infty)$. 5. $(-\infty, 1)$, $(1, +\infty)$. 6. $\frac{1}{6}$. 7. $\ln\frac{3}{2}$. 8. $\frac{1}{3}$. 9. $\frac{2}{\pi}$. 10. 0. 11. 0. 12. $\frac{2}{\pi}$. 13. 2. 14. 1/2. 15. 1. 16. $1/e$. 17. 1. 18. $x=a/2$, $y=a^2/4$ (max). 19. $x=1$, $y=1/3$ (max); $x=3$, $y=-1$ (min). 20. $x=1$, $y=1$ (max); $x=-1$, $y=-1$ (min). 21. $x=1/2$, $y=3/2$ (max). 22. $x=1/2$, $y=1/2e$ (max). 23. Sides of rectangle: $a\sqrt{2}$ and $a\sqrt{2}/2$. 24. $a\sqrt{2}$, $b\sqrt{2}$. 25. Altitude of cone is $4a/3$. 26. $a/6$, $2a/3$, $2a/3$. 27. $x=1/3$, $y=2/27$.

28. Domain of definition: $-\infty<x<+\infty$; $\lim\limits_{x\to-\infty} y=+\infty$, $\lim\limits_{x\to+\infty} y=-\infty$. Odd function. No points of discontinuity. Points of intersection with the coordinate axes: $(-\sqrt{3}, 0)$, $(0, 0)$ and $(\sqrt{3}, 0)$. Extremum points: $y_{\min}=-2$ for $x=-1$ and $y_{\max}=2$ for $x=1$. Inflection point $(0, 0)$.

29. Domain of definition $-\infty<x<+\infty$; $\lim\limits_{x\to-\infty} y=\lim\limits_{x\to+\infty} y=+\infty$. Non-negative function. No points of discontinuity. Zeros of function: $x_1=0$, and $x_2=2$. Extremum points: $y_{\min}=0$ for $x=0$, and $x=2$; $y_{\max}=1$ for $x=1$. Inflection points: $\left(1-\frac{\sqrt{3}}{3}, \frac{4}{9}\right)$ and $\left(1+\frac{\sqrt{3}}{3}, \frac{4}{9}\right)$.

30. Domain of definition: $-\infty<x<+\infty$; $\lim\limits_{x\to-\infty} y=\lim\limits_{x\to+\infty} y=1$. Even function. No points of discontinuity. Zero of function $x=0$. Extremum point $y_{\min}=0$ for $x=0$. Inflection points: $\left(\mp\frac{1}{\sqrt{3}}, \frac{1}{4}\right)$.

31. Domain of definition: $-\infty<x<0$ and $0<x<+\infty$; $\lim\limits_{x\to-\infty} y=-\infty$, $\lim\limits_{x\to-0} y=-\infty$, $\lim\limits_{x\to+0} y=+\infty$, $\lim\limits_{x\to+\infty} y=+\infty$. Odd function. Point of discontinuity $x=0$. No zeros; the function is negative for $x<0$ and positive for $x>0$. Extremum points: $y_{\max}=-2$ for $x=-1$ and $y_{\min}=2$ for $x=1$. No points of inflection; concave downward for $x<0$, and upward for $x>0$.

32. Domain of definition: $-\infty<x<-1$ and $-1<x<+\infty$; $\lim\limits_{x\to-\infty} y=1$, $\lim\limits_{x\to-1-0} y=+\infty$, $\lim\limits_{x\to-1+0} y=-\infty$, $\lim\limits_{x\to+\infty} y=1$. Point of discontinuity $x=-1$. Zero of function $x=0$. No extremum points; the function increases in the intervals $(-\infty, -1)$ and $(-1, +\infty)$. No inflection points; concave upward for $x<-1$ and downward for $x>-1$.

33. Domain of definition: $-\infty<x<-2$, $-2<x<2$ and $2<x<+\infty$; $\lim\limits_{x\to-\infty} y=0$, $\lim\limits_{x\to-2-0} y=-\infty$, $\lim\limits_{x\to-2+0} y=+\infty$, $\lim\limits_{x\to2-0} y=+\infty$, $\lim\limits_{x\to2+0} y=-\infty$, $\lim\limits_{x\to+\infty} y=0$. Even function. Points of discontinuity: $x_1=-2$, and $x_2=2$. Extremum point $y_{\min}=2$ for $x=0$. No inflection points; concave downward for $x<-2$ and $x>2$; concave upward for $-2<x<2$.

34. Domain of definition: $-\infty<x\leqslant 1$; $\lim\limits_{x\to-\infty} y=-\infty$, $y(1)=0$. No points of discontinuity. Zeros of function $x_1=0$ and $x_2=1$; the function is negative for $x<0$ and positive for $x>0$. No points of discontinuity. Extremum points: $y_{\max}=\frac{2}{9}\sqrt{3}\approx 0.38$ for $x=\frac{2}{3}$ and $y_{\min}=0$ (boundary) for $x=1$. No inflection points; concave downward.

35. Domain of definition: $0<x\leqslant 1$; $\lim\limits_{x\to+0} y=+\infty$, $y(1)=0$. Discontinuity

point $x=0$. Zero of function $x=1$; the function is non-negative. Decreasing function; boundary minimum $y=0$ for $x=1$. Inflection point $\left(\frac{3}{4}, \frac{1}{\sqrt{3}}\right)$.

36. Periodic function with period 2π; $y(0)=y(2\pi)=1$. No points of discontinuity. Zeros of function: $x_1=\frac{3\pi}{4}$ and $x_2=\frac{7\pi}{4}$. Extremum points: $y_{\max}=\sqrt{2}$ for $x=\frac{\pi}{4}$ and $y_{\min}=-\sqrt{2}$ for $x=\frac{5\pi}{4}$. Inflection points: $\left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$.

37. Domain of definition: $0<x<+\infty$; $\lim\limits_{x\to+0} y=0$ and $\lim\limits_{x\to+\infty} y=+\infty$. Removable singularity $x=0$. Zero of function $x=1$; the function is negative for $0<x<1$ and positive for $1<x<+\infty$. Extremum point $y_{\min}=-\frac{1}{e}\approx-0.37$ for $x=\frac{1}{e}$. No inflection points; concave upward.

Hint. Using l'Hospital's rule, it is proved that as $N\to+\infty$ the number N increases slower than its natural logarithm, i.e.

$$\lim_{N\to+\infty}\frac{\ln N}{N}=0.$$

Therefore

$$\lim_{x\to+0} y=\lim_{x\to+0} x\ln x=\lim_{x\to+0}\frac{-\ln\frac{1}{x}}{\frac{1}{x}}=0.$$

38. Domain of definition: $-\infty<x<+\infty$; $\lim\limits_{x\to-\infty} y=-\infty$, $\lim\limits_{x\to+\infty} y=0$. No points of discontinuity. Zero of function $x=0$; the function is negative for $x<0$ and positive for $x>0$. Extremum point $y_{\max}=\frac{1}{e}\approx 0.37$ for $x=1$. Inflection point $\left(2, \frac{2}{e^2}\right)$, where $\frac{2}{e^2}\approx 0.27$.

Hint. To find the limit $\lim\limits_{x\to+\infty} y$, take advantage of the hint to the preceding example. Namely,

$$\lim_{x\to+\infty} y=\lim_{x\to+\infty} xe^{-x}=\lim_{x\to+\infty}\frac{x}{e^x}=\lim_{x\to+\infty}\frac{\ln(e^x)}{e^x}=0.$$

39. Domain of definition: $-\infty<x<+\infty$; $\lim\limits_{x\to+\infty} y=-\infty$, $\lim\limits_{x\to-\infty} y=+\infty$. Odd function. No points of discontinuity. Zero of function $x=0$; the function is negative for $x<0$ and positive for $x>0$. No extremum points; an increasing function. Inflection point (0, 0); concave upward for $x<0$ and downward for $x>0$.

Chapter 12

1. $dy=6x\,dx$. **2.** $dy=x\cos x\,dx$. **3.** $dy=\frac{1+x^2}{(1-x^2)^2}dx$. **4.** $dy=-\frac{x\,dx}{\sqrt{1-x^2}}$. **5.** $dy=\frac{dx}{x}$. **6.** $dy=2x\,dx=2(2-t+t^2)(-1+2t)\,dt$. **7.** (a) $\Delta y=4$, $dy=2$; (b) $\Delta y=0.211$, $dy=0.2$. **8.** 30.301 m³, 30 m³. **9.** (a) 0.983; (b) 0.495; (c) 0.795. **10.** $\Delta y=0.07$. **11.** $dy=-2xe^{-x^2}dx$, $d^2y=(4x^2-2)e^{-x^2}dx^2$.

Chapter 13

1. $\frac{x^3}{3}-\frac{3x^2}{2}+x+C$. 2. $x-2\ln|x|-\frac{1}{x}+C$. 3. $-\frac{3}{\sqrt[3]{x}}+\frac{12}{5\sqrt[6]{x^5}}-$ $-\frac{3}{2x\sqrt[3]{x}}+C$. 4. $\frac{a^{2x}}{\ln a^2}+\frac{b^{2x}}{\ln b^2}+\frac{2(ab)^x}{\ln ab}+C$. 5. $-\frac{\cos 3x}{3}+\frac{\sin 5x}{5}+C$. 6. $\tan x-\cot x+C$. 7. $\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+C$. 8. $\frac{1}{2\sqrt{6}}\ln\left|\frac{\sqrt{2}+x\sqrt{3}}{\sqrt{2}-x\sqrt{3}}\right|+C$. 9. $\frac{1}{\sqrt{3}}\arcsin\left(x\sqrt{\frac{3}{2}}\right)+C$. 10. $\arcsin x+\ln(x+\sqrt{1+x^2})+C$. 11. $-x+$ $+\tan x+C$. 12. $\frac{1}{2}\ln(1+x^2)+C$. 13. $-\frac{2}{3}\sqrt{2-3x}+C$. 14. $-\left(x+\frac{x^2}{2}\right)-$ $-\ln|1-x|+C$. 15. $x-\arctan x+C$. 16. $\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|-x+C$. 17. $x+$ $+\ln(1+x^2)+C$. 18. $\frac{1}{4}\ln\left|\frac{2+x}{2-x}\right|+C$. 19. $\ln\left|\frac{x-3}{x-2}\right|+C$. 20. $\frac{x}{2}+\frac{1}{20}\sin 10x+$ $+C$. 21. $-\frac{1}{16}\cos 8x+\frac{1}{12}\cos 6x+C$. 22. $\frac{1}{4}\sin 2x+\frac{1}{16}\sin 8x+C$. 23. $\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+C$. 24. $2\sqrt{x}-2\ln(1+\sqrt{x})+C$. 25. $\frac{x}{\sqrt{1-x^2}}+$ $+C$. 26. $x-\ln(1+2e^x)+C$. 27. $x\arctan x-\frac{1}{2}\ln(1+x^2)+C$. 28. $\frac{x^2}{2}\ln x-$ $-\frac{x^2}{4}+C$. 29. $-x\cos x+\sin x+C$. 30. $(x-1)e^x+C$. 31. $-\frac{\ln x}{x}-\frac{1}{x}+C$. 32. $(x^2-2x+2)e^x+C$. 33. $\frac{1}{\sqrt{21}}\arctan\left(x\sqrt{\frac{3}{7}}\right)+C$. 34. $\frac{1}{2\sqrt{10}}\times$ $\times\ln\left|\frac{x\sqrt{5}-\sqrt{2}}{x\sqrt{5}+\sqrt{2}}\right|+C$. 35. $\frac{1}{2}\arctan\frac{x+1}{2}+C$. 36. $\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C$. 37. $\frac{1}{\sqrt{5}}\ln\left|\frac{2x-3-\sqrt{5}}{2x-3+\sqrt{5}}\right|+C$. 38. $\frac{1}{10}\ln\left|\frac{x-1}{x+9}\right|+C$. 39. $\ln\frac{|x+5|^5}{(x+3)^2}+C$. 40. $x-2\ln|x^2+4x+5|+3\arctan(x+2)+C$. 41. $\frac{3}{8}\sqrt[3]{(2x-3)^4}+C$. 42. $\frac{2}{3}\sqrt{(x+1)^3}-4\sqrt{x+1}+C$. 43. $\arcsin\frac{x}{\sqrt{2}}+C$. 44. $\ln|x+\sqrt{x^2-2}|+C$. 45. $\ln\left|x+\frac{1}{2}+\sqrt{x^2+x}\right|+C$. 46. $\arcsin\frac{x-1}{\sqrt{2}}+C$. 47. $\ln|x+2+$ $+\sqrt{x^2+4x+3}|+C$. 48. $-\frac{1}{4}\cos^4 x+C$. 49. $\frac{1}{8}x-\frac{1}{32}\sin 4x+C$. 50. $-\cos x+\frac{1}{3}\cos^3 x+C$. 51. $\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$. 52. $\frac{x}{2}\cos\alpha-$ $-\frac{1}{4}\sin(2x+\alpha)+C$. 53. $\tan x+\frac{1}{3}\tan^3 x+C$. 54. $-(x^2+2x+2)e^{-x}+C$. 55. $(x^2+3)e^x+C$. 56. $-\frac{x-1}{2}\cos 2x+\frac{1}{4}\sin 2x+C$. 57. $\frac{1}{13}e^{-2x}(3\sin 3x-$ $-2\cos 3x)+C$. 58. $\frac{x}{2}+\frac{x^2+1}{2}\operatorname{arccot} x+C$. 59. $x(\ln^2 x-2\ln x+2)+C$. 60. $x\arcsin x+\sqrt{1-x^2}+C$.

Chapter 14

1. 2. 2. 0.2. 3. 1. 4. 4.5. 5. (a) e^{y^2}; (b) $-e^{x^2}$. 6. 13/6. 7. (a) $-\pi$; (b) $-e^{-1}$. 8. (a) π; (b) $\pi/2$. 10. (a) 1; (b) $2\pi/3$; (c) $2\pi/3\sqrt{3}$.

Chapter 15

1. $\frac{1}{3}$. 2. $a^2 \ln 2$. 3. $\frac{4}{3}$. 4. πa^2. 5. 1. 6. $2\frac{2}{3}$. 7. $5\frac{1}{3}$. 8. $2-\frac{1}{\ln 2} \approx 0.56$. 9. $21\frac{1}{3}$. 9.1. $\frac{3}{8}\pi a^2$. 10. $17\frac{1}{3}$. 11. $\frac{1}{4}(e^2+1) \approx 2.10$. 11.1. $8a$. 12. a^2. 13. $\frac{\pi a^2}{4}$. 14. $\frac{a}{m}\sqrt{1+m^2}$. 15. $8a$. 16. $\frac{h}{6}[AB+ab+(A+a)(B+b)]$. 17. $4\pi R^3/3$. 18. $16\pi/15$. 19. $\pi^2/2$. 20. 2π. 21. $64\pi/15$. 22. $\pi a^2 h/2$. 23. 2π. 24. 25 m; 2.5 m/s. 25. 25 cal. 26. $2q_0 l/\pi$. 27. $\approx$ 17 400 J. 28. $2a^3/3$.

Chapter 16

1. $z = -\frac{1}{2}(1 + i\sqrt{3})$. 2. (a) Vertical strip; (b) half-plane; (c) angle; (d) interior of a unit circle. 3. The represented domain is a curvilinear triangle bounded by the curves: $v^2 = 4(1 - u)$, $v^2 = 4(1 + u)$, and $v = 0$ ($w = u + iv$).

Chapter 17

1. (a) 1; (b) 1. 2. $x = 280$, $y = -310$. 3. (1) $x = t$, $y = 1 - t$ ($-\infty < t < +\infty$) for $\alpha = 2$; (2) no solutions for $\alpha \neq 2$. 4. $x = (5 - 3\alpha)/(25 + \alpha)$, $y = 16/(25 + \alpha)$, if $\alpha \neq -25$; for $\alpha = -25$ the lines are parallel. 5. $x = 39t$, $y = 23t$, $z = 6t$ ($-\infty < t < +\infty$); $x = 39$, $y = 23$, $z = 6$ for $t = 1$. 6. (a) 0; (b) 12; (c) $2x^3$. 7. $x_1 = -1$, $x_2 = 2$. 8. $D = 84$, $D_x = 14$, $D_y = -84$, $D_z = 70$; $x = \frac{1}{6}$, $y = -1$, $z = \frac{5}{6}$. 9. $x = 17t$, $y = 2t$, $z = -7t$ ($-\infty < t < +\infty$). 10. $x = 0$, $y = 0$, $z = 0$ for $\alpha \neq 0$; $x = t$, $y = -t$, $z = -t$ ($-\infty < t < +\infty$) for $\alpha = 0$. 11. $x_1 = 55/127 \approx 0.4331$, $x_2 = -248/127 \approx \approx -1.9528$, $x_3 = -183/127 \approx -1.4403$, $x_4 = -70/127 \approx -0.5512$.

Chapter 18

1. $a = 2$; $\alpha = \frac{\pi}{3}$, $\beta = \frac{2\pi}{3}$, $\gamma = \frac{\pi}{4}$. 2. $F = 10$; $\alpha = \frac{\pi}{2}$, $\beta = 0$, $\gamma = \frac{\pi}{2}$. 3. $\frac{3}{\sqrt{2}}$. 4. 20. 5 $S = 8\sqrt{3}$; $\cos\varphi = \frac{1}{7}$, $\sin\varphi = \frac{4}{7}\sqrt{3}$. 6. Noncoplanar.

Chapter 19

1. (a) A combination of the coordinate planes Oyz and Oxz; (b) a combination of the coordinate plane Oxy and the bisecor plane of the dihedral angle between the coordinate planes Oxz and Oyz; (c) a pair of parallel planes $y = -2$ and $y = 1$; (d) a parabolic cylinder; (e) a straight line parallel to the x-axis; (f) the coordinate plane Oyz; (g) the z-axis; (h) the point O (0, 0, 0). 2. $p = 4$;

$\alpha=60°$, $\beta=120°$, $\gamma=45°$. 3. $\frac{x}{2}+\frac{y}{3}=1$. 4. $z=3$. 5. ≈ 1.31. 6. 60°, 120°, 135°. 7. $\left(0, -\frac{1}{2}, 0\right)$; $(-1/3, 0, -2/3)$; $(0, -1/2, 0)$. 8. $x^2+y^2+z^2-4x+4y-2z=0$. 9. $x=R\cos\alpha\cos\varphi$, $y=R\cos\alpha\sin\varphi$, $z=R\sin\alpha$. 10. $a=2$, $b=\sqrt{2}$, $c=2/\sqrt{3}$. 11. $3\sqrt{3/2}$, $\sqrt{3}$. 12. $\frac{x-a\cos t_0}{-a\sin t_0}=\frac{y-a\sin t_0}{a\cos t_0}=\frac{z-bt_0}{b}$; $\arccos\frac{b}{\sqrt{a^2+b^2}}$; $x^2+y^2=a^2$, $x=a\cos\frac{z}{b}$; $y=a\sin\frac{z}{b}$.

Chapter 20

1. (a) A strip $|y|\leqslant 1$; (b) exterior of the circle $x^2+y^2\geqslant 1$; (c) half-plane $x+y>0$. 2. (a) Parallel straight lines; (b) straight lines passing through the origin excluding this origin; (c) hyperbolas whose axes coincide with the coordinate axes Ox and Oy; (d) parabolas with a common axis Oy. 3. (a) Parallel planes; (b) circular cylinders with a common axis Ox. 4. (a) $du=2[(x-y)\,dx-(x+y)\,dy]$; (b) $du=\frac{y(y\,dx-x\,dy)}{(x^2+y^2)^{\frac{3}{2}}}$; (c) $du=\frac{y\,dx-x\,dy}{2|y|\sqrt{xy}}$. 5. $\frac{\partial u}{\partial l}=2\cos\alpha-\sin\alpha$, 2, $\frac{1}{\sqrt{2}}$, -1, $-\frac{3}{\sqrt{2}}$, -2, $-\frac{1}{\sqrt{2}}$, 1, $\frac{3}{\sqrt{2}}$, 2; $|\operatorname{grad} u(M_0)|=\sqrt{5}$. 6. $|\operatorname{grad} u(A)|\approx 50.5°/\text{km}$. 7. (a) $\frac{\partial^2 u}{\partial x^2}=-\frac{1+\sin^2 x}{4\sqrt{\sin^3 x}}$, $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{1}{y}$, $\frac{\partial^2 u}{\partial y^2}=-\frac{x}{y^2}$; (b) $\frac{\partial^2 u}{\partial x^2}=0$, $\frac{\partial^2 u}{\partial y^2}=0$, $\frac{\partial^2 u}{\partial z^2}=\frac{6xy}{z^4}$, $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{1}{z^2}$, $\frac{\partial^2 u}{\partial x\,\partial z}=-\frac{2y}{z^3}$, $\frac{\partial^2 u}{\partial y\,\partial z}=-\frac{2x}{z^3}$; (c) $\frac{\partial^2 u}{\partial x^2}=\frac{-1}{(x+y^2)^2}\cdot\frac{\partial^2 u}{\partial x\,\partial y}=-\frac{2y}{(x+y^2)^2}$, $\frac{\partial^2 u}{\partial y^2}=\frac{2(x-y^2)}{(x+y^2)^2}$; (d) $\frac{\partial^2 u}{\partial x^2}=-\frac{2}{9x\sqrt[3]{x^2}}$, $\frac{\partial^2 u}{\partial x\,\partial y}=0$, $\frac{\partial^2 u}{\partial y^2}=-\frac{2y}{(1+y^2)^2}$. 9. 24.8 N/dm^3 $\pm$ 2.6 N/dm^3. 11. $x=y=z=\sqrt[3]{a}$. 12. Dimensions of parallelepiped: $\frac{2a}{\sqrt{3}}$, $\frac{2a}{\sqrt{3}}$, $\frac{a}{\sqrt{3}}$. 13. Dimensions of parallelepiped $\frac{2r\sqrt{2}}{3}$, $\frac{2r\sqrt{2}}{3}$, $\frac{h}{3}$. 14. $y=198.77-5.06x$. 15. $x^2\approx 6x-\frac{26}{3}$. 16. $\min u=0=u(-0.6, -0.8)$; $\max u=10=u(0.6; 0.8)$.

Chapter 21

1. Diverges. 2. Diverges. 3. Converges. 4. Converges. 5. Diverges. 6. Converges. 7. Converges. 8. Converges. 9. Diverges. 10. Converges. 11. Converges. 12. Converges. 13. Converges. 14. Converges. 15. Diverges. 16. Converges. 17. $-1<x\leqslant 1$. 18. $-\infty<x<+\infty$. 19. $x=0$. 20. $-1\leqslant x<3$. 21. $a^x=1+x\ln a+\frac{x^2\ln^2 a}{2!}+\ldots$ $(-\infty<x<+\infty)$. 22. $\sin^2 x=\frac{2x^2}{2!}-\frac{2^3x^4}{4!}+\frac{2^5x^6}{6!}-\ldots$ $(-\infty<x<+\infty)$. 23. $\sqrt{x}=1+\frac{1}{2}(x-1)-\frac{1}{2\cdot 4}(x-1)^2+$

$+\frac{1\cdot3}{2\cdot4\cdot6}(x-1)^3+\ldots$ $(0\leqslant x\leqslant 2)$. 24. $e^{-x^2}=1-x^2+\frac{x^4}{2!}-\ldots$ $(-\infty<x<+\infty)$. 25. *Hint.* $\cos^2 x=\frac{1}{2}(1+\cos 2x)$ $\cos^2 x=1-\frac{2x^2}{2!}+\frac{2^3x^4}{4!}-\frac{2^5x^6}{6!}+\ldots$ $(-\infty<x<+\infty)$. 26. $\frac{1}{(1-x)^2}=1+2x+3x^2+\ldots$ $(-1<x<1)$. 27. $\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{1\cdot3}{2\cdot4}x^4+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6+\ldots$ $(-1<x<1)$. 28. (a) 1.649; (b) 0.309; (c) 1.037. 29. 0.5450. 30. $f(x)\approx x-\frac{x^2}{4}+\frac{x^3}{72}$. 31. (a) $\frac{x}{2}=\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots$ $(-\pi<x<\pi)$; (b) $x^2=\frac{\pi^2}{3}-4\left(\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\ldots\right)$ $(-\pi\leqslant x\leqslant\pi)$.

Chapter 22

3. $y=\frac{3}{4}x^2$. 4. (a) $x^2+y^2=C$; (b) $xy=C$; (c) $y=C+\ln x$; (d) $y=-2+Ce^x$; (e) $y=-\ln(C-e^x)$. 5. (a) $y^2=2x^2\ln Cx$; (b) $y=xe^{1+Cx}$. 6. $y=\frac{1}{2}(x^2-1)$. 7. (a) $y=\frac{x^3}{2}+Cx$; (b) $x=Ce^y-(y^2+2y+2)$. 8. $y=\sin x$. 9. $x^2+y^2=C$. 10. $y=e^{\frac{x-2}{4}}$. 11. $Q_t=Q_0\cdot\left(\frac{1}{2}\right)^{\frac{t}{1600}}$. 12. 100 g. 13. $y(2)=1.122$; exact value of $y(2)=1+e^{-2}=1.135$. 14. $y=C_1+C_2x-\sin x$. 15. $y=C_1\sin(x+C_2)$. 16. $y=C_1+\frac{(x+C_2)^2}{4C_1}$. 17. $y=1+\frac{x}{2}+\frac{x^3}{6}$. 18. (a) $y=1+x+\frac{1}{2}x^2+\frac{1}{4}x^3+\ldots$; (b) $y=x+\frac{1}{12}x^4+\frac{1}{504}x^7+\ldots$ 19. $y=C_1e^x+C_2e^{-2x}$. 20. $y=e^{-x}(C_1\cos x+C_2\sin x)$. 21. $y=e^{-\frac{x}{2}}\left(C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x\right)$. 22. $y=e^{\frac{x}{2}}(C_1+C_2x)$. 23. $y=2\cos x-\sin x$. 24. $y=\frac{1}{3}e^{2x}+C_1e^x+C_2e^{-x}$. 25. $y=\frac{1}{3}\sin x+C_1\cos 2x+C_2\sin 2x$. 26. $y=\frac{1}{6}x^2+\frac{5}{18}x+\frac{19}{108}+C_1e^{2x}+C_2e^{3x}$. 27. $y=1+x-e^x\cos x$. 28. $x=C_1e^{-\frac{pt}{2}}\cos\left(t\sqrt{q-\frac{p^2}{4}}+C_2\right)$, where $q=k_1/m$, $p=k_2/m$ and k_1, k_2 are proportionality factors.

Chapter 23

1. (a) $3\sqrt{5}/2$; (b) $(2\sqrt{2}-1)/3$; (c) $\pi R^5/4$; (d) $a[(\pi+4)\sqrt{2}-8]$. 2. $3\frac{1}{3}$. 3. πR. 4. $x_0=0$, $y_0=\frac{2}{\pi}R$. 5. $\frac{2^7}{3^6}\cdot\frac{848}{105}\approx 1.42$. 6. $I_x=\frac{256}{15}a^3$. 7. (a) 16/15; (b) 2π; (c) 2π. 8. $-9/2$. 9. (a) -2; (b) 0; (c) 3/2; (d) e^2-1. 10. $-2\pi ab$. 11. $A=\frac{k}{2}(a^2-b^2)$. 12. $-\left(\frac{\pi}{2}-1\right)$. 13. $\ln\frac{b}{a}$. 14. $31\frac{1}{2}$.

Chapter 24

1. (a) 3/20; (b) $(1-e^{-1})^2$; (c) $\pi/2$. 2. (a) $4(2\sqrt{2}-1)/15$; (b) 2/5; (c) 2π. 3. (a) $\int_0^1 dx \int_0^{2x} f(x, y)\,dy = \int_0^2 dy \int_{\frac{y}{2}}^1 f(x, y)\,dx$; (b) $\int_0^1 dx \int_0^1 f(x, y)\,dy + \int_1^2 dx \times$ $\times \int_0^{2-x} f(x, y)\,dy = \int_0^1 dy \int_0^{2-y} f(x, y)\,dx$; (c) $\int_0^1 dx \int_0^{\sqrt{1-x^2}} f(x, y)\,dy = \int_0^1 dy \times$ $\times \int_0^{\sqrt{1-y^2}} f(x, y)\,dx$; (d) $\int_{-2}^2 dx \int_{x^2}^4 f(x, y)\,dy = \int_0^4 dy \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\,dx$; (e) $\int_0^2 dx \times$ $\times \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} f(x, y)\,dy = \int_{-1}^1 dy \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} f(x, y)\,dx$; 4. (a) $\int_0^1 dy \int_y^{\sqrt{y}} f(x, y)\,dx$; (b) $\int_1^e dx \int_0^{\ln x} f(x, y)\,dy$; (c) $\int_0^1 dy \int_{\arcsin y}^{\pi-\arcsin y} f(x, y)\,dx$. 5. (a) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int_0^1 r^4 \sin\varphi \times$ $\times \cos^2\varphi\,dr$; (b) $\int_{-\arctan\frac{1}{2}}^{\arctan\frac{1}{2}} d\varphi \int_0^{2\sec\varphi} r^2\,dr$; (c) $\int_0^{\frac{\pi}{4}} d\varphi \int_0^{\frac{\sin\varphi}{\cos^2\varphi}} rf[r(\cos\varphi+\sin\varphi)]\,dr$.

6. (a) $\frac{1}{3}$; (b) $\frac{4}{21}$; (c) $2-\sqrt{2}\ln(1+\sqrt{2})$; (d) 16/5. 7. (a) $2\pi^2$; (b) $\pi/4$. 8. (a) 1/3; (b) $(3\pi-4)/6$; (c) 88/105; (d) 32/9; (e) $2abc/3$. 9. $I_x = I_y = \frac{\pi}{4}R^4$. 10. $x_0 = 0$, $y_0 = \frac{3}{5}a$. 11. 1/48. 12. 28/3.

Chapter 25

1. $A+A=A$, $AA=A$, $A+C=A$, $AC=A$. 2. $P(A)=\frac{1}{6}$, $P(B)=\frac{1}{2}$, $P(C)=\frac{2}{3}$. 3. $\geqslant 790$. 4. (a) $\frac{3}{7}$; (b) $\frac{1}{2}$. 5. (a) 80%; (b) $\approx 97\%$. 6. (a) 0.992; (b) 0.876; (c) 0.008. 8. $P_0=\frac{32}{243}$, $P_1=\frac{80}{243}$, $P_2=\frac{80}{243}$, $P_3=\frac{40}{243}$, $P_4=\frac{10}{243}$, $P_5=\frac{1}{243}$. 9. $\geqslant 31$. 10. $M=1.31$, $D=0.0049$, $\sigma=0.07$. 12. $\Phi(x)=0$ for $-\infty<x<0$, $\Phi(x)=\frac{1}{4}x$ for $0\leqslant x\leqslant 1$, $\Phi(x)=\frac{3}{4}x-\frac{1}{2}$ for $1\leqslant x\leqslant 2$, $\Phi(x)=1$ for $2<x<+\infty$, $M(X)=1\frac{1}{4}$, $D(X)=\frac{13}{48}$. 13. $\varphi(x)=\frac{1}{2l}$, $M(X)=0$, $D(X)=\frac{1}{3}l^2$, $\sigma(X)=\frac{l}{\sqrt{3}}\approx 0.58l$. 15. ≈ 0.384. 16. The number of standard packages must be approximately 9540.

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